Gazi University Journal of Science
GU J Sci
28(4):631-637 (2015)

On the Numerical Solution of Evolution Equation via Soliton Kernels

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Received: 05/01/2015 Accepted: 11/11/2015

ABSTRACT

In application of kernel-based methods, some particular types of PDEs need some special types of kernels for their approximations. For example some nonlinear evolution equations describing wave processes in dispersive and dissipative media. These models may have soliton like solutions for example KdV equation. In such a situation some special types of kernels may perform better than standards kernels for example soliton kernels.

Keywords: *Soliton kernels, Meshless technique, RBF-PS scheme, evolution equations.*

1. INTRODUCTION

For approximating PDEs in the fields of computer experiments, response surface modeling, finance and image processing, we need some special types of kernels (for more detail see [1, 2]). These special types of kernels may perform better than the available standard kernels. These kernels are more appropriate than standard kernels when looking at special solutions of PDEs. For example nonlinear evolution equations describing wave processes in dispersive and dissipative media may have soliton like solutions, so soliton kernels, would be more suited to approximate the solution of such types of PDEs. An example is the Korteweg-de Vries (KdV) equation,

$$v_t + \varepsilon v v_x + \mu v_{xxx} = 0, x \in [a,b], t > 0 \quad (1)$$

where ε and μ are real constant and the subscripts represent differentiation, e.g.

$v_t = \partial v / \partial t$. This nonlinear transport or wave equation has a special traveling soliton solution.

For solving such types of PDEs we have the soliton kernel

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$$\psi(x,y) = \operatorname{sech}^2(\|x-y\|) \tag{2}$$

which is positive definite. Korteweg-de Vries equation is a model to describe the behavior of shallow-water waves [3]. The Korteweg-de Vries (KdV) equation has a variety of application in many fields for example waves in a fluid mechanics, waves in a ion-acoustic waves in a plasma, waves in a liquid-gas, and wave in harmonic crystals. The kernel-based methods are based on interpolating functions expressed entirely in terms of nodes [4]. During the past two decades the meshless methods have been developed and effectively applied to solve many engineering and science problems [5-11]. Many radial kernels contain a free shape parameter and solution accuracy greatly depends on this parameter. For improved accuracy, the user has to choose an optimal value of the shape parameter. But the soliton kernel is free of shape parameter.

2. SOLITON KERNEL-BASED MESHLESS METHOD

Corresponding to a set of centers, $\{x_1, x_2, \ldots, x_N\} \subseteq \Omega$, the kernel-based approximation is given as

$$v(x,t) = \sum_{j=1}^{N} \alpha_j(t)\psi(x,x_j), x \in \Omega. \tag{3}$$

where the soliton kernel ψ is given as $\psi(x,x_j) : \kappa(\|x-x_j\|) = \operatorname{sech}^2(\|x-x_j\|), 1 \le j \le N$, and $r = \|x-x_j\|$ denotes the Euclidean distance between two points x and x_j is a function defined for $r \ge 0$. For the given set of interpolation points $\{x_1, x_2, \ldots, x_N\}$, the kernel-based interpolation is obtained by solving the system of equations,

$$v(x_i,t) = \sum_{j=1}^{N} \alpha_j(t)\psi(x_i,x_j), 1 \le i \le N. \tag{4}$$

The matrix form of the above system is given as

$$v = B\alpha\ , \tag{5}$$

where the entries of the interpolation matrix B are $\psi(x_i,x_j), 1 \le i, j \le N$, and

the vector of expansion coefficients is $[\alpha_1, \alpha_2, \ldots, \alpha_N]^T$. Using equation (4) the derivatives v_x may be obtained by differentiating the kernel functions and then evaluating at each point $x_i, 1 \le i \le N$, we have in matrix-vector notation

$$v_x = B_x\alpha, \tag{6}$$

where the entries of the matrix B_x are $\frac{d}{dx}\psi(x,x_j)_{x=x_i}$, $1 \le i, j \le N.$ The differentiation matrix can be obtained by solving equations (4)-(5) for the value of α. Thus, we have

$$v_x = B_x B^{-1} u = D_x u\ , \tag{7}$$

where $D_x = B_x B^{-1}$ is the differentiation matrix. It should be noted that the differentiation matrix depends on the invertibility of the matrix B. It is well known that the matrix B is always invertible for distinct set of collocation points. In a similar way, we can compute differentiation matrices of higher order.

