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Fixed Point, Coincidence Point and Common Fixed Point Theorems under Various Expansive Conditions in Parametric Metric Spaces and Parametric b-Metric Spaces

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ABSTRACT

In this article, we establish some fixed point, common fixed point and coincidence point theorems for expansive type mappings in parametric metric spaces and parametric b-metric spaces. The presented theorems extend, generalize and improve many existing results in the literature. Also, we introduce some examples the support the validity of our results.

Keywords: *Coincidence point, fixed point, common fixed point, parametric metric space, parametric b-metric space, weakly compatible maps.*

1. INTRODUCTION

Fixed point theory has fascinated many researchers since 1922 with the celebrated Banach's fixed point theorem. There exists a vast literature on the topic and is a very active field of research at present. Theorems concerning

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the existence and properties of fixed points are known as fixed point theorems. Such theorems are very important tool for proving the existence and eventually the uniqueness of the solutions to various mathematical models (integral and partial differential equations, variational inequalities).

The concept of metric spaces has been generalized in many directions. The notion of a b-metric space was studied by Czerwik in [25-26] and a lot of fixed point results for single and multivalued mappings by many authors have been obtained in (ordered) b-metric spaces (see, e.g., [27]-[28]). The concept of fuzzy set was introduced by Zadeh [1] in 1965. In 1975, Kramosil and Michalek [2] introduced the notion of fuzzy metric space, which can be regarded as a generalization of the statistical (probabilistic) metric space. This work has provided an important basis for the construction of fixed point theory in fuzzy metric spaces.

In 2004, Park introduced the notion of intuitionistic fuzzy metric space [12]. He showed that, for each intuitionistic fuzzy metric space (X, M, N, $*$, $\diamond$), the topology generated by the intuitionistic fuzzy metric (M, N) coincides with the topology generated by the fuzzy metric M. For more details on intuitionistic fuzzy metric space and related results we refer the reader to [12–20]. The study of expansive mappings is a very interesting research area in fixed point theory. In 1984, Wang et.al [37] introduced the concept of expanding mappings and proved some fixed point theorems in complete metric spaces. In 1992, Daffer and Kaneko [36] defined an expanding condition for a pair of mappings and proved some common fixed point theorems for two mappings in complete metric spaces. Chintaman and Jagannath [38] introduced several meaningful fixed point theorems for one expanding mapping.

In this paper, we present some new fixed point, coincidence point and common fixed point theorems under various expansive conditions in parametric metric spaces and parametric b-metric spaces. These results improve and generalize some important known results in [29-38]. Some related results and illustrative some examples to highlight the realized improvements are also furnished.

2. DEFINITIONS AND PRELIMINARIES

Throughout this paper $\mathbb{R}$ and $\mathbb{R}^+$ will represents the set of real numbers and nonnegative real numbers, respectively.

In 2014, Hussain et al. [16] defined and studied the concept of parametric metric space as follows.

Definition 2.1 Let X be a nonempty set and $\mathcal{P} : X \times X \times (0,+\infty) \to [0,+\infty)$ be a function. We say $\mathcal{P}$ is a parametric metric on X if,

(1) $\mathcal{P}(x,y,t) = 0$ for all $t > 0$ if and only if $x = y$;

(2) $\mathcal{P}(x,y,t) = \mathcal{P}(y,x,t)$ for all $t > 0$;

(3) $\mathcal{P}(x,y,t) \leq \mathcal{P}(x,z,t) + \mathcal{P}(z,y,t)$ for all $x,y,z \in X$ and all $t > 0$:

and one says the pair $(X,\mathcal{P})$ is a parametric metric space.

The following definitions are required in the sequel which can be found in [16].

Definition 2.2 Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a parametric metric space $(\mathrm{X},\mathcal{P})$.

1. $\{x_n\}_{n=1}^{\infty}$ is said to be convergent to $x \in X$, written as $\lim_{n\to\infty} x_n = x$, for all $t > 0$, if $\lim_{n\to\infty} \mathcal{P}(x_n,x,t) = 0$.
2. $\{x_n\}_{n=1}^{\infty}$ is said to be a Cauchy sequence in X if for all $t > 0$, if $\lim_{n,m\to\infty} \mathcal{P}(x_n,x_m,t) = 0$.
3. $(\mathrm{X},\mathcal{P})$ is said to be complete if every Cauchy sequence is a convergent sequence.

Definition 2.3 Let $(\mathrm{X},\mathcal{P})$ be a parametric metric space and $T: X \to X$ be a mapping. We say T is a continuous mapping at x in X, if for any sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} Tx_n = Tx$.

Example 2.4 Let X denote the set of all functions $f : (0,+\infty) \to \mathbb{R}$. Define $\mathcal{P} : X \times X \times (0,+\infty) \to [0,+\infty)$ by

$$\mathcal{P}(f,g,t) = |f(t) - g(t)|$$

$\forall\, f,g \in X$ and all $t > 0$. Then $\mathcal{P}$ is a parametric metric on X and the pair $(X,\mathcal{P})$ is a parametric metric space.

Very recently, Hussain et al. [21] introduced the concept of parametric b-metric space as follows.

Definition 2.5 Let X be a nonempty set, $s \geq 1$ be a real number and $\mathcal{P} : X \times X \times (0,+\infty) \to [0,+\infty)$ be a function. We say $\mathcal{P}$ is a parametric b-metric on X if,

(1) $\mathcal{P}(x,y,t) = 0$ for all $t > 0$ if and only if $x = y$;

(2) $\mathcal{P}(x,y,t) = \mathcal{P}(y,x,t)$ for all $t > 0$;

(3) $\mathcal{P}(x,y,t) \leq s[\mathcal{P}(x,z,t) + \mathcal{P}(z,y,t)]$ for all $x,y,z \in X$ and all $t > 0$, where $s \geq 1$.

and one says the pair $(X,\mathcal{P},)$ is a parametric metric space with parameter $s \geq 1$.

Obviously, for $s = 1$, parametric b-metric reduces to parametric metric.

The following definitions will be needed in the sequel which can be found in [21].

Definition 2.6 Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a parametric b-metric space $(\mathrm{X},\mathcal{P},s)$.

1. $\{x_n\}_{n=1}^{\infty}$ is said to be convergent to $x \in X$, written as $\lim_{n\to\infty} x_n = x$, for all $t > 0$, if $\lim_{n\to\infty} \mathcal{P}(x_n, x, t) = 0$.
2. $\{x_n\}_{n=1}^{\infty}$ is said to be a Cauchy sequence in X if for all $t > 0$, if $\lim_{n,m\to\infty} \mathcal{P}(x_n, x_m, t) = 0$.
3. $(X, \mathcal{P})$ is said to be complete if every Cauchy sequence is a convergent sequence.

Example 2.6 Let $X = [0, +\infty)$ and define $\mathcal{P} : X \times X \times (0, +\infty) \to [0, +\infty)$ by $\mathcal{P}(x, y, t) = t(x - y)^p$. Then $\mathcal{P}$ is a parametric b-metric with constant $s = 2^p$. In fact, we only need to prove (3) in Definition 2.5 as follows: let $x, y, z \in X$. Set $u = x - z, v = z - y$, so $u + v = x - y$. From the inequality $(a + b)^p \le (2\, max\{a, b\})^p \le 2^p(a^p + b^p), \forall\, a, b \ge 0$, we have

$$\begin{aligned}\mathcal{P}(x, y, t) &= t(x - y)^p \\ &= t(u + v)^p \\ &\le 2^p t(u^p + v^p) \\ &= 2^p(t(\,x - z)^p + t(z - y)^p) \\ &= s\big(\mathcal{P}(x, z, t) + \mathcal{P}(z, y, t)\big)\end{aligned}$$

with $s = 2^p > 1$.

Definition 2.7 Let $(X, \mathcal{P}, s)$ be a parametric b-metric space and $T: X \to X$ be a mapping. We say T is a continuous mapping at x in X, if for any sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} Tx_n = Tx$.

In general, a parametric b-metric function for $s > 1$ is not jointly continuous in all its variables.

3. FIXED POINT RESULTS IN PARAMETRIC METRIC SPACE

In this section, we first prove some unique fixed point results satisfying expansive condition by considering surjective self-mapping in the context of parametric metric space.

We begin with a simple but a useful lemma.

Lemma 3.1 Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a parametric metric space $(X, \mathcal{P})$ such that

(1) $$\mathcal{P}(x_n, x_{n+1}, t) \le \lambda\, \mathcal{P}(x_{n-1}, x_n, t)$$

where $\lambda \in [0, 1)$ and $n = 1, 2, \ldots$. Then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(X, \mathcal{P})$.

Proof Let $m > n \ge 1$. It follows that

(2) $$\begin{aligned}\mathcal{P}(x_n, x_m, t) &\le \mathcal{P}(x_n, x_{n+1}, t) + \mathcal{P}(x_{n+1}, x_{n+2}, t) \\ &\quad + \cdots \ldots \ldots \ldots + \mathcal{P}(x_{m-1}, x_m, t) \\ &\le (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1})\, \mathcal{P}(x_0, x_1, t) \\ &\le \frac{\lambda^n}{1-\lambda}\, \mathcal{P}(x_0, x_1, t)\end{aligned}$$

for all $t > 0$. Since $\lambda < 1$. Assume that $\mathcal{P}(x_0, x_1, t) > 0$. By taking limit as $m, n \to +\infty$ in above inequality we get

(3) $$\lim_{n,m\to\infty} \mathcal{P}(x_n, x_m, t) = 0.$$

Therefore, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Also, if $\mathcal{P}(x_0, x_1, t) = 0$, then $\mathcal{P}(x_n, x_m, t) = 0$ for all $m > n$ and hence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X.

