Gazi University Journal of Science
GU J Sci
29(1):115-122 (2016)

Voronovskaja Type Approximation Theorem For q - Szász- Beta-Stancu Type Operators

Ülkü DİNLEMEZ[1,♠], İsmet YÜKSEL[1]

[1]*Gazi University, Faculty of Science, Department of Mathematics, Teknikokullar, 06500, Ankara, TURKEY*

Received: 12/11/2015 *Accepted: 27/01/2016*

ABSTRACT

In this paper, we study on q −analogue of Szász-Beta-Stancu type operators. We give a Voronovskaja type theorem for q - Szász-Beta-Stancu type operators.

Key words: *q − Stancu type operators, Szász−beta type operators, Voronovskaja type operators.*

1. INTRODUCTION

In the paper, in [18], Mahmudov introduced King type q −Szász operators and obtained rate of global convergence in the frame of weighted spaces and a Voronovskaja type theorem for these operators. In [22], Yüksel and Dinlemez gave a Voronovskaja type theorem for q-analogue of a certain family Szász-Beta type operators. In [8], Govil and Gupta constructed the q −analogue of certain Beta-Szász-Stancu operators. They estimated the moments and established direct results in terms of modulus of continuity and an asymptotic formula for the q-operators. In [4], Dinlemez gave approximation properties of q - Szász-Beta-Stancu type operators and obtain a weighted approximation theorem for the operators. After then several interesting generalization about q −calculus were given in $[5, 6, 10, 11, 12, 13, 17, 19, 23]$. Our goal is to give a Voronovskaja type approximation theorem for these operators. We use without further explanation the basic notations and formulas, from the theory of q −calculus as set out in [1,2,3,14,15,16,20,21].

We need fix some notations and recall some definitations: Let $A > 0$ and f be a real valued continuous function defined on the interval $[0, \infty)$. For $0 < q \le 1, q$ −Szász-Beta-Stancu type operators are defined as

♠Corresponding author, e-mail: *ulku@gazi.edu.tr*

$$B_{n,q}^{(\alpha,\beta)}(f,x)=\sum_{k=0}^{\infty}s_{n,k}^{q}(x)\int_{0}^{\infty/A}b_{n,k}^{q}(t)f\left(\frac{[n]_q t+\alpha}{[n]_q t+\beta}\right)d_q(t), \tag{1.1}$$

where

$$s_{n,k}^{q}(x)=\left([n]_q x\right)^k\frac{e^{-[n]_q x}}{[k]_q!}$$

and

$$b_{n,k}^{q}(x)=\frac{q^{k^2}x^k}{B_q(k+1,n)(1+x)_q^{n+k+1}}.$$

If we write $q=1$ and $\alpha=\beta=0$ in (1.1), then the operators $B_{n,q}^{(\alpha,\beta)}$ are reduced to q −Szász-beta type operators studied in [9].

2. VORONOVSKAJA TYPE THEOREM

For the sake of brevity, the notations $F_s^q(n)=\prod_{i=0}^{s}[n-i]_q$ and $G_\beta^q(n)=\left([n]_q+\beta\right)$ will be used throughout the paper. Now we are ready to give the following lemma for the Korovkin test functions.

Lemma 1. Let $e_m(t)=t^m, m=0,1,2,3$ and 4. Then, we get

(i) $B_{n,q}^{(\alpha,\beta)}(e_0,x)=1,$

(ii) $B_{n,q}^{(\alpha,\beta)}(e_1,x)=\frac{[n]_q^2}{q^2G_\beta^q(n)F_1^q(n)}x+\frac{[n]_q}{qG_\beta^q(n)F_1^q(n)}+\frac{\alpha}{G_\beta^q(n)},$

(iii)
$$B_{n,q}^{(\alpha,\beta)}(e_2,x)=\frac{[n]_q^4}{q^6\left(G_\beta^q(n)\right)^2F_2^q(n)}x^2+\left\{\frac{[n]_q^3}{q^5\left(G_\beta^q(n)\right)^2F_2^q(n)}+\frac{(1+[2]_q)[n]_q^3}{q^4\left(G_\beta^q(n)\right)^2F_2^q(n)}+\frac{2\alpha[n]_q^2}{q^2\left(G_\beta^q(n)\right)^2F_1^q(n)}\right\}x$$
$$+\frac{[2]_q[n]_q^2}{q^3\left(G_\beta^q(n)\right)^2F_2^q(n)}+\frac{2\alpha[n]_q}{q\left(G_\beta^q(n)\right)^2F_1^q(n)}+\frac{\alpha^2}{\left(G_\beta^q(n)\right)^2},$$

