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Fixed Point Theorem Through Ω-distance of Suzuki Type Contraction Condition

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ABSTRACT

In this article, we utilize the notion of Ω-distance in the sense of Saadati et al [R. Saadati, S.M. Vaezpour, P. Vetro and B.E. Rhoades, Fixed point theorems in generalized partially ordered G-metric spaces, Mathematical and Computer Modeling, 52, 797-801, 2010] to introduce and prove some fixed point results of self-mapping under contraction conditions of the form Ω-Suzuki-contractions.

Key Words: *Ω-Distance, Fixed Point Theory, G-Metric Space.*

1. INTRODUCTION

G-metric space was introduced by Mustafa and Sims [1] in 2006, which is a generalization of metric space. Since 2006, many researchers have worked on G-metric spaces; see for example [2]-[10].

Samet et al in [11] and [12] proved that many results in G-metric spaces can be derived from known results of the corresponding usual metric space. Moreover, the notion of Ω-distance related to a complete G-metric space was considered by Saadati *et.al.* [13] in 2010.

Recently, many researchers studied several fixed point results using Ω-distance mappings; see for example, [14]-[17]. It is worth mentioning that the interesting method of Samet et. al. [11] and [12] doesn't work in the fixed point results involving Ω-distance.

In this paper, we prove new results of fixed point theorem using the map Ω in a complete G-metric space under contractive conditions of the form Ω-Suzuki-contraction.

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Definition 1.1. [1]. *Let X be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$ be a function thatsatisfies the following conditions:*

(G1) $G(x, y, z) = 0$ *if* $x = y = z$;

(G2) $G(x, x, y) > 0$ *for all* $x, y \in X$ *with* $x \neq y$;

(G3) $G(x, y, y) \leq G(x, y, z)$ *for all* $x, y, z \in X$ *with* $y \neq z$;

(G4) $G(x, y, z) = G(p\{x, y, z\})$, *for any permutation of* x, y, z;

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ *for all* $x, y, z, a \in X$.

Then the function G is called a generalized metric space, or more specifically G-metric on X, and the pair (X, G) *is called a G-metric space.*

The notion of convergence and Cauchy sequences in the setting of a G-metric space are given as follows:

Definition 1.2. [1]. *Let* (X, G) *be a G-metric space, and let* (x_n) *be a sequence of points of X. We say that* (x_n) *is G-convergent to x if for any* $\epsilon > 0$, *there exists* $k \in \mathbb{N}$ *such that* $G(x, x_n, x_m) < \epsilon$ *for all* $n, m \geq k$.

Definition 1.3. [1]. *Let* (X, G) *be a G-metric space. A sequence* (x_n) *in X is said to be G-Cauchy if for every* $\epsilon > 0$, *there exists* $k \in N$ *such that* $G(x_n, x_m, x_l) < \epsilon$ *for all* $n, m, l \geq k$.

Definition 1.4. [5]. *A G-metric space* (X, G) *is said to be G-complete or complete G-metric space if every G-Cauchy sequence in* (X, G) *is G-convergent in* (X, G).

In 2010, Saadati *et. al.* [13] introduced the notion of Ω-distance related to a complete G-metric space and proved many results.

Definition 1.5. [13]. *Let* (X, G) *be a G-metric space. Then a function* $\Omega : X \times X \times X \to [0, \infty)$ *is called an Ω-distance on X if the following conditions are satisfied:*

(a) $\Omega(x, y, z) \leq \Omega(x, a, a) + \Omega(a, y, z)$ *for all* $x, y, z, a \in X$,

(b) *for any* $x, y \in X$, *the functions* $\Omega(x, y, .)$, $\Omega(x, ., y) : X \to [0, \infty)$ *are lower semi continuous,*

(c) *for each* $\epsilon > 0$, *there exists* $\delta > 0$ *such that if* $\Omega(x, a, a) \leq \delta$ *and* $\Omega(a, y, z) \leq \delta$, *then* $\Omega(x, y, z) \leq \epsilon$.