Now, our first main results as follows.

Theorem 3.2 Let $(X, \mathcal{P})$ be a complete parametric metric space and $T: X \to X$ be a surjection. Suppose that there exist $a, b, c \ge 0$ with $a + b + c > 1$ such that

(4) $$\mathcal{P}(Tx, Ty, t) \ge a\, \mathcal{P}(x, y, t) + b\, \mathcal{P}(x, Tx, t) + c\, \mathcal{P}(y, Ty, t)$$

$\forall\, x, y \in X$ with $x \ne y$ and all $t > 0$. Then T has a fixed point in X.

Proof Under the assumption. It is clear that T is injective. Let G be the inverse mapping of T. Choose $x_0 \in X$, set $x_1 = G(x_0)$, $x_2 = G(x_1) = G^2(x_0), \ldots \ldots \ldots, x_{n+1} = G(x_n) = G^{n+1}(x_0) \ldots \ldots$. Without loss of generality, we assume that $x_{n-1} \ne x_n$ for all $n = 1, 2, \ldots$ (otherwise, if there exists some n_0 such that $x_{n_0 - 1} = x_{n_0}$, then x_{n_0} is a fixed point of T).

It fallows that from condition (4)

(5) $$\begin{aligned}\mathcal{P}(x_{n-1}, x_n, t) &= \mathcal{P}(TT^{-1}x_{n-1}, TT^{-1}x_n, t) \\ &\ge a\, \mathcal{P}(T^{-1}x_{n-1}, T^{-1}x_n, t) \\ &\quad + b\, \mathcal{P}(T^{-1}x_{n-1}, TT^{-1}x_{n-1}, t) \\ &\quad + c\, \mathcal{P}(T^{-1}x_n, TT^{-1}x_n, t) \\ &= a\, \mathcal{P}(Gx_{n-1}, Gx_n, t) \\ &\quad + b\, \mathcal{P}(Gx_{n-1}, x_{n-1}, t) \\ &\quad + c\, \mathcal{P}(Gx_n, x_n, t) \\ &= a\, \mathcal{P}(x_n, x_{n+1}, t) + b\, \mathcal{P}(x_n, x_{n-1}, t) \\ &\quad + c\, \mathcal{P}(x_{n+1}, x_n, t)\end{aligned}$$

Hence

(6) $$(1 - b)\, \mathcal{P}(x_{n-1}, x_n, t) \ge (a + c)\, \mathcal{P}(x_{n+1}, x_n, t)$$

If $a + c = 0$, then $b > 0$. The above inequality implies that a negative number is greater then or equal to zero. This is impossible. So, $a + c \ne 0$ and $(1 - b) > 0$. Therefore,

(7) $$\mathcal{P}(x_{n+1}, x_n, t) \le k\, \mathcal{P}(x_{n-1}, x_n, t)$$

where $k = \frac{1-b}{a+c} < 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $t > 0$. Repeating (7) n-times, we get

(8) $$\mathcal{P}(x_{n+1}, x_n, t) \le k^n\, \mathcal{P}(x_0, x, t)$$

for all $t > 0$. By Lemma 3.1, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since $(X, \mathcal{P})$ is a complete parametric metric space, there exists $x^\star \in X$ such that $x_n \to x^\star$ as $n \to \infty$.

Now since T is surjective map. So there exists a point y in X such that $x^\star = Ty$. Consider

$$(9)\quad \mathcal{P}(x_n, x^\star, t) = \mathcal{P}(Tx_{n+1}, Ty, t) \geq a\,\mathcal{P}(x_{n+1}, y, t) + b\,\mathcal{P}(x_{n+1}, Tx_{n+1}, t) + c\,\mathcal{P}(y, Ty, t) = a\,\mathcal{P}(x_{n+1}, y, t) + b\,\mathcal{P}(x_{n+1}, x_n, t) + c\,\mathcal{P}(y, x^\star, t)$$

which implies that as $n \to +\infty$

$$(10)\qquad 0 \geq (a+c)\,\mathcal{P}(y, x^\star, t)$$

Hence $y = x^\star$. This gives that $x^\star$ is a fixed point of T. This completes the proof.

Now we give an example illustrating Theorem 3.2.

Example 3.3 Let $X = [0, +\infty)$ be endowed with parametric metric,

$$\mathcal{P}(x, y, t) = \begin{cases} t\max\{x, y\}, & x \neq y \\ 0, & x = y \end{cases}$$

for all $x, y \in X$ and $t > 0$. Define $T: X \to X$ by $Tx = \frac{5}{2}x$. Obviously, T is continuous surjective map on X. So for $a = 4, b = -2$ and $c = \frac{3}{4}$ all the conditions of Theorem 3.2 are satisfied. Therefore $x^\star = 0$ is the unique fixed point of T.

Setting $b = c$ and $a = k$ in Theorem 3.2, we can obtain the following result.

Corollary 3.4 Let $(X, \mathcal{P})$ be a complete parametric metric space and $T: X \to X$ be a surjection. Suppose that there exist a constant $k > 1$ such that

$$(11)\qquad \mathcal{P}(Tx, Ty, t) \geq k\mathcal{P}(x, y, t)$$

$\forall\, x, y \in X$ and all $t > 0$. Then T has a unique fixed point in X.

Proof From Theorem 3.2, it follows that T has a fixed point $x^\star$ in X by setting $b = c = 0$ and $a = k$ in condition (4).

Uniqueness. Suppose that $x^\star \neq y^\star$ is also another fixed point of T, then from condition (11), we obtain

$$(12)\qquad \mathcal{P}(x^\star, y^\star, t) = \mathcal{P}(Tx^\star, Ty^\star, t) \geq k\mathcal{P}(x^\star, y^\star, t)$$

which implies $\mathcal{P}(x^\star, y^\star, t) = 0$, that is $x^\star = y^\star$. This completes the proof.

Corollary 3.5 Let $(X, \mathcal{P})$ be a complete parametric metric space and $T: X \to X$ be a surjection. Suppose that there exist a positive integer n and a real number $k > 1$ such that

$$(13)\qquad \mathcal{P}(T^n x, T^n y, t) \geq k\,\mathcal{P}(x, y, t)$$

$\forall\, x, y \in X$ and all $t > 0$. Then T has a unique fixed point in X.

Proof From Corollary 3.4, T^n has a fixed point $x^\star$. But $T^n(Tx^\star) = T(T^n x^\star) = Tx^\star$, So $Tx^\star$ is also a fixed point of T^n. Hence $Tx^\star = x^\star$, $x^\star$ is a fixed point of T. Since the fixed point of T is also fixed point of T^n, the fixed point of T is unique.

Theorem 3.6 Let $(X, \mathcal{P})$ be a complete parametric space and $T: X \to X$ be a continuous surjection. if there exist a constant $k > 1$ such that for any $x, y \in X$, there is

$$(14)\qquad \mathcal{M}_t(x, y) \in \{\mathcal{P}(x, y, t), \mathcal{P}(x, Tx, t), \mathcal{P}(y, Ty, t)\}$$

satisfying

$$(15)\qquad \mathcal{P}(Tx, Ty, t) \geq k\,\mathcal{M}_t(x, y)$$

$\forall\, x, y \in X$ and all $t > 0$. Then T has a unique fixed point in X.

Proof Similar to the proof of the Theorem 3.2, we can obtain a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_{n-1} = Tx_n$. Without loss of generality, we assume that $x_{n-1} \neq x_n$ for all $n = 1, 2, \ldots$.(otherwise, if there exists some n_0 such that $x_{n_0-1} \neq x_{n_0}$, then x_{n_0} is a fixed point of T). It follows that from condition (15)

$$(16)\qquad \mathcal{P}(x_{n-1}, x_n, t) = \mathcal{P}(Tx_n, Tx_{n+1}, t) \geq k\,\mathcal{M}_t(x_n, x_{n+1})$$

where

$$\mathcal{M}_t(x_n, x_{n+1}) = \{\mathcal{P}(x_n, x_{n+1}, t), \mathcal{P}(x_n, x_{n-1}, t)\}$$

Now we have to consider the following two cases:

- If $\mathcal{M}_t(x_n, x_{n+1}) = \mathcal{P}(x_n, x_{n-1}, t)$ then from (14), we have
$$\mathcal{P}(x_{n-1}, x_n, t) \geq k\mathcal{P}(x_n, x_{n-1}, t)$$

which implies $\mathcal{P}(x_{n-1}, x_n, t) = 0$ that is $x_{n-1} = x_n$. This is a contradiction.

- If $\mathcal{M}_t(x_n, x_{n+1}) = \mathcal{P}(x_n, x_{n+1}, t)$ then from (14), we have
$$\mathcal{P}(x_{n-1}, x_n, t) \geq k\,\mathcal{P}(x_n, x_{n+1}, t)$$

Proceeding like Theorem 2.2, we obtain that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in complete parametric metric space $(X, \mathcal{P})$. So the sequence $\{x_n\}_{n=1}^{\infty}$ converges to a point $x^\star \in X$. Since T is continuous, it is clear that $x^\star$is a fixed point of T. This is completes of the proof.