(iv)
$$B_{n,q}^{(\alpha,\beta)}(e_3,x)=\frac{[n]_q^6}{q^{12}\left(G_\beta^q(n)\right)^3F_3^q(n)}x^3+\left\{\frac{\{[5]_q+[2]_q^2q\}[n]_q^5}{q^{11}\left(G_\beta^q(n)\right)^3F_3^q(n)}+\frac{3[n]_q^4\alpha}{q^6\left(G_\beta^q(n)\right)^3F_2^q(n)}\right\}x^2$$
$$+\left\{\frac{\{[2]_q^2[4]_q+[2]_qq^2\}[n]_q^4}{q^9\left(G_\beta^q(n)\right)^3F_3^q(n)}+\frac{3[2]_q^2[n]_q^3\alpha}{q^5\left(G_\beta^q(n)\right)^3F_2^q(n)}+\frac{3[n]_q^2\alpha^2}{q^2\left(G_\beta^q(n)\right)^3F_1^q(n)}\right\}x$$
$$+\frac{[2]_q[3]_q[n]_q^3}{q^6\left(G_\beta^q(n)\right)^3F_3^q(n)}+\frac{3[2]_q[n]_q^2\alpha}{q^3\left(G_\beta^q(n)\right)^3F_2^q(n)}+\frac{\alpha^3}{\left(G_\beta^q(n)\right)^3},$$

(v)
$$B_{n,q}^{(\alpha,\beta)}(e_4,x)=\frac{[n]_q^8}{q^{20}\left(G_\beta^q(n)\right)^4F_4^q(n)}x^4+\left\{\frac{\{[7]_q+[5]_qq+[2]_q^2q^2\}[n]_q^7}{q^{19}\left(G_\beta^q(n)\right)^4F_4^q(n)}+\frac{4\alpha[n]_q^6}{q^{12}\left(G_\beta^q(n)\right)^4F_3^q(n)}\right\}x^3$$
$$+\left\{\frac{4\alpha\{[5]_q+[2]_q^2q\}[n]_q^5}{q^{11}\left(G_\beta^q(n)\right)^4F_3^q(n)}+\frac{\{[5]_q[6]_q+[2]_q^2[6]_qq+[2]_q^2[4]_qq^2+[2]_qq^4\}[n]_q^7}{q^{17}\left(G_\beta^q(n)\right)^4F_4^q(n)}+\frac{6\alpha^2[n]_q^4}{q^6\left(G_\beta^q(n)\right)^4F_2^q(n)}\right\}x^2$$

$$+\left\{\frac{\{[2]_q^2[4]_q[5]_q+[2]_q[5]_q q^2+[2]_q[3]_q q^3\}[n]_q^5}{q^{14}\left(G_\beta^q(n)\right)^4 F_4^q(n)}+\frac{4\alpha\{[2]_q^2[4]_q+[2]_q q^2\}[n]_q^4}{q^9\left(G_\beta^q(n)\right)^4 F_3^q(n)}+\frac{6\alpha^2[2]_q^2[n]_q^3}{q^5\left(G_\beta^q(n)\right)^4 F_2^q(n)}+\frac{4\alpha^3[n]_q^2}{q^2\left(G_\beta^q(n)\right)^4 F_1^q(n)}\right\}x+$$
$$\frac{[2]_q[3]_q[4]_q[n]_q^4}{q^{10}\left(G_\beta^q(n)\right)^4 F_4^q(n)}+\frac{4\alpha[2]_q[3]_q[n]_q^3}{q^6\left(G_\beta^q(n)\right)^4 F_3^q(n)}+\frac{6\alpha^2[2]_q[n]_q^2}{q^3\left(G_\beta^q(n)\right)^4 F_2^q(n)}+\frac{4\alpha^3[n]_q}{q\left(G_\beta^q(n)\right)^4 F_1^q(n)}+\frac{4\alpha^4}{\left(G_\beta^q(n)\right)^4}.$$

Proof. Using q −Gamma and q −Beta functions in $[1,2]$, we obtain the estimate,

$$q^{k^2}\int_0^{\frac{\infty}{A}}\frac{1}{B(k+1,n)}\frac{t^k}{(1+t)_q^{n+k+1}}t^m d_q t=\frac{[m+k]_q![n-m-1]_q!\,q^{\frac{\{2k^2-(k+m)(k+m+1)\}}{2}}}{[k]_q![n-1]_q!}. \tag{2.1}$$