Using the above differentiation matrices, the numerical scheme corresponding to equations (1) is given as

$$v' + \varepsilon v * D_x v + \mu D_{xxx} v = 0\ . \tag{8}$$

This equation may be written as

$$v' = -\varepsilon v * D_x v - \mu D_{xxx} v. \tag{9}$$

Equation (9) is of the form

$$v' = F(v)\ . \tag{10}$$

Now this ODE system can be discretize in time using any ODE solver like ode45, ode113, ode23. The initial solution vector is v_0. Any good ODE solver will select an appropriate time step δt to overcome the stiffness of ODE system. The fourth-order Runge-Kutta scheme may also be used for solving the generated ODE system.

3. NUMERICAL EXPERIMENTS

In this section we have applied the soliton kernel-based meshless method as discussed above for the numerical solution of the KdV equation. The accuracy and efficiency of the method is verified in terms of the error norms L_∞ and L_2 defined by

$$L_\infty = \|v^{ex} - v^{ap}\|_\infty = \max\left|(v^{ex})_i - (v^{ap})_i\right|, \tag{11}$$

$$L_2 = \left\| v^{ex} - v^{ap} \right\|_2 = \sqrt{h \sum_{i=0}^{N} \left| (v^{ex})_i - (v^{ap})_i \right|^2},$$

where $h = (b-a)/N$, and the invariants of motion defined by

$$I_1 = \int_a^b v dx \cong h \sum_{i=0}^{N} (v^n)_i, \; I_2 = \int_a^b v^2 dx \cong h \sum_{i=0}^{N} (v^n)^2{}_i, \quad (12)$$

2.1 Problem 1

Equation (1) has a soliton type solution [12],

$$v(x,t) = \frac{c}{2} \operatorname{sech}^2 \left(\frac{\sqrt{c}}{2}(x - ct) - 7 \right). \qquad (13)$$

We approximate the solution of (1) by using soliton kernels (2). We choose the spatial domain [0, 40] to approximate the solution of equation (1). The problem is solved for the time domain

[0, 5], where the parameters $\varepsilon = 6$ and $\mu = 1$ have been used. The initial solution v_0 and boundary conditions are extracted from (13). The time integration was carried out by the RK4 method. The results of the present method are compared with other methods in the literature [12]. It is found that the application of soliton kernels for solving the PDEs having soliton type solutions is very is beneficial in terms of computations time as well as in terms of accuracy as evident from Figure 1 and Table 1.