4. COMMON FIXED POINT RESULTS IN PARAMETRIC METRIC SPACE

In this section, now we establish that common fixed points for a pair of two weakly compatible self-mappings satisfying expansive condition are proved in the frame of parametric metric spaces.

In [22] Jungck introduced the concept of commuting maps. In [23] Jungck introduced the concept of compatible mappings which generalize the concept of commuting maps. Jungck in [24] further generalized the concept of compatible maps as follows.

Definition 4.1 Let S and T be two self-mappings on a nonempty set X. Then S and T are said to be weakly compatible if they commute at all of their coincidence points; that is, $Sx = Tx$ for some $x \in X$ and then $STx = TSx$.

Now, let us prove our result.

Theorem 4.2 Let $(X, \mathcal{P})$ be a complete parametric metric space. Let S and T be a weakly compatible self-mappings of X and $T(X) \subseteq S(X)$. Suppose that there exist $k > 1$ such that

(17) $\mathcal{P}(Sx, Sy, t) \geq k\,\mathcal{P}(Tx, Ty, t),\ \forall\, x, y \in X$.

If one of the subspaces T(X) or S(X) is complete, Then S and T have a unique common fixed point in X.

Proof Let $x_0 \in X$. Since $T(X) \subseteq S(X)$, choose x_1 such that $y_1 = Sx_1 = Tx_0$. In general choose x_{n+1} such that $y_{n+1} = Sx_{n+1} = Tx_n$, then from condition (17),

$$\begin{aligned}(18)\quad \mathcal{P}(y_{n+1}, y_{n+2}, t) &= \mathcal{P}(Tx_n, Tx_{n+1}, t)\\ &\leq \tfrac{1}{k}\,\mathcal{P}(Sx_n, Sx_{n+1}, t)\\ &= \tfrac{1}{k}\,\mathcal{P}(Tx_{n-1}, Tx_n, t)\\ &= \tfrac{1}{k}\,\mathcal{P}(y_n, y_{n+1}, t)\\ &= \lambda\,\mathcal{P}(y_n, y_{n+1}, t)\end{aligned}$$

where $\lambda = \frac{1}{k} < 1$. Repeating (18) $(n+1)$-times, we obtain

(19) $\mathcal{P}(y_{n+1}, y_{n+2}, t) \leq \lambda^{n+1}\,\mathcal{P}(y_0, y_1, t)$

Hence for $n > m$, we have

$$\begin{aligned}(20)\quad \mathcal{P}(y_n, y_m, t) &\leq \mathcal{P}(y_n, y_{n+1}, t) + \mathcal{P}(y_n, y_{n+1}, t)\\ &\quad + \cdots \ldots \ldots + \mathcal{P}(y_{m-1}, y_m, t)\\ &\leq (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1})\mathcal{P}(y_0, y_1, t)\\ &\leq \frac{\lambda^n}{1-\lambda}\,\mathcal{P}(y_0, y_1, t)\end{aligned}$$

for all $t > 0$. By taking $n, m \to \infty$ in the above inequality, we get $\lim_{m,n\to\infty} \mathcal{P}(y_n, y_m, t) = 0$. Therefore, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy Sequence. Since $(X, \mathcal{P})$ is a complete parametric metric space, there exists $x^\star \in X$ such that $y_n \to x^\star$ as $n \to +\infty$. Hence

(21) $\lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = x^\star$.

Since T(X) or S(X) is complete and $T(X) \subseteq S(X)$, there exists a point $p \in X$ such that $Sp = x^\star$. Now from (17), for all $t > 0$,

(22) $\mathcal{P}(Tp, Tx_n, t) \leq \frac{1}{k}\mathcal{P}(Sp, Sx_n, t)$

Proceeding to the limit as $n \to +\infty$ in (22), we have

(23) $\mathcal{P}(Tp, x^\star, t) \leq \frac{1}{k}\mathcal{P}(Sp, x^\star, t)$

for all $t > 0$, which implies that $Tp = x^\star$. Therefore $Tp = Sp = x^\star$. Since T and S are weakly compatible, therefore $STp = TSp$, that is $Sx^\star = Tx^\star$.

Now we show that $x^\star$ is a fixed point of S and T. From (17), we have

(24) $\mathcal{P}(Sx^\star, Sx_n, t) \geq k\,\mathcal{P}(Tx^\star, Tx_n, t)$

Proceeding to the limit as $n \to \infty$ in (24), we have

(25) $\mathcal{P}(Sx^\star, x^\star, t) \geq k\,\mathcal{P}(Tx^\star, x^\star, t)$

for all $t > 0$, which implies that $Sx^\star = x^\star$. Hence $Sx^\star = Tx^\star = x^\star$.

Uniqueness. Suppose that $x^\star \neq y^\star$ is also another common fixed point of S and T. Then

(26) $\mathcal{P}(Sx^\star, Sy^\star, t) \geq k\,\mathcal{P}(Tx^\star, Ty^\star, t)$

for all $t > 0$, which implies that $x^\star = y^\star$. This completes the proof.

Example 4.3 Let $X = [0, +\infty)$ be endowed with parametric metric,

$$\mathcal{P}(x, y, t) = \begin{cases} t\max\{x, y\}, & x \neq y \\ 0, & x = y \end{cases}$$

for all $x, y \in X$ and all $t > 0$. Define $S, T: X \to X$ by $S(x) = \frac{x}{2}$, $T(x) = \frac{x}{6}$ for all $x \in X$. Then $T(X) \subseteq S(X)$ and S(X) is complete. Further

$$\begin{aligned}\mathcal{P}(Sx, Sy, t) &= t\max\{Sx, Sy\}\\ &= t\max\left\{\frac{x}{2}, \frac{y}{2}\right\}\\ &= \frac{1}{2}t\max\{x, y\}\\ &\geq k\,t\max\left\{\frac{x}{6}, \frac{y}{6}\right\}\\ &= k\,\mathcal{P}(Tx, Ty, t)\end{aligned}$$

for $1 < k < 3$ and (17) is satisfied. Moreover mappings are weakly compatible at $x = 0$. Thus all conditions of Theorem 4.2 are satisfied and $x^\star = 0$ is the unique common fixed point of S and T.

Now, we prove the following common fixed point theorem, which is generalization of Theorem 2.2 of W. Shatanwi and F. Awawdeh [42] in the setting of parametric metric space.

Theorem 4.4 Let $T, S: X \to X$ be two surjective mappings of a complete parametric metric space $(X, \mathcal{P})$. Suppose that T and S satisfying the following inequalities

(27) $\mathcal{P}(\mathrm{T}(\mathrm{Sx}), \mathrm{Sx}, \mathrm{t}) + \mathrm{k}\mathcal{P}(\mathrm{T}(\mathrm{Sx}), \mathrm{x}, \mathrm{t}) \geq \mathrm{a}\, \mathcal{P}(\mathrm{Sx}, \mathrm{x}, \mathrm{t})$

and

(28) $\mathcal{P}(\mathrm{S}(\mathrm{Tx}), \mathrm{Tx}, \mathrm{t}) + \mathrm{k}\mathcal{P}(\mathrm{S}(\mathrm{Tx}), \mathrm{x}, \mathrm{t}) \geq \mathrm{b}\mathcal{P}(\mathrm{Tx}, \mathrm{x}, \mathrm{t})$

for all $x \in X$, all $t > 0$ and some nonnegative real numbers a, b and k with $a > 1 + 2k$ and $b > 1 + 2k$. If T or S is continuous. Then T and S have a common fixed point.

Proof Let x_0 be an arbitrary point in X. since T is surjective, there exists $x_1 \in X$ such that $x_0 = Tx_1$. Also since S is surjective, there exists $x_2 \in X$ such that $x_2 = Sx_1$. Continuing this process, we construct a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that

(29) $x_{2n} = Tx_{2n+1}$ and $x_{2n+1} = Sx_{2n+2}$

for all $n \in \mathbb{N} \cup \{0\}$. Now for $n \in \mathbb{N} \cup \{0\}$, we have

(30) $$\mathcal{P}(T(Sx_{2n+2}), Sx_{2n+2}, t) + k\mathcal{P}(T(Sx_{2n+2}), x_{2n+2}, t) \geq a\mathcal{P}(Sx_{2n+2}, x_{2n+2}, t)$$

Thus we have

(31) $$\mathcal{P}(x_{2n}, x_{2n+1}, t) + k\mathcal{P}(x_{2n}, x_{2n+2}, t) \geq a\mathcal{P}(x_{2n+1}, x_{2n+2}, t)$$

Since

$$\mathcal{P}(x_{2n}, x_{2n+2}, t) \leq \mathcal{P}(x_{2n}, x_{2n+1}, t) + \mathcal{P}(x_{2n+1}, x_{2n+2}, t)$$

Hence from (31),

(32) $$\mathcal{P}(x_{2n+1}, x_{2n+2}, t) \leq \frac{1+k}{a-k}\mathcal{P}(x_{2n}, x_{2n+2}, t)$$

On other hand, we have

(33) $$\mathcal{P}(S(Tx_{2n+1}), Tx_{2n+1}, t) + k\,\mathcal{P}(S(Tx_{2n+1}), x_{2n+1}, t) \geq b\mathcal{P}(Tx_{2n+1}, x_{2n+1}, t)$$