the proof of (i) − (iii) are given in [4]. Then, using (2.1) and the equality

$$[n]_q=[s]_q+q^s[n-s]_q,\ 0\le s\le n, \tag{2.2}$$

We get

$$B_{n,q}^{(\alpha,\beta)}(e_3,x)=\frac{[n]_q^3}{\left(G_\beta^q(n)\right)^3 F_3^q(n)}\sum_{k=3}^{\infty}e^{-[n]_q x}\frac{\left([n]_q x\right)^k}{[k-3]_q!}q^{\frac{k^2-7k-12}{2}}$$
$$+\frac{[n]_q^3}{\left(G_\beta^q(n)\right)^3 F_3^q(n)}\sum_{k=2}^{\infty}e^{-[n]_q x}\frac{\left([n]_q x\right)^k([5]_q+[2]_q^2 q)}{[k-2]_q!}q^{\frac{k^2-5k-16}{2}}$$
$$+\frac{[n]_q^3}{\left(G_\beta^q(n)\right)^3 F_3^q(n)}\sum_{k=1}^{\infty}e^{-[n]_q x}\frac{\left([n]_q x\right)^k([2]_q^2[4]_q+[2]_q q^2)}{[k-1]_q!}q^{\frac{k^2-3k-14}{2}}$$
$$+\frac{[n]_q^3}{\left(G_\beta^q(n)\right)^3 F_3^q(n)}\sum_{k=0}^{\infty}e^{-[n]_q x}\frac{\left([n]_q x\right)^k[2]_q[3]_q}{[k-1]_q!}q^{\frac{k^2-k-12}{2}}$$
$$=\frac{[n]_q^6}{q^{12}\left(G_\beta^q(n)\right)^3 F_3^q(n)}x^3+\left\{\frac{\{[5]_q+[2]_q^2 q\}[n]_q^5}{q^{11}\left(G_\beta^q(n)\right)^3 F_3^q(n)}+\frac{3\alpha[n]_q^4}{q^6\left(G_\beta^q(n)\right)^3 F_2^q(n)}\right\}x^2$$
$$+\left\{\frac{\{[2]_q^2[4]_q+[2]_q q^2\}[n]_q^4}{q^9\left(G_\beta^q(n)\right)^3 F_3^q(n)}+\frac{3\alpha[2]_q^2[n]_q^3}{q^5\left(G_\beta^q(n)\right)^3 F_2^q(n)}+\frac{3\alpha^2[n]_q^2}{q^2\left(G_\beta^q(n)\right)^3 F_1^q(n)}\right\}x$$
$$+\frac{[2]_q[3]_q[n]_q^3}{q^6\left(G_\beta^q(n)\right)^3 F_3^q(n)}+\frac{3\alpha[2]_q[n]_q^2}{q^3\left(G_\beta^q(n)\right)^3 F_2^q(n)}+\frac{\alpha^3}{\left(G_\beta^q(n)\right)^3},$$

and so we have proof of (iv). Finally we make same process in (iv) we get (v) easily.

Using the de.nation of $[n]_{q_n}$, it is not diffucult to see that if $\lim_{n\to\infty}q_n^n=\lambda$ than $\lim_{n\to\infty}[n]_{q_n}=\infty$, where $0<q_n<1$ and $q_n\to 1$.

We need the following lemma for the Voronovskaja theorem.

Lemma 2. Let $(q_n)\subset(0,1)$ a sequence such that $q_n\to 1$ and $q_n^n\to\lambda$ as $n\to\infty$.

Then for any $n\in \boldsymbol{IN}$ we have the following limits

(i) $\lim_{n\to\infty}[n]_{q_n}B_{n,q_n}^{(\alpha,\beta)}((t-x)^2;x)=(2-2\beta-\lambda)x^2+2x$,

(ii) $\lim_{n\to\infty}[n]_{q_n}^2 B_{n,q_n}^{(\alpha,\beta)}((t-x)^4;x)=(3\lambda^2-12\lambda+12)x^4+(24-12\lambda)x^3+2x^2$.

Proof. i) Using the equality (2.2) for q_n , we get desired result

$$\lim_{n\to\infty}[n]_{q_n} B_{n,q_n}^{(\alpha,\beta)}((t-x)^2;x)$$

$$= \lim_{n\to\infty}\left\{\frac{(1-q_n^n)(q_n^2+q_n^3-q_n^4)[n-2]_{q_n}^3}{\left(G_\beta^{q_n}(n)\right)^2 F_1^{q_n}(n)} + \frac{(q_n^4+4q_n^3-12q_n^2+8-2\beta q_n^2)[n]_{q_n}[n-2]_{q_n}^2}{\left(G_\beta^{q_n}(n)\right)^2 F_1^{q_n}(n)}\right\}x^2$$

$$+\lim_{n\to\infty}\left\{\frac{[2]_{q_n}^2[n]_{q_n}^4}{q_n^5\left(G_\beta^{q_n}(n)\right)^2 F_2^{q_n}(n)}+\frac{2\alpha[n]_{q_n}^3}{q_n^2\left(G_\beta^{q_n}(n)\right)^2 F_1^{q_n}(n)} - \frac{2[n]_{q_n}^2}{q_n G_\beta^{q_n}(n)F_1^{q_n}(n)}-\frac{2\alpha[n]_{q_n}}{G_\beta^{q_n}(n)}\right\}x=(2-2\beta-\lambda)x^2+2x.$$