Definition 1.6. [13]. *Let (X,G) be a G-metric space and Ω be an Ω-distance on X. Then we say that X is Ω-bounded if there exists* $M > 0$ *such that* $\Omega(x, y, z) \leq M$ *for all* $x, y, x \in X$.

The following lemma plays an important role in the development of the results in this article.

Lemma 1.1. [13]. *Let X be a metric space with metric G and Ω be an Ω-distance on X. Let* (x_n), (y_n) *be sequences in X, and* (α_n),(β_n) *be sequences in* $[0, \infty)$ *converging to zero. Then for all* $x, y, z, a \in X$, *we have the following:*

(1) *If* $\Omega(y, x_n, x_n) \leq \alpha_n$ *and* $\Omega(x_n, y, z) \leq \beta_n$ *for* $n \in \mathbb{N}$, *then* $\Omega(y, y, z) < s$ *and hence* $y = z$;

(2) *If* $\Omega(y_n, x_n, x_n) \leq \alpha_n$ *and* $\Omega(x_n, y_m, z) \leq \beta_n$ *for all* $m > n \in \mathbb{N}$, *then* $\Omega(y_n, y_m, z) \to 0$ *and hence* $y_n \to z$;

(3) *If* $\Omega(x_n, x_m, x_l) \leq \alpha_n$ *then the sequence* (x_n) *is a G-Cauchy sequence, for all* $m, n, l \in \mathbb{N}$ *with* $n \leq m \leq l$,;

(4) *If* $\Omega(x_n, a, a) \leq \alpha_n$ *for any* $n \in \mathbb{N}$, *then* (x_n) *is a G-Cauchy sequence.*

2. MAIN RESULT

Definition 2.7. [19] *A nondecreasing continuous function* $\varphi : [0, \infty) \to [0, \infty)$ *is called an altering distance function if the following condition holds;* $\varphi(t) = 0$ *if and only if* $t = 0$.

Definition 2.8. *A mapping* $T : X \to X$ *of a G-metric space (X,G) is called an Ω-Suzuki-contraction if there exists* $k \in [0, 1)$ *and an altering distance function φ such that for all* $x, y, z \in X$ *and* $p, q \in \mathbb{N}$ *with* $q \geq p$, *the following condition holds*

if $(1 - k)\,\Omega(x, T^p x, T^q x) \leq \Omega(x, y, z)$, *then* $\varphi\Omega(Tx, Ty, Tz) \leq k\,\varphi\Omega(x, y, z)$.

Theorem 2.2. *Let* (X, G) *be a complete G-metric space and Ω be an Ω-distance on X such that X is Ω-bounded. Let* $T : X \to X$ *be an Ω-Suzuki-contraction mapping that satisfies the following condition:*

for all $u \in X$ *if* $Tu \neq u$, *then*

$$\inf\{\Omega(x, Tx, u) : x \in X\} > 0. \tag{2.1}$$

Then T has a fixed point in X. Moreover, for any fixed Point $z \in X$ *of T, we have* $\Omega(z, z, z) = 0$.

Proof. Let $x_0 \in X$ and define a sequence (x_n) in X inductively by setting $x_n = Tx_{n-1}$, $n \in \mathbb{N}$.

For $p = q = 1$, since $(1 - k)\,\Omega(x, Tx, Tx) \leq \Omega(x, Tx, Tx)$ holds for every $x \in X$, we have

$$\varphi\Omega(Tx, T^2x, T^2x) \leq k\varphi\Omega(x, Tx, Tx). \tag{2.2}$$

Substituting $x = x_{n-1}$ in the inequality (2.2), gives us

$$\varphi\Omega(x_n, x_{n+1}, x_{n+1}) = \varphi\Omega(Tx_{n-1}, Tx_n, Tx_n) \leq k\,\varphi\Omega(x_{n-1}, x_{n-1}, x_n). \tag{2.3}$$