Thus, we have

(34) $$\mathcal{P}(x_{2n-1}, x_{2n}, t) + k\mathcal{P}(x_{2n-1}, x_{2n+`1}, t) \geq b\mathcal{P}(x_{2n}, x_{2n+1}, t)$$

Since

$$\mathcal{P}(x_{2n-1}, x_{2n+`1}, t) \leq \mathcal{P}(x_{2n-1}, x_{2n}, t) + \mathcal{P}(x_{2n}, x_{2n+1}, t)$$

Hence from (34), we have

(35) $$\mathcal{P}(x_{2n}, x_{2n+1}, t) \leq \frac{1+k}{b-k}\mathcal{P}(x_{2n-1}, x_{2n}, t)$$

Let

$$\lambda = \max\left\{\frac{1+k}{a-k}, \frac{1+k}{b-k}\right\}$$

Then by combining (32) and (35), we have

(36) $$\mathcal{P}(x_n, x_{n+1}, t) \leq \lambda\mathcal{P}(x_{n-1}, x_n, t)$$

for all $n \in \mathrm{N} \cup \{0\}$ and for all $t > 0$. Repeating (34) n-times, we get

(37) $$\mathcal{P}(x_n, x_{n+1}, t) \leq \lambda^n\mathcal{P}(x_0, x_1, t)$$

for all $n \in \mathbb{N} \cup \{0\}$ and for all $t > 0$. By Lemma 3.1, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the complete parametric metric space $(X, \mathcal{P})$. Then there exists $x^\star \in X$ such that $x_n \to x^\star$ as $n \to +\infty$. Therefore $x_{2n+1} \to x^\star$ and $x_{2n+2} \to x^\star$ as $n \to +\infty$. Without loss of generality, we may assume that T is continuous, then $Tx_{2n+1} \to Tx^\star$ as $n \to +\infty$. But $Tx_{2n+1} = x_{2n} \to x^\star$ as $n \to +\infty$. Thus, we have $Tx^\star = x^\star$. Since S is surjective, there exists $u \in X$ such that $Su = x^\star$. Now

(38) $$\mathcal{P}(T(Su), Su, t) + k\mathcal{P}(T(Su), u, t) \geq a\mathcal{P}(Su, u, t)$$

implies that $k\mathcal{P}(x^\star, u, t) \geq a\mathcal{P}(x^\star, u, t)$. Thus

(39) $$\mathcal{P}(x^\star, u, t) \leq \frac{k}{a}\mathcal{P}(x^\star, u, t)$$

Since $a > k$, we conclude that $\mathcal{P}(x^\star, u, t) = 0$. So $x^\star = u$. Hence $Tx^\star = Sx^\star = x^\star$. Therefore $x^\star$ is a common fixed point of T and S.

By taking $b = a$ in above Theorem 4.4, we have the following result.

Corollary 4.5 Let $T, S: X \to X$ be two surjective mappings of a complete parametric metric space $(X, \mathcal{P})$. Suppose that T and S satisfying the following inequalities

(40) $$\mathcal{P}(T(Sx), Sx, t) + k\mathcal{P}(T(Sx), x, t) \geq a\, \mathcal{P}(Sx, x, t)$$

and

(41) $$\mathcal{P}(S(Tx), Tx, t) + k\mathcal{P}(S(Tx), x, t) \geq a\mathcal{P}(Tx, x, t)$$

for all $x \in X$, all $t > 0$ and some nonnegative real numbers a and k with $a > 1 + 2k$. If T or S is continuous. Then T and S have a common fixed point.

By taking $S = T$ in above Corollary 4.5, we have the following result.

Corollary 4.7 Let $T: X \to X$ be a surjective mapping of a complete parametric metric space $(X, \mathcal{P})$. Suppose that T satisfying the following inequality

(42) $$\mathcal{P}(T(Tx), Tx, t) + k\mathcal{P}(T(Tx), x, t) \geq a\, \mathcal{P}(Tx, x, t)$$

for all $x \in X$, all $t > 0$ and some nonnegative real numbers a and k with $a > 1 + 2k$. If T is continuous. Then T has a fixed point.

Now, we give an example.

Example 4.8 Let $X = \mathbb{R}^+$ be endowed with parametric metric $\mathcal{P}(x, y, t) = t|x - y|$ for all $x, y \in X$ and all $t > 0$. Then $(X, \mathcal{P})$ is a complete parametric metric space. Define $T: X \to X$ by $T(x) = 2x$ for all $x \in X$. Then T is a surjection on X. Note that

$$\mathcal{P}(T(Tx), Tx, t) + \mathcal{P}(T(Tx), x, t)$$

$$= t|4x - 2x| + t|4x - x|$$

$$\geq 4t|2x - x|$$

$$= a\,\mathcal{P}(Tx, x, t)$$

where $k = 1$ and $a = 4$. Clearly $4 = a > 1 + 2k = 3$. Then (42) is satisfied. Thus all conditions of Corollary 2.7 are satisfied and $x^\star = 0 \in X$ is a fixed point of T.

5. COINCIDENCE POINT RESULTS IN PARAMETRIC METRIC SPACE

In this section, we establish a new theorem of coincidence point for expansive mappings in parametric metric spaces under a set of conditions, which is generalization of Theorem 2.1 of W. Shatanwi and F. Awawdeh [42] in the setting of parametric metric space.

Theorem 5.1 Let $(X, \mathcal{P})$ be a parametric metric space. Let $T, f: X \to X$ be mappings satisfying

(43) $\mathcal{P}(Tx, Ty, t) \geq a\,\mathcal{P}(fx, fy, t) + b\,\mathcal{P}(fx, Tx, t)$

$$+c\,\mathcal{P}(fy, Ty, t)$$

for all $x, y \in X$ and all $t > 0$, where $a, b, c \geq 0$ with $a + b + c > 1$. Suppose the following hypotheses:

1) $b > 1$ or $c < 1$;
2) $f(X) \subseteq T(X)$;
3) $T(X)$ is a complete subspace of X.

Then T and f have a coincidence point.

Proof Let $x_0 \in X$. Since $f(X) \subseteq T(X)$, we choose $x_1 \in X$ such that $Tx_1 = fx_0$. Again we can choose $x_2 \in X$ such that $Tx_2 = fx_1$. Continuing in the same way, we construct a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $Tx_{n+1} = fx_n, \forall\, n \in \mathbb{N} \cup \{0\}$. If $fx_{m-1} = fx_m$ for $m \in N$, then $Tx_m = fx_m$. Thus x_m is a coincidence point of T and f.

Now assume that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. We have the following two cases:

Case (1) Suppose $b < 1$. By (43), we have

(44) $\mathcal{P}(fx_{n-1}, fx_n, t) = \mathcal{P}(Tx_n, Tx_{n+1}, t)$

$$\geq a\,\mathcal{P}(fx_n, fx_{n+1}, t)$$
$$+b\,\mathcal{P}(fx_n, Tx_n, t)$$
$$+c\,\mathcal{P}(fx_{n+1}, Tx_{n+1}, t)$$
$$= a\,\mathcal{P}(fx_n, fx_{n+1}, t)$$
$$+b\,\mathcal{P}(fx_n, fx_{n-1}, t)$$
$$+c\,\mathcal{P}(fx_{n+1}, fx_n, t)$$

Thus, we have

$$(1 - b)\mathcal{P}(fx_{n-1}, fx_n, t) \geq (a + c)\mathcal{P}(fx_{n+1}, fx_n, t)$$

Hence

(45) $\mathcal{P}(fx_{n+1}, fx_n, t) \leq \frac{1-b}{a+c}\,\mathcal{P}(fx_{n-1}, fx_n, t)$

Case (2) Suppose$c < 1$. Also from (43), we have

(46) $\mathcal{P}(fx_n, fx_{n-1}, t) = \mathcal{P}(Tx_{n+1}, Tx_n, t)$

$$\geq a\,\mathcal{P}(fx_{n+1}, fx_n, t)$$
$$+b\,\mathcal{P}(fx_{n+1}, Tx_{n+1}, t)$$
$$+c\,\mathcal{P}(fx_n, Tx_n, t)$$
$$= a\,\mathcal{P}(fx_n, fx_{n+1}, t)$$
$$+b\,\mathcal{P}(fx_{n+1}, fx_n, t)$$
$$+c\,\mathcal{P}(fx_n, fx_{n-1}, t)$$

Thus, we have

$$(1 - c)\mathcal{P}(fx_{n-1}, fx_n, t) \geq (a + b)\mathcal{P}(fx_{n+1}, fx_n, t)$$

Hence

(47) $\mathcal{P}(fx_{n-1}, fx_n, t) \leq \frac{1-c}{a+b}\,\mathcal{P}(fx_{n+1}, fx_n, t)$

In case (1), we let $\lambda_1 = \frac{1-b}{a+c}$ and in case (2), we let $\lambda_2 = \frac{1-c}{a+b}$. Define $\lambda = \max\{\lambda_1, \lambda_2\}$. Thus in both cases, we have $\lambda < 1$. Hence

(48) $\mathcal{P}(fx_{n+1}, fx_n, t) \leq \lambda\mathcal{P}(fx_{n-1}, fx_n, t)$

Repeating (48), n-times, we obtain

(49) $\mathcal{P}(fx_{n+1}, fx_n, t) \leq \lambda^n\mathcal{P}(fx_0, fx_1, t)$

So for $m > n$, we have

(50) $\mathcal{P}(fx_n, fx_m, t) \leq \mathcal{P}(fx_n, fx_{n+1}, t) + \mathcal{P}(fx_{n+1}, fx_{n+2}, t)$

$$+ \cdots \dots \dots \dots \dots \dots . + \mathcal{P}(fx_{m-1}, fx_m, t)$$
$$\leq (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1})\mathcal{P}(x_0, x_1, t)$$
$$\leq \frac{\lambda^n}{1-\lambda}\mathcal{P}(fx_0, fx_1, t)$$

for all $t > 0$. By taking limit as $n, m \to +\infty$ in above inequality (50), we get $\lim_{n,m\to+\infty} \mathcal{P}(fx_n, fx_m, t) = 0$ for all $t > 0$. Therefore $\{Tx_n\}_{n=1}^{\infty}$ is a Cauchy sequence in TX. Since TX is a complete subspace of X, there is $x^\star \in X$ such that $\{Tx_n\}_{n=1}^{\infty}$ converges $Tx^\star$ as $n \to +\infty$. Hence fx_n converges to $Tx^\star$ as $n \to +\infty$. Since $a + b + c > 1$, we have a, b and c are not all 0. So we have the following cases.