(ii) From Lemma 1 and using the linearity property of the $B_{n,q_n}^{(\alpha,\beta)}$ operators for $n > 4$;

we obtain

$$B_{n,q_n}^{(\alpha,\beta)}((t-x)^4;x)=S_1(n,q_n)x^4+S_2(n,q_n)x^3+S_3(n,q_n)x^2+S_4(n,q_n)x+S_5(n,q_n), \tag{2.3}$$

where

$$S_1(n,q_n)=\frac{[n]_{q_n}^8}{q_n^{20}\left(G_\beta^{q_n}(n)\right)^4 F_4^{q_n}(n)}-\frac{4[n]_{q_n}^6}{q_n^{12}\left(G_\beta^{q_n}(n)\right)^3 F_3^{q_n}(n)}+\frac{6[n]_{q_n}^4}{q_n^6\left(G_\beta^{q_n}(n)\right)^2 F_2^{q_n}(n)}-\frac{4[n]_{q_n}^2}{q_n^2 G_\beta^{q_n}(n)F_1^{q_n}(n)}+1,$$

$$S_2(n,q_n)=\frac{[n]_{q_n}^7\left([7]_{q_n}+[5]_{q_n}q_n+[2]_{q_n}^2q_n^2\right)}{q_n^{19}\left(G_\beta^{q_n}(n)\right)^4 F_4^{q_n}(n)}+\frac{4\alpha[n]_{q_n}^6}{q_n^{12}\left(G_\beta^{q_n}(n)\right)^4 F_3^{q_n}(n)}$$

$$-\frac{4\left([5]_{q_n}+[2]_{q_n}^2q_n\right)[n]_{q_n}^5}{q_n^{11}\left(G_\beta^{q_n}(n)\right)^3 F_3^{q_n}(n)}-\frac{12\alpha[n]_{q_n}^4}{q_n^6\left(G_\beta^{q_n}(n)\right)^3 F_2^{q_n}(n)}+\frac{6[n]_{q_n}^3}{q_n^5\left(G_\beta^{q_n}(n)\right)^2 F_2^{q_n}(n)}+\frac{6\left(1+[2]_{q_n}\right)[n]_{q_n}^3}{q_n^4\left(G_\beta^{q_n}(n)\right)^2 F_2^{q_n}(n)}$$
$$+\frac{12\alpha[n]_{q_n}^2}{q_n^2\left(G_\beta^{q_n}(n)\right)^2 F_1^{q_n}(n)}-\frac{4[n]_{q_n}}{q_n G_\beta^{q_n}(n)F_1^{q_n}(n)}-\frac{4\alpha}{G_\beta^{q_n}(n)},$$

$$S_3(n,q_n)=\frac{[n]_{q_n}^6\left([5]_{q_n}[6]_{q_n}+[2]_{q_n}^2[6]_{q_n}q_n+[2]_{q_n}^2[4]_{q_n}q_n^2+[2]_{q_n}q_n^4\right)}{q_n^{17}\left(G_\beta^{q_n}(n)\right)^4 F_4^{q_n}(n)}+\frac{4\alpha[n]_{q_n}^5\left([5]_{q_n}+q_n[2]_{q_n}^2\right)}{q_n^{11}\left(G_\beta^{q_n}(n)\right)^4 F_3^{q_n}(n)}+\frac{6\alpha^2[n]_{q_n}^4}{q_n^6\left(G_\beta^{q_n}(n)\right)^4 F_2^{q_n}(n)}-$$
$$\frac{4\left([2]_{q_n}^2[4]_{q_n}+q_n^2[2]_{q_n}\right)[n]_{q_n}^4}{q_n^9\left(G_\beta^{q_n}(n)\right)^3 F_3^{q_n}(n)}-\frac{12\alpha[2]_{q_n}^2[n]_{q_n}^3}{q_n^5\left(G_\beta^{q_n}(n)\right)^3 F_2^{q_n}(n)}-\frac{12\alpha^2[n]_{q_n}^2}{q_n^2\left(G_\beta^{q_n}(n)\right)^3 F_2^{q_n}(n)}+\frac{6[2]_{q_n}[n]_{q_n}^2}{q_n^3\left(G_\beta^{q_n}(n)\right)^2 F_2^{q_n}(n)}+\frac{12\alpha[n]_{q_n}}{q_n\left(G_\beta^{q_n}(n)\right)^2 F_1^{q_n}(n)}+$$
$$\frac{6\alpha^2}{\left(G_\beta^{q_n}(n)\right)^2},$$