Since $k < 1$ and φ is an altering distance function, the sequence $(\Omega(x_n, x_{n+1}, x_{n+1})$: $n \in \mathbb{N})$ is a non- increasing sequence of nonnegative real numbers. Therefore, there is $r \geq 0$ such that

$$\lim_{n\to\infty} \Omega(x_n, x_{n+1}, x_{n+1}) = r.$$

Taking the limit as $n \to \infty$ in 2.3, implies that $\varphi r \leq k\varphi r$ and thus $r = 0$, since $k < 1$. Hence

$$\lim_{n\to\infty} \Omega(x_n, x_{n+1}, x_{n+1}) = 0. \tag{2.4}$$

Moreover, for $p = 1$, and $q \geq 1$, since $(1-k)\Omega(x, Tx, T^q x) \leq \Omega(x, Tx, T^q x)$ holds for every $x \in X$, then

$$\varphi\Omega(Tx, T^2x, T^{q+1}x) \leq k\,\varphi\Omega(x, Tx, T^q x). \tag{2.5}$$

For $n, s \in \mathbb{N}$ with $s \geq 1$, substituting $x = x_{n-1}$ in (2.5), implies that

$\varphi\Omega(x_n,x_{n+1},x_{n+s}) = \varphi\Omega(Tx_{n-1},Tx_n,Tx_{n+s-1})$

$\leq k\ \varphi\Omega(x_{n-1},x_n,x_{n+s-1}).$ (2.6)

Since $k < 1$ and φ is an altering distance function, the sequence $(\Omega(x_n, x_{n+1}, x_{n+s})$: $n \in \mathbb{N})$ is a non-increasing sequence of nonnegative real numbers. Therefore, there is $r \geq 0$ such that

$$\lim_{n\to\infty} \Omega(x_n,x_{n+1},x_{n+s}) = r.$$

Applying the limit as $n \to \infty$ to the inequality 2.6, gives us $\varphi r \leq k\ \varphi r$. Since $k < 1$, we have $r = 0$ and hence

$$\lim_{n\to\infty} \Omega(x_n,x_{n+1},x_{n+s}) = 0, \text{ for all } s \geq 1. \quad (2.7)$$

Considering the Definition 1.5, implies that

$\Omega(x_n, x_m, x_l) \leq \Omega(x_n, x_{n+1}, x_{n+1}) + \Omega(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + \Omega(x_{m-1}, x_m, x_l),$

for all $l,m,n \in \mathbb{N}$ with $l \geq m \geq n$, $m = n+s$ and $l = m + t$.

By taking the limit of the above inequality as $n \to \infty$, we get

$$\lim_{n,m,l\to\infty} \Omega(x_n,x_m,x_l) = 0.$$

Lemma 1.1 implies that (x_n) is a G-Cauchy sequence and hence (x_n) converges to an element $u \in X$. For all $\epsilon > 0$, since (x_n) is a G-Cauchy sequence, there exists $N \in \mathbb{N}$ such that $\Omega(x_n,x_m,x_l) < \epsilon$, for all $n,m,l \geq N$. Thus,

$$\lim_{l\to\infty} \inf \Omega(x_n,x_m,x_l) \leq \epsilon, \text{ for all } n,m \geq N.$$

The lower semi-continuity of Ω implies that

$$\Omega(x_n,x_m,u) \leq \lim_{l\to\infty} \inf \Omega(x_n,x_m,x_l) \leq \epsilon, \text{for all } n,m \geq N.$$

Considering $m = n+1$ in (2.8), gives us $\Omega(x_n,x_{n+1},u) \leq \epsilon$, for all $n \geq N$.

Assume that $Tu \neq u$. Then 2.1 implies that

$0 < \inf\{\Omega(x,Tx,u) : x \in X\} \leq \inf\{\Omega(x_n,x_{n+1},u) :$

$n \geq N\} \leq \epsilon$, for all $\epsilon > 0$ which is a contradiction.