Case (1) If $a \neq 0$, then

$$\mathcal{P}(Tx_n, Tx^\star, t) \geq a\,\mathcal{P}(fx_n, fx^\star, t)$$
$$+b\,\mathcal{P}(fx_n, Tx_n, t)$$
$$+c\,\mathcal{P}(fx^\star, Tx^\star, t)$$
$$\geq a\,\mathcal{P}(fx_n, fx)$$

Hence

(51) $\mathcal{P}(fx_n, fx^\star, t) \le \frac{1}{a}\,\mathcal{P}(Tx_n, Tx^\star, t)$

Since $\frac{1}{a}\,\mathcal{P}(Tx_n, Tx^\star, t) \to 0$ as $n \to +\infty$. Thus $fx_n \to fx^\star$ as $n \to +\infty$. By uniqueness of limit, we have $Tx^\star = fx^\star$. Therefore T and f have a coincidence point.

Case (2) If $b \ne 0$, then

$$\begin{aligned}\mathcal{P}(Tx^\star, Tx_n, t) &\ge a\,\mathcal{P}(fx_n, fx^\star, t)\\ &+b\,\mathcal{P}(fx^\star, Tx^\star, t)\\ &+c\,\mathcal{P}(fx_n, Tx_n, t)\\ &\ge b\,\mathcal{P}(fx^\star, Tx^\star, t)\end{aligned}$$

Hence

(52) $\mathcal{P}(fx^\star, Tx^\star, t) \le \frac{1}{b}\,\mathcal{P}(Tx^\star, Tx_n, t)$

As similar proof of case (1), we can show that $fx^\star = Tx^\star$. thus f and T have a coincidence point.

Case (3) If $c \ne 0$, then

$$\begin{aligned}\mathcal{P}(Tx_n, Tx^\star, t) &\ge a\,\mathcal{P}(fx_n, fx^\star, t)\\ &+b\,\mathcal{P}(fx_n, Tx_n, t)\\ &+c\,\mathcal{P}(Tx^\star, fx^\star, t)\\ &\ge c\,\mathcal{P}(fx^\star, Tx^\star, t)\end{aligned}$$

Hence

(53) $\mathcal{P}(fx^\star, Tx^\star, t) \le \frac{1}{c}\,\mathcal{P}(Tx_n, Tx^\star, t)$

for all $t > 0$. As similar proof of case (1), we can show that $fx^\star = Tx^\star$. thus f and T have a coincidence point.

Setting $c = 0$ in Theorem 5.1, we can obtain the following result.

Corollary 5.2 Let $(X, \mathcal{P})$ be a parametric metric space. Let $T, f: X \to X$ be mappings satisfying

(53) $\mathcal{P}(Tx, Ty, t) \ge a\,\mathcal{P}(fx, fy, t) + b\,\mathcal{P}(fx, Tx, t)$

for all $x, y \in X$ and all $t > 0$, where $a, b \ge 0$ with $a + b > 1$ and $b > 1$. Suppose the following hypotheses:

1) $f(X) \subseteq T(X)$;
2) $T(X)$ is a complete subspace of X.

Then T and f have a coincidence point.

Setting $b = c = 0$ in Theorem 5.1, we can obtain the following corollary.

Corollary 5.3 Let $(X, \mathcal{P})$ be a parametric metric space. Let $T, f: X \to X$ be mappings satisfying

(54) $\mathcal{P}(Tx, Ty, t) \ge a\,\mathcal{P}(fx, fy, t)$

for all $x, y \in X$ and all $t > 0$, where $a > 1$. Suppose the following hypotheses:

1) $f(X) \subseteq T(X)$;
2) $T(X)$ is a complete subspace of X.

Then T and f have a coincidence point.

Setting $T = S$ in Theorem 5.1, we have the following corollary.

Corollary 5.4 Let $(X, \mathcal{P})$ be a parametric metric space. Let $T: X \to X$ be a surjective mapping satisfying

(55) $\mathcal{P}(Tx, Ty, t) \ge a\,\mathcal{P}(x, y, t) + b\,\mathcal{P}(x, Tx, t) + c\,\mathcal{P}(y, Ty, t)$

for all $x, y \in X$ and all $t > 0$, where $a, b, c \ge 0$ with $a + b + c > 1$. Suppose $b > 1$ or $c < 1$. Then T has a fixed point.

Setting $b = c = 0$ in Corollary 5.4, we can obtain the following corollary.

Corollary 5.5 Let $(X, \mathcal{P})$ be a parametric metric space. Let $T: X \to X$ be a surjective mapping satisfying

(56) $\mathcal{P}(Tx, Ty, t) \ge a\,\mathcal{P}(x, y, t)$

for all $x, y \in X$ and all $t > 0$, where $a > 1$. Then T has a fixed point.

Corollary 5.6 Let $(X, \mathcal{P})$ be a parametric metric space. Let $T: X \to X$ be a surjective mapping satisfying

(57) $\mathcal{P}(Tx, Ty, t) \ge a\,\mathcal{P}(x, y, t) + b\,\mathcal{P}(x, Tx, t)$

for all $x, y \in X$ and all $t > 0$, where $a, b \ge 0$ with $a + b > 1$. Suppose $b > 1$. Then T has a fixed point.

Corollary 5.7 Let $(X, \mathcal{P})$ be a parametric metric space. Let $T: X \to X$ be a surjective mapping satisfying

(58) $\mathcal{P}(Tx, Ty, t) \ge a\,\mathcal{P}(x, y, t) + b\,\mathcal{P}(y, Ty, t)$

for all $x, y \in X$ and all $t > 0$, where $a, b \ge 0$ with $a + b > 1$. Suppose $b > 1$. Then T has a fixed point.

Now, we introduce an example to support the validity of Corollary 5.3.

Example 5.8 Let $X = [0,1]$ be endowed with parametric metric $\mathcal{P}(x, y, t) = t|x - y|$ for all $x, y \in X$ and all $t > 0$. Then $(X, \mathcal{P})$is a complete parametric metric space. Define $T, f: X \to X$ by $T(x) = \frac{x}{4}$ and $f(x) = \frac{x}{16}$ for all $x \in X$. Then T and f are surjection on X. Then $T(X) \subseteq S(X)$ and $S(X)$ is complete. Further

$$\begin{aligned}\mathcal{P}(Tx, Ty, t) &= t\left|\frac{x}{4} - \frac{y}{4}\right|\\ &= 4t\left|\frac{x}{16} - \frac{y}{16}\right|\\ &\ge 3\mathcal{P}(fx, fy, t)\\ &= a\,\mathcal{P}(fx, fx, t)\end{aligned}$$

for all $x, y \in X$ and all $t > 0$,where $a = 3 > 1$. Then (54) is satisfied. Thus all conditions of Corollary 5.3 are satisfied and $x^\star = 0$ is a coincidence point of T and f.

6. FIXED POINT RESULTS IN PARAMETRIC B-METRIC SPACE

In this section, we establish some unique fixed point results satisfying expansive condition by considering surjective self-mapping in the context of parametric b-metric space. Our results generalize and extend some old and recent fixed point result in the literature.

We begin with following some lemmas.

Lemma 6.1 Let $(X, \mathcal{P}, s)$ be a b-metric space with the coefficient $s \geq 1$ and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. if $\{x_n\}_{n=1}^{\infty}$ converges to x and also $\{x_n\}_{n=1}^{\infty}$ converges to y, then $x = y$. That Is the limit of $\{x_n\}_{n=1}^{\infty}$ is unique.

Proof Since $x_n \to x$ and $x_n \to y$ as $n \to +\infty$, that is $\lim_{n\to\infty} \mathcal{P}(x_n, x, t) = 0$ and $\lim_{n\to 0} \mathcal{P}(x_n, y, t) = 0$. By using triangular inequality, we have

$$\mathcal{P}(x, y, t) \leq s[\mathcal{P}(x, x_n, t) + \mathcal{P}(x_n, y, t)]$$
$$= s[\mathcal{P}(x_n, x, t) + \mathcal{P}(x_n, y, t)]$$

for all $t > 0$. By taking limit as $n \to \infty$, we get $d(x, y) = 0$ and so $x = y$.