$$S_4(n,q_n)=\frac{[n]_{q_n}^5\left([2]_{q_n}^2[4]_{q_n}[5]_{q_n}+[2]_{q_n}[5]_{q_n}q_n^2+[2]_{q_n}[3]_{q_n}q_n^3\right)}{q_n^{14}\left(G_\beta^{q_n}(n)\right)^4F_4^{q_n}(n)}+\frac{4\alpha\left([2]_{q_n}^2[4]_{q_n}+[2]_{q_n}q_n^2\right)[n]_{q_n}^4}{q_n^9\left(G_\beta^{q_n}(n)\right)^4F_3^{q_n}(n)}+\frac{6\alpha^2[2]_{q_n}^2[n]_{q_n}^3}{q_n^5\left(G_\beta^{q_n}(n)\right)^4F_2^{q_n}(n)}+$$
$$\frac{4\alpha^3[n]_{q_n}^2}{q_n^2\left(G_\beta^{q_n}(n)\right)^4F_1^{q_n}(n)}-\frac{4[2]_{q_n}[3]_{q_n}[n]_{q_n}^3}{q_n^6\left(G_\beta^{q_n}(n)\right)^3F_3^{q_n}(n)}-\frac{12\alpha[2]_{q_n}[n]_{q_n}^2}{q_n^3\left(G_\beta^{q_n}(n)\right)^2F_2^{q_n}(n)}-\frac{4\alpha^3}{\left(G_\beta^{q_n}(n)\right)^3},$$

$$S_5(n,q_n)=\frac{[2]_{q_n}[3]_{q_n}[4]_{q_n}[n]_{q_n}^4}{q_n^{10}\left(G_\beta^{q_n}(n)\right)^4F_4^{q_n}(n)}+\frac{4\alpha[2]_{q_n}[3]_{q_n}[n]_{q_n}^3}{q_n^6\left(G_\beta^{q_n}(n)\right)^4F_3^{q_n}(n)}+\frac{6\alpha^2[2]_{q_n}[n]_{q_n}^2}{q_n^3\left(G_\beta^{q_n}(n)\right)^4F_2^{q_n}(n)}+\frac{4\alpha^3[n]_{q_n}}{q_n\left(G_\beta^{q_n}(n)\right)^4F_1^{q_n}(n)}+\frac{4\alpha^4}{\left(G_\beta^{q_n}(n)\right)^4}.$$

It is obvious that

$$\lim_{n\to\infty}[n]_{q_n}^2\{S_4(n,q_n)+S_5(n,q_n)\}=0. \tag{2.4}$$

Using the equality (2.1) we will get the following limits. First of all we have

$$\lim_{n\to\infty}[n]_{q_n}^2S_1(n,q_n)=\lim_{n\to\infty}\left\{(1-q_n^n)^2\frac{C_1(q_n)[n-4]_{q_n}^8}{q_n^{20}\left(G_\beta^{q_n}(n)\right)^4F_4^{q_n}(n)}+(1-q_n^n)\frac{C_2(q_n)[n]_{q_n}[n-4]_{q_n}^7}{q_n^{20}\left(G_\beta^{q_n}(n)\right)^4F_4^{q_n}(n)}+\frac{C_3(q_n)[n]_{q_n}^2[n-4]_{q_n}^6}{q_n^{20}\left(G_\beta^{q_n}(n)\right)^4F_4^{q_n}(n)}\right\}+\lim_{n\to\infty}\frac{H_1\left(n,[n-4]_{q_n},\alpha,\beta\right)}{q_n^{20}\left(G_\beta^{q_n}(n)\right)^4F_4^{q_n}(n)}=3\lambda^2-12\lambda+12, \tag{2.5}$$

where

$C_1(q_n)=q_n^{40}-2q_n^{39}-5q_n^{38}-2q_n^{37}+q_n^{36}+4q_n^{35}+3q_n^{34}+2q_n^{33}+q_n^{32},$

$C_2(q_n,\beta)=-7q_n^{40}+15q_n^{39}+34q_n^{38}+(8-4\beta)q_n^{37}+(8\beta-8)q_n^{36}+(8\beta-44)q_n^{35}$
$-(4\beta+52)q_n^{34}-(4\beta+24)q_n^{33}+(4-4\beta)q_n^{32}+32q_n^{31}+24q_n^{30}+16q_n^{29}+8q_n^{28},$

$C_3(q_n,\beta)=21q_n^{40}-48q_n^{39}-99q_n^{38}+(24\beta-5)q_n^{37}+(26-52\beta)q_n^{36}+(182-44\beta)q_n^{35}$
$+(6\beta^2+36\beta+238)q_n^{34}-(12\beta^2-12\beta-62)q_n^{33}+(60\beta-72)q_n^{32}+(6\beta^2+36\beta-246)q_n^{31}$
$-(24\beta+224)q_n^{30}-(24\beta-112)q_n^{29}-24\beta q_n^{28}+112q_n^{27}+84q_n^{26}+56q_n^{25}+28q_n^{24}.$

Secondly,

$$\lim_{n\to\infty}[n]_{q_n}^2S_2(n,q_n)$$
$$=\lim_{n\to\infty}\left\{(1-q_n^n)\frac{D_1(q_n)[n]_{q_n}[n-4]_{q_n}^7}{q_n^{19}\left(G_\beta^{q_n}(n)\right)^4F_4^{q_n}(n)}+\frac{D_2(q_n)[n]_{q_n}^2[n-4]_{q_n}^6}{q_n^{19}\left(G_\beta^{q_n}(n)\right)^4F_4^{q_n}(n)}\right\}+\lim_{n\to\infty}\frac{H_2\left(n,[n-4]_{q_n},\alpha,\beta\right)}{q_n^{19}\left(G_\beta^{q_n}(n)\right)^4F_4^{q_n}(n)}$$
$$=24-12\lambda, \tag{2.6}$$