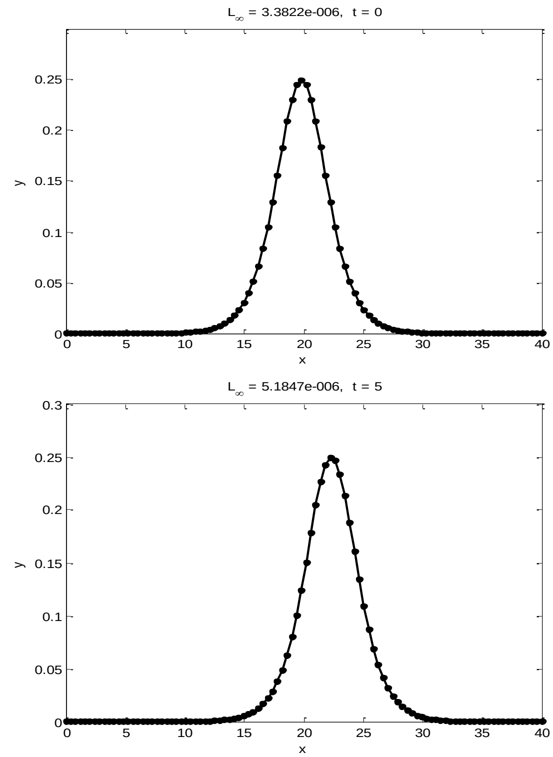


Figure 1: Single solitary wave: $\delta t = 0.001$, $\varepsilon = 6$, $N = 100$, $x \in [0,40]$.

Table 1: Single solitary wave: $\delta t = 0.001$, $\varepsilon = 6$, $\mu = 1$, $N = 200$, $x \in [-10,40]$.

Method	t	L_∞	L_2	C. time
RK4 (Soliton)	1	1.280e-006	5.249e-006	5.32
	2	1.934e-006	6.513e-006	9.86
	3	2.354e-006	7.905e-006	14.57
	4	2.903e-006	9.926e-006	19.08
	5	3.888e-006	1.302e-005	24.28
[32]	1	1.804e-005	6.236e-005	14.00
	2	3.037e-005	1.126e-004	20.00
	3	4.008e-005	1.553e-004	25.00
	4	4.834e-005	1.940e-004	30.00
	5	5.609e-005	2.294e-004	36.00

2. Problem 2

In this problem, we consider the solution of (1) in the form [13]

$$u(x,t) = 2\operatorname{sech}^2(x-4t), \qquad (14)$$

Here again we used soliton kernels (2) to approximate the solution of (1). We select the spatial domain $[-10,40]$, and the time domain $[0,5]$. We compute the solution for the choice of $\varepsilon = 6$, and $\mu = 1$. The boundary and initial conditions have been extracted from analytical solution (13). The time integration was carried out by the RK4 method. The results of the present method are compared with other methods in the literature [13]. Her the again the soliton kernel-based

method performed better.

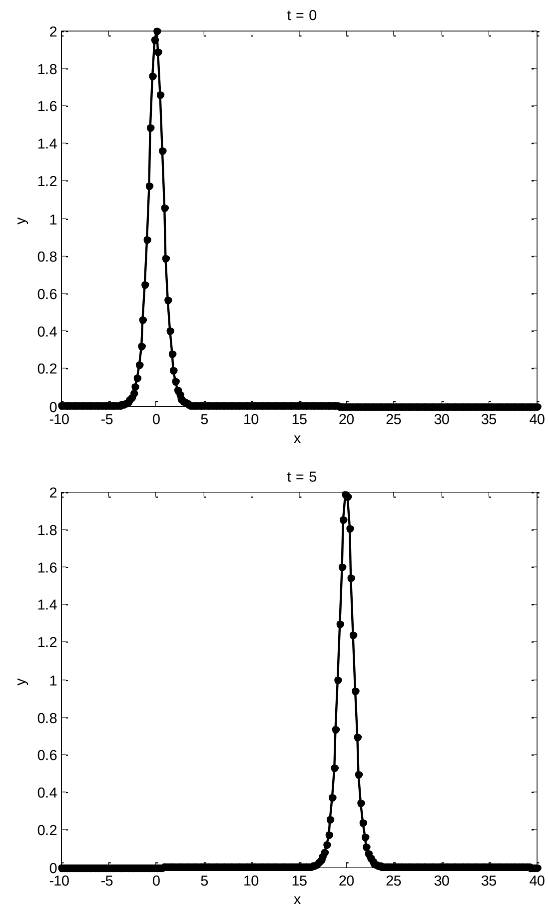


Figure 2: Single solitary wave: $\delta t = 0.001$, $\varepsilon = 6$, $\mu = 1$, $N = 250$, $x \in [-10,40]$.