Lemma 6.2 Let $(X, \mathcal{P}, s)$ be a b-metric space with the coefficient $s \geq 1$ and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. if $\{x_n\}_{n=1}^{\infty}$ converges to x. Then

(59) $$\frac{1}{s}\mathcal{P}(x, y, t) \leq \lim_{n\to+\infty} \mathcal{P}(x_n, y, t) \leq s\, \mathcal{P}(x, y, t)$$

$\forall\, y \in X$ and all $t > 0$.

Proof From triangular inequality, we have

(60) $$\frac{1}{s}\mathcal{P}(x, y, t) - \lim_{n\to+\infty} \mathcal{P}(x_n, x, t)$$
$$\leq \lim_{n\to+\infty} \mathcal{P}(x_n, y, t)$$
$$\leq s\,(\mathcal{P}(x, y, t) + \lim_{n\to+\infty} \mathcal{P}(x_n, x, t))$$

and so

$$\frac{1}{s}\mathcal{P}(x, y, t) \leq \lim_{n\to+\infty} \mathcal{P}(x_n, y, t) \leq s\, \mathcal{P}(x, y, t)$$

$\forall\, y \in X$ and all $t > 0$.

Lemma 6.3 Let $(X, \mathcal{P}, s)$ be a b-metric space with the coefficient $s \geq 1$ and let $\{x_k\}_{k=0}^{n} \subset X$. Then

(61) $$\mathcal{P}(x_n, x_0, t) \leq s\mathcal{P}(x_0, x_1, t) + s^2\mathcal{P}(x_2, x_3, t)$$
$$+ \cdots \ldots \ldots \ldots + s^{n-1}\mathcal{P}(x_{n-2}, x_{n-1}, t)$$
$$+ s^{n-1}\mathcal{P}(x_{n-1}, x_n, t)$$

From Lemma 6.3, we deduce the following result.

Lemma 6.4 Let $(X, \mathcal{P}, s)$ be a parametric metric space with the coefficient $s \geq 1$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points of X such that

(62) $$\mathcal{P}(x_n, x_{n+1}, t) \preceq \lambda\, \mathcal{P}(x_{n-1}, x_n, t)$$

where $\lambda \in \left[0, \frac{1}{s}\right)$ and $n = 1, 2, \ldots$. Then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(X, \mathcal{P}, s)$.

Proof Let $m > n \geq 1$. It follows that

(63) $$\mathcal{P}(x_n, x_m, t) \leq s\mathcal{P}(x_n, x_{n+1}, t) + s^2\mathcal{P}(x_{n+1}, x_{n+2}, t)$$
$$+ \cdots \ldots \ldots \ldots + s^{m-n}\mathcal{P}(x_{m-1}, x_m, t)$$
$$\leq (s\lambda^n + s^2\lambda^{n+1} + \cdots + s^{m-n}\lambda^{m-1})\, \mathcal{P}(x_0, x_1, t)$$
$$\leq s\lambda^n(1 + s\lambda + \ldots + (s\lambda)^{m-n-1})\, \mathcal{P}(x_0, x_1, t)$$
$$\leq \frac{s\lambda^n}{1 - s\lambda}\, \mathcal{P}(x_0, x_1, t)$$

for all $t > 0$. Since $s\lambda < 1$. Assume that $\mathcal{P}(x_0, x_1, t) > 0$. By taking limit as $m, n \to +\infty$ in above inequality we get

(64) $$\lim_{n,m\to\infty} \mathcal{P}(x_n, x_m, t) = 0.$$

Therefore, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Also, if $\mathcal{P}(x_0, x_1, t) = 0$, then $\mathcal{P}(x_n, x_m, t) = 0$ for all $m > n$ and hence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X.

Now, we have the following fixed point theorem in parametric b-metric space.

Theorem 6.5 Let $(X, \mathcal{P}, s)$ be a complete parametric b-metric space with the coefficient $s \geq 1$. Assume that the mapping $T: X \to X$ is surjection and satisfies

(65) $$\mathcal{P}(Tx, Ty, t) \geq \lambda\mathcal{P}(x, y, t)$$

$\forall\, x, y \in X$ and all $t > 0$ where $\lambda > s$. Then T has a fixed point.

Proof Let $x_0 \in X$, since T is surjection on X, then there exists $x_1 \in X$ such that $x_0 = Tx_1$. By continuing this process, we get

(66) $$x_n = Tx_{n+1},\ \forall\, n \in \mathbb{N} \cup \{0\}.$$

In case $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N} \cup \{0\}$, then it is clear that x_{n_0} is a fixed point of T. Now assume that $x_n \neq x_{n-1}$ for all n. Consider,

(67) $$\mathcal{P}(x_{n-1}, x_n, t) = \mathcal{P}(Tx_n, Tx_{n+1}, t)$$

Now by (67) and definition of the sequence

$$\mathcal{P}(x_{n-1}, x_n, t) = \mathcal{P}(Tx_n, Tx_{n+1}, t)$$
$$\geq \lambda\mathcal{P}(x_n, x_{n+1}, t)$$

and so

(68) $$\mathcal{P}d(x_n, x_{n+1}, t) \leq \frac{1}{\lambda}\mathcal{P}(x_{n-1}, x_n, t)$$
$$= h\, \mathcal{P}(x_{n-1}, x_n, t)$$

for all $n \in \mathbb{N} \cup \{0\}$ and all $t > 0$ where $h = \frac{1}{\lambda} < \frac{1}{s}$. Repeating (68) n-times, we get

(69) $$\mathcal{P}(x_{n+1}, x_n, t) \leq h^n \mathcal{P}(x_0, x, t)$$

for all $t > 0$. By Lemma 6.4, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Since X is a complete parametric b-metric space, there exists $x^\star \in X$ such that $x_n \to x^\star$ as $n \to +\infty$. Now since T is surjective map. So there exists a point p in X such that $x^\star = Tp$. Consider from (65), we have

(70) $$\mathcal{P}(x_n, x^\star, t) = \mathcal{P}(Tx_{n+1}, Tp, t) \geq \lambda \mathcal{P}(x_{n+1}, p, t)$$

for all $t > 0$. Taking limit as $n \to +\infty$ in the above inequality, we get

$$0 = \lim_{n\to+\infty} \mathcal{P}(x_n, x^\star, t) \geq \lambda \lim_{n\to\infty} \mathcal{P}(x_{n+1}, p, t)$$

which implies that

(71) $$\lim_{n\to+\infty} \mathcal{P}(x_{n+1}, p, t) = 0.$$

for all $t > 0$. Thus $x_{n+1} \to p$ as $n \to +\infty$. By lemma 6.1, we get $x^\star = p$. Hence $x^\star$ is a fixed point of T.

Finally, assume $x^\star = y^\star$ is also another fixed point of T. From (65), we get for all $t > 0$

(72) $$\mathcal{P}(x^\star, y^\star, t) = \mathcal{P}(Tx^\star, Ty^\star, t) \geq \lambda \mathcal{P}(x^\star, y^\star, t)$$

This is true only when $\mathcal{P}(x^\star, y^\star, t) = 0$. So $x^\star = y^\star$. Hence T has a unique fixed point in X.

Example 6.6 Let $X = [0, +\infty)$ and define $\mathcal{P} : X \times X \times (0, +\infty) \to [0, +\infty)$ by

$$\mathcal{P}(x, y, t) = \begin{cases} t(x-y)^2 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

$\forall x, y \in X$ and all $t > 0$. It is obvious that $\mathcal{P}$ is a parametric b-metric on X with $s = 2 > 1$ and $(X, \mathcal{P})$ is complete. Also, $\mathcal{P}$ is not a parametric metric on X. Define a mapping $T: X \to X$ by

$$Tx = \begin{cases} 6x & \text{if } x \in [0,1), \\ 5x+1 & \text{if } x \in [1,2), \\ 4x+3 & \text{if } x \in [2, \infty). \end{cases}$$

Clearly T is a surjection on X. Now we consider following cases.