Where

$D_1(q_n)=(4\alpha-2)q_n^{36}-(8\alpha+10)q_n^{35}-(8\alpha+8)q_n^{34}+(4\alpha+3)q_n^{33}+(4\alpha+9)q_n^{32}+(4\alpha+10)q_n^{31}$
$+6q_n^{30}+3q_n^{29},$

$$D_2(q_n) = (12-24\alpha)q_n^{36} + (52\alpha+58)q_n^{35} + (44\alpha+38)q_n^{34} - (12\alpha\beta+36\alpha+33)q_n^{33}$$
$$-(24\alpha\beta+12\alpha+20\beta+81)q_n^{32} - (60\alpha-4\beta+120)q_n^{31}$$
$$-(12\alpha\beta+36\alpha+12\beta+74)q_n^{30} + (24\alpha-8\beta+12)q_n^{29}$$
$$+(24\alpha-4\beta+60)q_n^{28} + (24\alpha+70)q_n^{27},$$

and also degree of $H_2\left(n,[n-4]_{q_n},\alpha,\beta\right)$ according to $[n]_{q_n}[n-4]_{q_n}$ is lower then $\left(G_\beta^{q_n}(n)\right)^4 F_4^{q_n}(n)$.

Finally,

$$\lim_{n\to\infty}[n]_{q_n}^2 S_3(n,q_n)$$
$$= \lim_{n\to\infty}[n]_{q_n}^2 \frac{E_1(q_n)[n-4]_{q_n}^6}{q_n^{17}F_4^{q_n}(n)\left(G_\beta^{q_n}(n)\right)^4} + \lim_{n\to\infty}\frac{H_3\left(n,[n-4]_{q_n},q_n,\alpha,\beta\right)}{q_n^{17}F_4^{q_n}(n)\left(G_\beta^{q_n}(n)\right)^4} = 12$$

(2.7)

where

$$E_1(q_n) = -3q_n^{33} - 3q_n^{32} + (6\alpha^2-7)q_n^{31} - (12\alpha^2+20\alpha+9)q_n^{30} - (4\alpha-2)q_n^{29}$$
$$+(6\alpha^2+12\alpha+10)q_n^{28} + (8\alpha+11)q_n^{27} + (4\alpha+7)q_n^{26} + 3q_n^{25} + q_n^{24},$$

and degrees of $H_1\left(n,[n-4]_{q_n},\alpha,\beta\right)$, $H_2\left(n,[n-4]_{q_n},\alpha,\beta\right)$, $H_3\left(n,[n-4]_{q_n},\alpha,\beta\right)$ according to $[n]_{q_n}[n-4]_{q_n}$ is lower then $F_4^{q_n}(n)\left(G_\beta^{q_n}(n)\right)^4$. Multiplying (2.3) by $[n]_{q_n}^2$, taking limits and combining the limits among (2.4)-(2.7) we reach the desired result.

The weighted Korovkin- type theorems was proved by Gadzhiev [7]. We give the Gadzhiev's results in weighted spaces. Let $\rho(x) = 1 + x^2$. $B_\rho[0,\infty)$ denotes the set of all functions f, from $[0,\infty)$ to IR, satisfying growth condition $|f(x)| \le N_f\rho(x)$, where N_f is a constant depending only on f. $B_\rho[0,\infty)$ is a normed space with the norm $\|f\|_\rho = \left\{sup\frac{|f(x)|}{\rho(x)} : x \in IR\right\}$. $C_\rho^*[0,\infty)$ denotes the subspace of continuous functions in $B_\rho[0,\infty)$ for which $\lim_{|x|\to\infty}\frac{|f(x)|}{\rho(x)}$ exists finitely.

We now give a Voronovskaja type theorem for $B_{n,q_n}^{(\alpha,\beta)}$ operators.