Method	t	L_∞	L_2	I_1	I_2
RK4(Soliton)	1.0	5.611e-005	4.869e-004	3.98399	5.31200
	2.0	7.694e-005	6.648e-004	3.98397	5.31200
	3.0	5.291e-005	4.536e-004	3.98396	5.31200
	4.0	3.541e-005	2.087e-004	3.98400	5.31200
	5.0	7.655e-005	5.293e-004	3.98399	5.31200
(MQ) [31]	1.0	4.172e-005	6.467e-005	3.9999	5.3333
	2.0	3.565e-005	7.400e-005	3.9999	5.3333
	3.0	3.931e-005	7.960e-005	3.9999	5.3333
	4.0	3.523e-005	8.580e-005	3.9999	5.3333
	5.0	3.822e-005	9.393e-005	3.9999	5.3333
(IMQ) [31]	1.0	4.094e-003	5.932e-003	4.0064	5.3334

2.0	3.992e-003	7.296e-003	4.0081	5.3334
3.0	4.138e-003	7.715e-003	4.0083	5.3334
4.0	3.902e-003	7.898e-003	4.0074	5.3334
5.0	3.873e-003	8.474e-003	4.0057	5.3334

Table 2: Single solitary wave: $\delta t = 0.001$, $\varepsilon = 6$, $\mu = 1$, $N = 250$, $x \in [-10,40]$.

2.3 Problem 3

Now we consider the nonlinear system

$$u_t = -u_x - \alpha uv, \quad v_t = v_x - \alpha uv. \qquad (15)$$

For the sake of comparison [14], we take $a = -0.5, b = 0.5, \nu = 100$, and the initial conditions

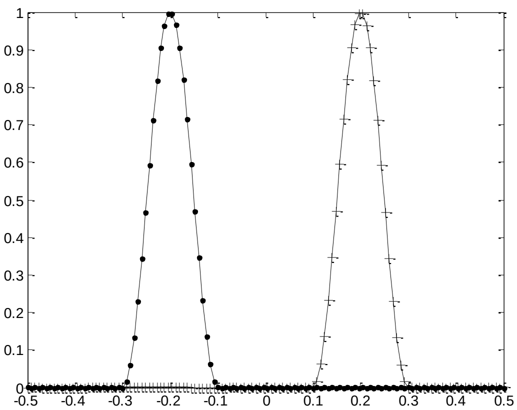
$$u(x,0) = \begin{cases} 0.5\{1+\cos(10\pi x)\}, x \in [-0.3,-0.1]. \\ 0, otherwise \end{cases} \qquad (16)$$

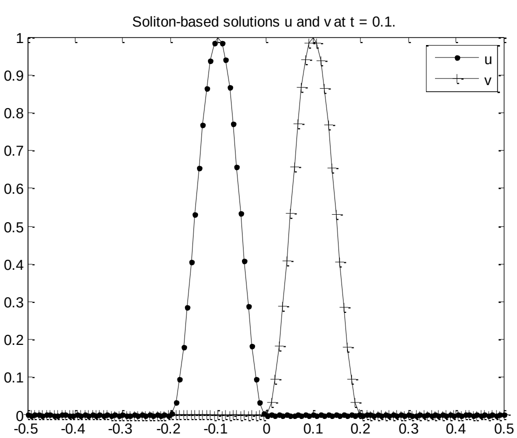
$$v(x,0) = \begin{cases} 0.5\{1+\cos(10\pi x)\}, x \in [0.1,0.3]. \\ 0, otherwise \end{cases} \qquad (17)$$

and with the boundary conditions

$$u(a,t) = u(b,t) = 0, v(a.t) = v(b,t) = 0. \qquad (18)$$

Here we apply the present method for solving the couple system of PDEs. For time integration we used RK4 scheme. The initial solutions $u(x,0)$ and $v(x,0)$ selected in the form of solitons, initially located at positions $x = -0.2$ and $x = 0.2$. For time $t > 0$, the nonlinearity uv causes the waves to move towards each other without any change in shape. The waves u and v collides at time $t = 0.1$, which result change in waves shapes. At time $t = 0.3$ the waves overlap each other and again separated at time $t = 0.3$. The linear term becomes dominant and the waves lose their symmetry and experience a decrease in the amplitude due to nonlinear interaction. The evolution of two waves *u*, and *v* are shown graphically. These graphical results are agreed well with the results obtained by quasi-linear interpolation method [14].