- Let $x, y \in [0,1)$, then

$$\begin{aligned} \mathcal{P}(Tx, Ty, t) &= t(6x-6y)^2 \\ &= 36t(x-y)^2 \\ &\geq 3t(x-y)^2 \\ &= 3\mathcal{P}(x, y, t) \end{aligned}$$

- Let $x, y \in [1,2)$, then

$$\begin{aligned} \mathcal{P}(Tx, Ty, t) &= t\big((5x+1)-(5y+1)\big)^2 \\ &= 25t(x-y)^2 \\ &\geq 3t(x-y)^2 \\ &= 3\mathcal{P}(x, y, t) \end{aligned}$$

- Let $x, y \in [2, \infty)$, then

$$\begin{aligned} \mathcal{P}(Tx, Ty, t) &= t\big((4x+3)-(4y+3)\big)^2 \\ &= t(4x-4y)^2 \\ &= 16t(x-y)^2 \\ &\geq 3t(x-y)^2 \\ &= 3\mathcal{P}(x, y, t) \end{aligned}$$

- Let $x \in [0,1)$ and $y \in [1,2)$, then

$$\begin{aligned} \mathcal{P}(Tx, Ty, t) &= t\big(6x-(5y+1)\big)^2 \\ &\geq t(6x-5y)^2 \\ &\geq t(5x-5y)^2 \\ &= 25t(x-y)^2 \\ &\geq 3t(x-y)^2 \\ &= 3\mathcal{P}(x, y, t) \end{aligned}$$

- Let $x \in [0,1)$ and $y \in [2, \infty)$, then

$$\begin{aligned} \mathcal{P}(Tx, Ty, t) &= t\big(6x-(4y+3)\big)^2 \\ &\geq t(6x-4y)^2 \\ &\geq t(4x-4y)^2 \\ &= 16t(x-y)^2 \\ &\geq 3t(x-y)^2 \\ &= 3\mathcal{P}(x, y, t) \end{aligned}$$

- Let $x \in [1,2)$ and $y \in [2, \infty)$, then

$$\begin{aligned} \mathcal{P}(Tx, Ty, t) &= t\big((5x+1)-(4y+3)\big)^2 \\ &= t(6x-4y-2)^2 \\ &\geq t(5x-4y)^2 \\ &\geq t(4x-4y)^2 \\ &= 16t(x-y)^2 \\ &\geq 3t(x-y)^2 \\ &= 3\mathcal{P}(x, y, t) \end{aligned}$$

Hence in all above cases

$$\mathcal{P}(Tx, Ty, t) \geq \lambda \mathcal{P}(x, y, t)$$

$\forall x, y \in X$ and all $t > 0$ where $\lambda = 3 > 2 = s$. The conditions of Theorem 6.5, are satisfied and T has a unique fixed point $x^\star = 0 \in X$.

Now, motivated by the work in [40], we give the following.

Let $\Psi_{\mathcal{B}}^{L}$ denote the class of those function $\mathcal{B}: (0, \infty) \to (L^2, \infty)$ which satisfy the condition $\mathcal{B}(t_n) \to (L^2)^+ \Rightarrow t_n \to 0$, where $L > 0$.

Theorem 6.7 Let $(X, \mathcal{P}, s)$ be a complete parametric b-metric space. Assume that the mapping $T: X \to X$ is surjection and satisfies

(73) $\quad \mathcal{P}(Tx, Ty, t) \geq \mathcal{B}(\mathcal{P}(x, y, t))\mathcal{P}(x, y, t)$

$\forall x, y \in X$ and all $t > 0$ where $\mathcal{B} \in \Psi_{\mathcal{B}}^{s}$. Then T has a fixed point.

Proof Let $x_0 \in X$. Since T is surjective, choose $x_1 \in X$ such that $Tx_1 = x_0$. Inductively, we can define a sequence $\{x_n\}_{n=1}^{\infty} \in X$ such that

(74) $\quad x_n = Tx_{n+1}, \ \forall\, n \in \mathbb{N} \cup \{0\}.$

In case $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N} \cup \{0\}$, then it is clear that x_{n_0} is a fixed point of T. Now assume that $x_n \neq x_{n-1}$ for all n. Consider

$$\mathcal{P}(x_{n-1}, x_n, t) = \mathcal{P}(Tx_n, Tx_{n+1}, t)$$

Now by (73) and definition of the sequence

(75)
$$\begin{aligned}\mathcal{P}(x_{n-1}, x_n, t) &= \mathcal{P}(Tx_n, Tx_{n+1}, t)\\ &\geq \mathcal{B}(\mathcal{P}(x_n, x_{n+1}, t))\mathcal{P}(x_n, x_{n+1}, t)\\ &\geq s^2\mathcal{P}(x_n, x_{n+1}, t)\\ &\geq \mathcal{P}(x_n, x_{n+1}, t)\end{aligned}$$

Thus the sequence $\{d(x_n, x_{n+1})\}_{n=1}^{\infty}$ is a decreasing sequence in $\mathbb{R}^+$ and so there exists $r \geq 0$ such that

(76) $\quad \lim_{n\to\infty} \mathcal{P}(x_n, x_{n+1}, t) = r$

for all $t > 0$. Let us prove that $r = 0$. Suppose to the contrary that $r > 0$. By (73) we can deduce that

(77)
$$\begin{aligned}s^2\frac{\mathcal{P}(x_{n-1}, x_n, t)}{\mathcal{P}(x_n, x_{n+1}, t)} &\geq \frac{\mathcal{P}(x_{n-1}, x_n, t)}{\mathcal{P}(x_n, x_{n+1}, t)}\\ &\geq \mathcal{B}(\mathcal{P}(x_n, x_{n+1}, t)) \geq s^2\end{aligned}$$

By taking limit as $n \to +\infty$ in the above inequality, we have

(78) $\quad \lim_{n\to+\infty} \mathcal{B}(\mathcal{P}(x_n, x_{n+1}, t)) = s^2$

Hence by definition of $\mathcal{B}$, we have

(79) $\quad r = \lim_{n\to+\infty} \mathcal{P}(x_n, x_{n+1}, t) = 0$

which is a contradiction. That is $r = 0$. Now, we shall show that

(80) $\quad \lim_{n,m\to+\infty} \sup \mathcal{P}(x_n, x_m, t) = 0$

Suppose to the contrary that $\lim_{n,m\to\infty} \sup \mathcal{P}(x_n, x_m, t) > 0$.

By (73), we have

$$\begin{aligned}\mathcal{P}(x_n, x_m, t) &= \mathcal{P}(Tx_{n+1}, Tx_{m+1}, t)\\ &\geq \mathcal{B}(\mathcal{P}(x_{n+1}, x_{m+1}, t))\mathcal{P}(x_{n+1}, x_{m+1}, t)\end{aligned}$$

That is,

$$\frac{\mathcal{P}(x_n, x_m, t)}{\mathcal{B}(\mathcal{P}(x_{n+1}, x_{m+1}, t))} \geq \mathcal{P}(x_{n+1}, x_{m+1}, t)$$

By triangular inequality, we have

$$\begin{aligned}\mathcal{P}(x_n, x_m, t) &\leq s\mathcal{P}(x_n, x_{n+1}, t) + s^2\mathcal{P}(x_{n+1}, x_{m+1}, t)\\ &\quad + s^2\mathcal{P}(x_{m+1}, x_m, t)\\ &\leq s\mathcal{P}(x_n, x_{n+1}, t) + s^2\frac{\mathcal{P}(x_n, x_m, t)}{\mathcal{B}(\mathcal{P}(x_{n+1}, x_{m+1}, t))}\\ &\quad + s^2\mathcal{P}(x_{m+1}, x_m, t)\end{aligned}$$

Therefore,

(81)
$$\mathcal{P}(x_n, x_m, t) \leq \left(1 - \frac{s^2}{\mathcal{B}(\mathcal{P}(x_{n+1}, x_{m+1}, t))}\right)^{-1} \left(s\mathcal{P}(x_n, x_{n+1}, t) + s^2\mathcal{P}(x_{m+1}, x_m, t)\right)$$

By taking limit as $n, m \to +\infty$ in the above inequality, since $\lim_{n,m\to+\infty} \sup \mathcal{P}(x_n, x_m, t) > 0$ and $r = 0 = \lim_{n\to+\infty} \mathcal{P}(x_n, x_{n+1}, t)$, then we obtain

(82) $\quad \lim_{n,m\to+\infty} \left(1 - \frac{s^2}{\mathcal{B}(\mathcal{P}(x_{n+1}, x_{m+1}, t))}\right)^{-1} = +\infty$

which implies that

(83) $\quad \lim_{m,n\to+\infty} \sup\, \mathcal{B}(\mathcal{P}(x_{n+1}, x_{m+1}, t)) = (s^2)^+$

and so by definition of $\mathcal{B}$, we have

(84) $\quad \lim_{m,n\to+\infty} \sup\, \mathcal{P}(x_{n+1}, x_{m+1}, t) = 0$

which is a contradiction. Hence,

$$\lim_{m,n\to+\infty} \sup\, \mathcal{P}(x_n, x_m, t) = 0$$

Since $\lim_{m,n\to+\infty} \sup\, \mathcal{P}(x_n, x_m, t) = 0$. So, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since X is a complete parametric b-metric space, the sequence $\{x_n\}_{n=1}^{\infty}$ in X converges to $x^\star \in X$. so that

(85) $\quad \lim_{n\to+\infty} \mathcal{P}(x_n, x^\star, t) = 0$

As T is surjective, so there exists $p \in X$ such that $x^\star = Tp$. Let us prove that $x^\star = p$. Suppose to the contrary that $x^\star \neq p$. Then by (73), we have

(86)
$$\begin{aligned}\mathcal{P}(x_n, x^\star, t) &= \mathcal{P}(Tx_{n+1}, Tp, t)\\ &\geq \mathcal{B}\left(\mathcal{P}(x_{n+1}, p, t)\right)\mathcal{P}(x_{n+1}, p, t)\end{aligned}$$

By Taking limit as $n \to +\infty$ in the above inequality and applying Lemma 6.2, we obtain

$$(87)\ 0 = \lim_{n\to+\infty} \mathcal{P}(x_n, x^\star, t)$$
$$\geq \lim_{n\to+\infty} \mathcal{B}\left(\mathcal{P}(x_{n+1}, p, t)\right) \lim_{n\to\infty} \mathcal{P}(x_{n+1}, p, t)$$
$$\geq \frac{1}{s} \lim_{n\to+\infty} \mathcal{B}\left(\mathcal{P}(x_{n+1}, x^\star, t)\right) \mathcal{P}(x^\star, p, t)$$

and hence,

$$(88)\qquad \lim_{n\to+\infty} \mathcal{B}\left(\mathcal{P}(x_{n+1}, x^\star, t)\right) = 0$$

which is a contradiction. Indeed,

$$\lim_{n\to+\infty} \mathcal{B}\big(\mathcal{P}(x_{n+1}, x_n, t)\big) \geq s^2.$$

Since $\mathcal{B}(t) > s^2$ for all $t \in [0, \infty)$, therefore $x^\star = p$. Hence $x^\star = Tp = Tx^\star$.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

AUTHOR'S CONTRIBUTIONS

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

REFERENCES

[1] L. A. Zadeh, "Fuzzy sets," Information and Computation, vol. 8, pp. 338–353, 1965.