Theorem 1. Let $(q_n) \subset (0,1)$ a sequence such that $q_n \to 1$ and $q_n^n \to \lambda$ as $n \to \infty$. For any $f \in C_\rho^*[0,\infty)$ such that $f', f'' \in C_\rho^*[0,\infty)$. We have the limit

$$\lim_{n\to\infty}[n]_{q_n}\left(B_{n,q_n}^{(\alpha,\beta)}(f,x) - f(x)\right) = ((2-\lambda-\beta)x + 1 + \alpha)f'(x) + (\left(1-\frac{\lambda}{2}\right)x^2 + x)f''(x).$$

Proof. By Taylor's expansion of *f*, we have

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \varepsilon(t,x)(t-x)^2,$$

where $\varepsilon(t,x) \to 0$ as $t \to x$. From linearity of the operators $B_{n,q_n}^{(\alpha,\beta)}$ we have

$$B_{n,q_n}^{(\alpha,\beta)}(f,x) = f(x) + f'(x)B_{n,q_n}^{(\alpha,\beta)}\left((t-x),x\right) + \frac{1}{2}f''(x)B_{n,q_n}^{(\alpha,\beta)}((t-x)^2,x) + B_{n,q_n}^{(\alpha,\beta)}(\varepsilon(t,x)(t-x)^2,x).$$

Using Lemma 1 and making necessary process, we abtain

$$B_{n,q_n}^{(\alpha,\beta)}(f,x) - f(x) = f'(x)\left\{\frac{\left(1+q_n[n-1]_{q_n}\right)^2 - q_n^2\left(1+\beta+q_n[n-1]_{q_n}\right)[n-1]_{q_n}}{q_n^2F_1^{q_n}(n)G_\beta^{q_n}(n)}x\right.$$
$$\left.+\frac{[n]_{q_n}+q_n\alpha[n-1]_{q_n}}{q_nF_1^{q_n}(n)G_\beta^{q_n}(n)}\right\} + \frac{1}{2}f''(x)\left\{\left(\frac{[n-2]_{q_n}^4(q_n^{11}-2q_n^{10}+q_n^8)}{q_n^6F_2^{q_n}(n)\left(G_\beta^{q_n}(n)\right)^2}\right.\right.$$
$$\left.+\frac{(q_n^{10}+2\left([2]_{q_n}+\beta\right)q_n^9 - 2\left(3[2]_{q_n}+\beta\right)q_n^8 + 4[2]_{q_n}q_n^6)[n-2]_{q_n}^3}{q_n^6F_2^{q_n}(n)\left(G_\beta^{q_n}(n)\right)^2}\right)x^2$$

$$+\frac{(q_n^6+2(1+\alpha)q_n^7-(1+2\alpha)q_n^8)[n-2]_{q_n}^3}{q_n^5F_2^{q_n}(n)\left(G_\beta^{q_n}(n)\right)^2}x+\frac{[2]_{q_n}[n]_{q_n}^2}{q_n^3F_2^{q_n}(n)\left(G_\beta^{q_n}(n)\right)^2}$$

$$\left.+\frac{2\alpha[n]_{q_n}}{q_n\,F_1^{q_n}(n)\left(G_\beta^{q_n}(n)\right)^2}+\frac{\alpha^2}{\left(G_\beta^{q_n}(n)\right)^2}\right\}+B_{n,q_n}^{(\alpha,\beta)}(\varepsilon(t,x)(t-x)^2,x).$$

For third term on the right side, using Cauchy-Schwarz inequality we have

$$\lim_{n\to\infty}[n]_{q_n}B_{n,q_n}^{(\alpha,\beta)}(\varepsilon(t,x)(t-x)^2,x)\le\sqrt{\lim_{n\to\infty}B_{n,q_n}^{(\alpha,\beta)}(\varepsilon^2(t,x),x)}\sqrt{\lim_{n\to\infty}[n]_{q_n}^2B_{n,q_n}^{(\alpha,\beta)}((t-x)^4,x)}$$

Since $\lim_{n\to\infty}B_{n,q_n}^{(\alpha,\beta)}(\varepsilon^2(t,x),x)=0$ and from Lemma 2 (ii) $\lim_{n\to\infty}[n]_{q_n}^2B_{n,q_n}^{(\alpha,\beta)}((t-x)^4,x)$ is finite. We yield

$$\lim_{n\to\infty}[n]_{q_n}B_{n,q_n}^{(\alpha,\beta)}(\varepsilon(t,x)(t-x)^2,x)=0.$$

Thus we get

$$\lim_{n\to\infty}[n]_{q_n}\left(B_{n,q_n}^{(\alpha,\beta)}(f(t),x)-f(x)\right)$$

$$=f'(x)\left\{\lim_{n\to\infty}\frac{(q_n^2-q_n^3)[n-1]_{q_n}+\left(2q_n-(1+\beta)\right)[n-1]_{q_n}[n]_{q_n}}{q_n^2F_1^{q_n}(n)G_\beta^{q_n}(n)}x\ +1+\alpha\right\}$$

$$+\frac{1}{2}f''(x)\lim_{n\to\infty}\left\{\left((1-q_n^n)\frac{(q_n^8+q_n^9-q_n^{10})[n-2]_{q_n}^4}{q_n^6F_2^{q_n}(n)\left(G_\beta^{q_n}(n)\right)^2}+\frac{\left(q_n^{10}+2\left([2]_{q_n}+\beta\right)q_n^9\right.}{q_n^6F_2^{q_n}(n)\left(G_\beta^{q_n}(n)\right)^2}\right.\right.$$