Therefore $Tu = u$. Let $z = Tz$. Then by (2.2), we have

$\Omega(z,z,z) = \Omega(Tz,T^2z,T^2z) \leq k\ \varphi\Omega(z,Tz,Tz) =$

$k\ \varphi\Omega(z,z,z).$

Since $k < 1$ and φ is an altering distance function,we have $\Omega(z,z,z) = 0$.

Definition 2.9. *A mapping $T : X \to X$ of a G-metric space (X,G) is called a generalized Ω-Suzuki-contraction if there exists $k \in [0, 1)$ and an altering distance function φ such that the following condition holds:*

If for all $p,q \in \mathbb{N}$ with $q \geq p$,

$(1 - k)\ \Omega(x, T^px, T^qx) \leq \Omega(x,y,z)$

then we have

$\Omega(Tx,Ty,Tz) \leq k \max\{\Omega(x,Tx,Tx), \Omega(y,Ty,Ty), \Omega(z,Tz,Tz)\}$

for all $x,y,z \in X$.

Lemma 2.3. *Let $T : X \to X$ be a generalized Ω-Suzuki-contraction. Then*

$\Omega(Tx,T^2x,T^2x) \leq k\ \Omega(x,Tx,Tx)$ *for all $x \in X$.* (2.9)

Proof. Assume $p = q = 1$. Since $(1 - k)\Omega(x, Tx, Tx) \leq \Omega(x,Tx,Tx)$ holds for every $x \in X$, then we have

If $\max\{\Omega(x, Tx, Tx), \Omega(x, T^2x, T^2x)\} = \Omega(x, T^2x, T^2x)$, then $\Omega(x,T^2x,T^2x) \leq k\Omega(x,T^2x,T^2x)$ which is a contradiction, since $k < 1$. Therefore, $\max\{\Omega(x,Tx, Tx),\Omega(x,T^2x,T^2x)\} = \Omega(x,Tx,Tx)$ and hence

$\Omega(Tx,T^2x,T^2x) \leq k\ \Omega(x,Tx,Tx)$ for all $x \in X$. (2.10)

Lemma 2.4. *Let $q \geq 1$ and $T : X \to X$ be a generalized Ω-Suzuki-contraction. Then*

$\Omega(T^qx,\ T^{q+1}x,\ T^{q+1}x) \leq k^q\ \Omega\ (x, Tx, Tx)$ *for all $x \in X$.*

Proof. By substituting x in Lemma (2.3) by $T^{q-1}x$, we get

$$\begin{aligned}\Omega(T^qx,\ T^{q+1}x,\ T^{q+1}x) &= \Omega(T\ (T^{q-1}x),\ T\ (T^qx),\ T(T^qx)) \\ &\leq k\ \Omega(T^{q-1}x,T^qx,T^qx) \\ &\quad . \\ &\quad . \\ &\quad . \\ &\leq k^q\ \Omega(x, Tx, Tx).\end{aligned}$$

Thus (2.8)

$\Omega(T^qx,T^{q+1}x,T^{q+1}x) \leq k^q\ \Omega(x,Tx,Tx).$ (2.11)

Theorem 2.5. *Let (X, G) be a complete G-metric space and Ω be an Ω-distance on X such that X is Ω-bounded. Let T be a self-mapping on X that satisfies the following conditions:*

(1) *T is a generalized Ω-Suzuki-contraction;*

(2) *if for all $u \in X$, $Tu \neq u$, then*

$\inf\{\Omega(x,Tx,u) : x \in X\} > 0.$ (2.12)

Then T has a fixed point in X.

Proof. Let $x_0 \in X$ and define a sequence (x_n) in X inductively by taking $x_n = Tx_{n-1}$ for $n \in \mathbb{N}$.