[2] I. Kramosil and J. Michalek, "Fuzzy metrics and statistical metric spaces," Kybernetika, vol. 11, no. 5, pp. 336–344, 1975.

[3] M. Grabiec, "Fixed points in fuzzy metric spaces," Fuzzy Sets and Systems, vol. 27, no. 3, pp. 385–389, 1988.

[4] A. George and P. Veeramani, "On some results in fuzzy metric spaces," Fuzzy Sets and Systems, vol. 64, no. 3, pp. 395–399, 1994.

[5] C. Di Bari and C. Vetro, "Fixed points, attractors and weak fuzzy contractive mappings in a fuzzy metric space," Journal of Fuzzy Mathematics, vol. 13, no. 4, pp. 973–982, 2005.

[6] D. Gopal, M. Imdad, C. Vetro, and M. Hasan, "Fixed point theory for cyclic weak ϕ-contraction in fuzzy metric spaces," Journal of Nonlinear Analysis and Application, vol. 2012, Article ID jnaa-00110, 11 pages, 2012.

[7] P. Salimi, C. Vetro, and P. Vetro, "Some new fixed point results in non-Archimedean fuzzy metric spaces," Nonlinear Analysis: Modelling and Control, vol. 18, no. 3, pp. 344–358, 2013.

[8] Y. Shen, D. Qiu, and W. Chen, "Fixed point theorems in fuzzy metric spaces," Applied Mathematics Letters, vol. 25, no. 2, pp. 138–141, 2012.

[9] C. Vetro, "Fixed points in weak non-Archimedean fuzzy metric spaces," Fuzzy Sets and Systems, vol. 162, pp. 84–90, 2011.

[10] C. Vetro, D. Gopal, and M. Imdad, "Common fixed point theorems for (ϕ, ψ)-weak contractions in fuzzy metric spaces," Indian Journal of Mathematics, vol. 52, no. 3, pp. 573–590, 2010.

[11] C. Vetro and P. Vetro, "Common fixed points for discontinuous mappings in fuzzy metric spaces," Rendiconti del Circolo Matematico di Palermo, vol. 57, no. 2, pp. 295–303, 2008.

[12] J. H. Park, "Intuitionistic fuzzy metric spaces," Chaos, Solitons and Fractals, vol. 22, no. 5, pp. 1039–1046, 2004.

[13] C. Alaca, D. Turkoglu, and C. Yildiz, "Fixed points in intuitionistic fuzzy metric spaces," Chaos, Solitons & Fractals, vol.29, no. 5, pp.1073–1078, 2006.

[14] D. Coker, "An introduction to intuitionistic fuzzy topological spaces," Fuzzy Sets and Systems, vol. 88, no. 1, pp. 81–89, 1997.

[15] J. S. Park, Y. C. Kwun, and J. H. Park, "A fixed point theorem in the intuitionistic fuzzy metric spaces," Far East Journal of Mathematical Sciences, vol. 16, no. 2, pp. 137–149, 2005.

[16] M. Rafi and M. S. M. Noorani, "Fixed point theorem on intuitionistic fuzzy metric spaces," Iranian Journal of Fuzzy Systems, vol. 3, no. 1, pp. 23–29, 2006.

[17] B. Schweizer and A. Sklar, "Statistical metric spaces," Pacific Journal of Mathematics, vol. 10, pp. 313–334, 1960.

[18] S. Manro, S. Kumar and S.S. Bhatia, Common fixed point theorems for weakly compatible maps satisfying common (E.A) like property in intuitionistic fuzzy metric spaces using implicit relation, *Journal of the Indian Math. Soc*., 81(1-2) (2014), 123-133.

[19] C. Ionescu, S. Rezapour, and M. E. Samei, "Fixed points of some new contractions on intuitionistic fuzzy metric spaces," Fixed Point Theory and Applications, vol. 2013, article 168, 2013.

[20] N. Hussain, S. Khaleghizadeh, P. Salimi, and Afrah A. N. Abdou, "A New Approach to Fixed Point Results in Triangular Intuitionistic Fuzzy Metric

Spaces", Abstract and Applied Analysis, Volume 2014, Article ID 690139, 16 pages.

[21] N. Hussain, P. Salimi, V. Parvaneh, Fixed point results for various contractions in parametric and fuzzy b-metric spaces, J. Nonlinear Sci. Appl. 8 (2015), 719-739.

[22] Jungck, G., Commuting mappings and fixed points. Amer. Math. Monthly, 83(1976), 261-263.

[23] G. Jungck, "Compatible mappings and common fixed points," International Journal of Mathematics and Mathematical Sciences, vol. 9, no.4, pp. 771-779, 1986.

[24] G. Jungck, "Fixed points for non-continuous non-self mappings on non-metric space," Far East Journal of Mathematical Sciences, vol. 4, pp. 199-212, 1996.

[25] S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inf. Univ. Ostravensis, 1 (1993), 5-11.

[26] S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces, Atti Sem. Mat. Fis. Univ. Modena, 46 (1998), 263-276.

[27] M. A. Alghamdi, N. Hussain, P. Salimi, Fixed point and coupled fixed point theorems on b-metric-like spaces, J. Inequal. Appl., 2013 (2013), 25 pages.

[28] N. Hussain, V. Parvaneh, J. R. Roshan, Z. Kadelburg, Fixed points of cyclic weakly (ψ, φ, L, A, B)-contractive mappings in ordered b-metric spaces with applications, Fixed Point Theory Appl., 2013 (2013), 18pages.

[29] A. S. Saluja, A. K. Dhakda and D. Magarde, Some fixed point theorems for expansive type mapping in dislocated metric space, Mathematical Theorey and Modeling, 3(2013).

[30] R. D. Daheriya, Rashmi Jain, and Manoj Ughade "Some Fixed Point Theorem for Expansive Type Mapping in Dislocated Metric Space", ISRN Mathematical Analysis, Volume 2012, Article ID 376832, 5 pages, doi:10.5402/2012/376832.

[31] Yan Han and Shaoyuan Xu, Some new theorems of expanding mappings without continuity in cone metric spaces, *Fixed Point Theory and Applications*, 2013, 2013:3.

[32] Wasfi Shatanawi and Fadi Awawdeh, Some fixed and coincidence point theorems for expansive maps in cone metric spaces, Fixed Point Theory and Applications 2012, 2012:19.

[33] Xianjiu Huang, Chuanxi Zhu, Xi Wen, Fixed point theorems for expanding mappings in partial metric spaces, An. St. Univ. Ovidius Constant_a Vol. 20(1), 2012, 213-224.

[34] Z. Mustafa, F. Awawdeh and W. Shatanawi, Fixed Point Theorem for Expansive Mappings in *G*-Metric Spaces, Int. J. Contemp. Math. Sciences, Vol. 5, 2010, no. 50, 2463-2472.

[35] A. Muraliraj and R. Jahir Hussain, Coincidence and Fixed Point Theorems for Expansive Maps in $d-$Metric Spaces, Int. Journal of Math. Analysis, Vol. 7, 2013, no. 44, 2171 – 2179.

[36] P. Z. Daffer, H. Kaneko, *On expansive mappings*, Math. Japonica. 37 (1992), 733-735.

[37] S. Z. Wang, B. Y. Li, Z. M. Gao, K. Iseki, *Some fixed point theorems for expansion mappings*, Math. Japonica. 29 (1984), 631-636.

[38] Aage, CT, Salunke, JN: Some fixed point theorems for expansion onto mappings on cone metric spaces. Acta Math. Sin. Engl. Ser. 27(6), 1101-1106 (2011).

[39] G. Jungck, "Commuting mappings and fixed points," The American Mathematical Monthly, vol. 83, no.4, pp. 261-263, 1976.

[40] M. A. Alghamdi, N. Hussain and P. Salimi, Fixed point and coupled fixed point theorems on b-metric-like spaces, Journal of Inequalities and Applications, 2013, 2013:402.

[41] R. D. Daheriya, Rashmi Jain, and Manoj Ughade, Unified fixed point theorems for non-surjective expansion mappings by using class Ψ on 2- metric spaces, Bessel J. Math., 3(3), (2013), 271-291.

[42] W. Shatanwi and F. Awawdeh, Some fixed and coincidence point theorems for expansive maps in cone metric spaces, Fixed point theory and apllicaions, 2012, 2012;19, doi:10.1186/1687-1812-2012-19.

[43] V. Ozturk and D. Turkoglu, Common fixed point theorems for mapping satisfying (E.A.) property in b-metric spaces, Journal of Nonlinear Sciences and Applications 8(6), 2015, 1127-1133.