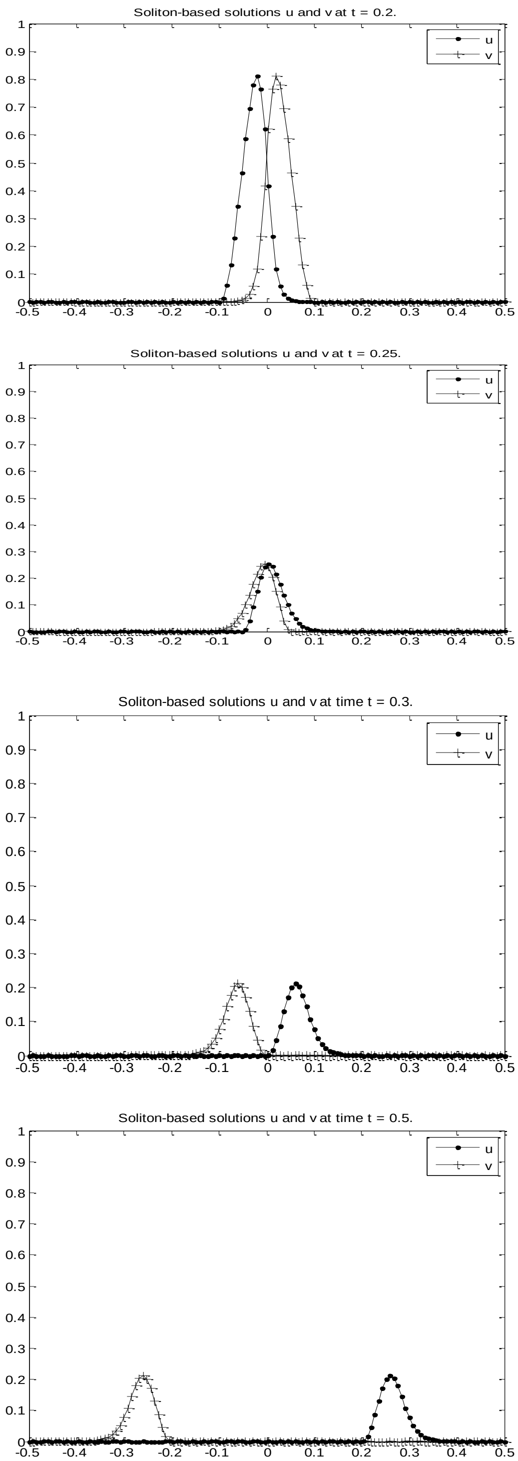


Figure 3: Nonlinear interaction: $\delta t = 0.0001$, $N = 126$, $x \in [-0.5, 0.5]$.

4. CONCLUDING REMARKS

In the present work, the soliton kernel-based meshless method is investigate for the applicability of soliton kernels for numerically approximating PDEs with soliton type solutions. To show how good and accurate the present numerical scheme is, we computed the error norms and invariants of the model and compared the results of the present method with the available methods in the literature. The soliton-based method has been found to be very accurate. The soliton kernel is free of shape parameter and suited to approximate the PDEs which have soliton type solutions.

5. ACKNOWLEDGEMENTS

The third author is thankful to HEC Pakistan for financial support through Grant No. 112-252337-2PS1-127 (50021443). We are also thankful to the reviewers for their constructive comments.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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