Substitute $x = x_{n-1}$ in (2.10), implies that

$$\begin{aligned}\Omega(x_n,x_{n+1},x_{n+1}) &= \Omega(Tx_{n-1},Tx_n,Tx_n) \\ &\leq k\ \Omega(x_{n-1},x_{n-1},x_n) \\ &\quad . \\ &\quad . \\ &\quad . \\ &\leq k^n\ \Omega(x_0,x_1,x_1).\end{aligned}$$

Since X is Ω-bounded, there exists $M > 0$ such that $\Omega(x,y,z) \leq M$ for all $x,y,z \in X$. Hence

$\Omega(x_n, x_{n+1}, x_{n+1}) \leq k^n M.$

By taking the limit as $n \to \infty$ for both sides, we get

$$\lim_{n\to\infty} \Omega(x_n, x_{n+1}, x_{n+1}) = 0. \quad (2.13)$$

since $k < 1$. Also, for $p = 1$, and $q \geq 1$, since $(1-k)\Omega(x, Tx, T^q x) \leq \Omega(x, Tx, T^q x)$ holds for every $x \in X$, we have

$$\Omega(Tx, T^2x, T^{q+1}x) \leq k \max\{\Omega(x,Tx,Tx), \Omega(Tx, T^2x, T^2x), \Omega(T^q x, T^{q+1}x, T^{q+1}x)\}$$

$$= k \max\{\Omega(x,Tx,Tx), \Omega(T^q x, T^{q+1}x, T^{q+1}x)\}.$$

But from 2.11, we have $\Omega(T^q x, T^{q+1}x, T^{q+1}x) \leq k^q \Omega(x, Tx, Tx)$ and thus,

$$\Omega(Tx, T^2x, T^{q+1}x) \leq k \max\{\Omega(x,Tx,Tx), k^q \Omega(x,Tx,Tx)\}.$$

Since $k < 1$, we have

$$\Omega(Tx, T^2x, T^{q+1}x) \leq k\, \Omega(x, Tx, Tx). \quad (2.14)$$

For $n, s \in \mathbb{N}$ with $s \geq 1$ substitute $x = x_{n-1}$ in (2.14), implies that

$$\Omega(x_n, x_{n+1}, x_{n+s}) = \Omega(Tx_{n-1}, T^2 x_{n-1}, T x_{n+s-1}) \leq k\, \Omega(x_{n-1}, x_n, x_n).$$

Taking the limit as $n \to \infty$ for both sides and using 2.13, we get

$$\lim_{n\to\infty} \Omega(x_n, x_{n+1}, x_{n+s}) = 0. \quad (2.15)$$

The Definition 1.5 implies that

$$\Omega(x_n, x_m, x_l) \leq \Omega(x_n, x_{n+1}, x_{n+1}) + \Omega(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + \Omega(x_{m-1}, x_m, x_l),$$

for all $l, m, n \in \mathbb{N}$ with $l \geq m \geq n$, $m = n + s$ and $l = m + t$.

Applying the limit as $n \to \infty$ and using 2.13 and 2.15, we get that

$$\lim_{n,m,l\to\infty} \Omega(x_n, x_m, x_l) = 0.$$

Lemma 1.1 implies that (x_n) is a G-Cauchy sequence and so (x_n) converges to some $u \in X$. Since (x_n) is a G-Cauchy sequence, then for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\Omega(x_n, x_m, x_l) \leq \epsilon$, for all $n, m, l \geq N$. Thus

$$\lim_{l\to\infty} \inf \Omega(x_n, x_m, x_l) \leq \epsilon.$$

Since Ω is lower semi-continuous, we have

$$\Omega(x_n, x_m, u) \leq \lim_{l\to\infty} \inf \Omega(x_n, x_m, x_l) \leq \epsilon, \quad (2.16)$$

for all $n, m \geq N$.

Considering $m = n + 1$ in (2.16), we get $\Omega(x_n, x_{n+1}, u) \leq \epsilon$, for all $n \geq N$. Suppose that $Tu \neq u$. Then Condition 2.12 implies that

$0 < \inf\{\Omega(x, Tx, u) \colon x \in X\} \leq \inf\{\Omega(x_n, x_{n+1}, u) \colon n \geq N\} \leq \epsilon$, for all $\epsilon > 0$ which is a contradiction. Therefore $Tu = u$.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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