$$\left.-\frac{2\left(3[2]_{q_n}+\beta\right)q_n^8-4[2]_{q_n}q_n^6)[n-2]_{q_n}^3[n]_{q_n}}{q_n^6F_2^{q_n}(n)\left(G_\beta^{q_n}(n)\right)^2}\right)x^2+\left(\frac{(q_n^6+2(1+\alpha)q_n^7}{q_n^5F_4^{q_n}(n)\left(G_\beta^{q_n}(n)\right)^2}\right.$$

$$\left.\left.-\frac{(1+2\alpha)q_n^8)[n-2]_{q_n}^3[n]_{q_n}}{q_n^5F_2(n)\left(G_\beta^{q_n}(n)\right)^2}\right)x\right\}$$

$$=\left((2-\lambda-\beta)x+1+\alpha\right)f'(x)+\left(\left(1-\frac{\lambda}{2}\right)x^2\ +x\right)f''(x)$$

and the proof is completed.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES

[1] A. Aral, V. Gupta and R. P. Agarwal, Applications of q −calculus in operator theory, Springer, New York, 2013.

[2] De Sole, A. and Kac, V. G., On integral representations of q −gamma and q −beta functions, Atti. Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl., 16(1) (2005) 11-29.

[3] De Vore R. A. and Lorentz, G. G., Constructive Approximation, Springer, Berlin 1993.

[4] Dinlemez, Ü. Convergence of the q −Stancu-Szasz-beta type operators, J. Inequal. Appl. 2014, 2014:354, 8 pp.

[5] Doğru, O. and Gupta, V., Monotonicity and the asymptotic estimate of Bleimann Butzer and Hahn

operators based on q -integers, Georgian Math. J. 12 (2005) (3) 415-422.

[6] Doğru, O. and Gupta, V., Korovkin-type approximation properties of bivariate $q-$Meyer-König and Zeller operators, Calcolo 43 (1) (2006) 51-63.

[7] Gadzhiev, A. D., Theorems of the type of P. P. Korovkin type theorems, Math. Zametki 20(5) (1976) 781-786; English Translation, Math. Notes, 20(5/6) (1976) 996-998.

[8] Govil, N. K. and Gupta, V., $q-$Beta-Szász-Stancu operators. Adv. Stud. Contemp. Math. 22(1) (2012) 117-123.

[9] Gupta, V. G., Srivastava, S. and Sahai, A., On simultaneous approximation by Szász-beta operators, Soochow J. Math. 21(1) (1995) 1-11.

[10] Gupta, V. and Heping, W., The rate of convergence of q-Durrmeyer operators for

$0 < q < 1$, Math. Methods Appl. Sci. 31(16) (2008) 1946-1955.

[11] Gupta, V. and Aral, A., Convergence of the $q-$analogue of Szász-beta operators, Appl. Math. Comput., 216 (2) (2010) 374-380.

[12] Gupta, V. and Karslı, H., Some approximation properties by Szász -Mirakyan-Baskakov- Stancu operators, Lobachevskii J. Math. 33(2) (2012) 175-182.

[13] Gupta, V. and Mahmudov, N. I., Approximation properties of the $q-$Szasz-Mirakjan-Beta operators, Indian J. Industrial and Appl. Math. 3(2) (2012) 41-53.

[14] Jackson, F. H., On $q-$definite integrals, quart. J. Pure Appl. Math., 41(15) (1910) 193-203.

[15] Kac, V. G. and Cheung, P., Quantum calculus, Universitext. Springer-Verlag, New York, 2002.

[16] Koelink, H. T. and Koornwinder, T. H., $q-$special functions, a tutorial, Deformation theory and quantum groups with applications to mathematical physics (Amherst, MA, 1990) 141,142, Contemp. Math., 134, Amer. Math. Soc., Providence, RI, 1992.

[17] Lupaş, A. A., $q-$analogue of the Bernstein operator, Seminar on numerical and statistical calculus, University of Cluj-Napoca 9 (1987) 85-92.

[18] Mahmudov, N. I., $q-$Szász operators which preserve x^2 . Slovaca 63(5) (2013) 1059-1072

[19] Phillips, G. M., Bernstein polynomials based on the q-integers, Ann. Numer. Math. 4 (1997) 511-518.

[20] Yüksel, İ., Approximation by $q-$Phillips operators, Hacet. J. Math. Stat. 40 (2011) no. 2, 191-201.

[21] Yüksel, İ., Direct results on the $q-$mixed summation integral type operators, J. Appl. Funct. Anal. 8(2) (2013) 235-245.

[22] Yüksel, İ. and Dinlemez, Ü., Voronovskaja type approximation theorem for $q-$Szász-beta operators. Appl. Math. Comput. 235 (2014) 555-559.

[23] Yüksel, İ. and Dinlemez, Ü., Weighted approximation by the $q-$ Szász -Schurer-beta type operators, Gazi University Journal of Science. 28(2) (2015) 231-238.