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Common Fixed Point Theorems Via (ψ, α, β)-Weak Contractions

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ABSTRACT

In this paper, we prove some common fixed point theorems for a pair of weakly compatible mappings satisfying (ψ, α, β)-weak contractions in fuzzy metric spaces employing a control function. Our results improve and generalize several previously known relevant results of the existing literature. Some illustrative examples are also furnished to substantiate our main results.

Keywords:fuzzy metric space; control functions; coincidence point; weakly compatible mappings; common fixed point.

1. INTRODUCTION

The notion of fuzzy sets was introduced by Zadeh [21] which proved a turning point in the development of Mathematics as the advent of Fuzzy Set Theory sets out the fuzzyfication of almost entire Mathematics. The strength of Fuzzy Mathematics lies in its thought provoking applications. Fuzzy Mathematics has a wide range of applications in applied sciences which include neural network theory, stability theory, mathematical programming, modeling theory, engineering sciences, medical sciences (medical genetics, nervous system), image processing, control theory, communication etc.

After the development of core part of fuzzy set theory, the notion of Fuzzy metric space was introduced by several authors in several ways. The noted paper due to Kramosil and Michalek [13] is very relevant to our presentation. Grabiec [8] extended Banach Contraction Principle to fuzzy metric space. In their noted article, George and Veermani [7] modified the notion of fuzzy metric space with the help of continuous t-norm.

In recent years, several researchers utilized weak contractions to generalize Banach Contraction Principle while Boyd and Wong [3] introduced the notion of ϕ-contractions for the same. In 1997, Alber and Guerre-

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Delabriere [2] introduced the idea ofφ-weak contraction which is indeed a generalization of Φ-contractions. On the other hand, Khan et al. [12] employed the altering distance functions to prove some interesting fixed point theorems. An altering distance function is a control function which alters the metric distance between two points enabling one to deal with relatively new classes of fixed point problems. But, the presence of control function sometimes creates certain difficulties in proving the existence of fixed point under contractive conditions. Altering distances have already been generalized to a two variable function and now in [5] a generalization to a three-variable function has been introduced and utilized to prove fixed point results in metric spaces.In 2011, Abbas et al. [1] introduced the notion of ψ-weak contraction in the framework of fuzzy metric spaces and utilize the same to prove some fixed point theorems for a pair of self mappings. Thereafter, Vetro et al. [19] firstly studied (ϕ,ψ)-weak contraction in fuzzy metric space and established a common fixed point theorem for a sequence of self mappings.

In this paper, as an extension of (ϕ,ψ)-weak contraction, we introduce (ψ,α,β)-weak contraction in fuzzy metric spaces and utilize the same to prove coincidence and common fixed point theorem for a pair of weakly compatible mappings. Some illustrative examples are also furnished to substantiate our main results.

2. PRELIMINARIES

Definition 2.1 [21] A fuzzy set A in X is a function with domain X with values in $[0,1]$.

Definition 2.2 [16] A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if $([0,1],*)$ is a topological abelian monoid with unit 1 such that $a*b\le c*d$ whenever $a\le c$and $b\le d,\forall\, a,b,c,d\in[0,1]$.

Examples of t-norm are as follows:

(i) $a*b=ab$,
(ii) $a*b=\min\{a,b\}$.

Definition 2.3 [7] The 3-tuple $(X,M,*)$ is called a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2\times[0,\infty)$ satisfying the following conditions: for all $x,y,z\in X$ and $s,t>0$

(FM-1) $M(x,y,t)>0$and $M(x,y,0)=0$,
(FM-2) $M(x,y,t)=1$iff $x=y$,
(FM-3) $M(x,y,t)=M(y,x,t)$,
(FM-4) $M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$,
(FM-5) $M(x,y,t):(0,\infty)\to[0,1]$ is continuous.

Definition 2.4 [8] Let $(X,M,*)$ be a fuzzy metric space. Then a sequence$\{x_n\}$ is said to be

(i) convergent to a point $x\in X$ if $\lim_{n\to\infty}M(x_n,x,t)=1$ for all $t>0$.
(ii) G-Cauchy sequence (i.e., Cauchy sequence in sense of Grabiec [8]) if $\lim_{n\to\infty}M(x_{n+p},x_n,t)=1$ for all $t>0$ and each $p>0$.

Definition 2.5 Let $(X,M,*)$be a fuzzy metric space and f and $g$be self mappings defined on X. A point x in X is called a coincidence point of f and g iff $fx=gx$while$w=fx=gx$ is called a point of coincidence of $f$and g.

Definition 2.6 [11] Apair of self mappings(f,g) of a non-empty set $X$is said to be weakly compatible if they commute at the coincidence points i.e., $fu=gu$ for some $u$in X, then $fgu=gfu$.

Definition 2.7 [12] An altering distance function is a function $\psi:[0,\infty)\to[0,\infty)$ which is

(i) monotonically increasing and continuous and
(ii) $\psi(t)=0$ iff $t=0$.

3. MAIN RESULTS

Our main result runs as follows:

Theorem 3.1 Let $(X,M,*)$ be a fuzzy metric space and $f,g:X\to X$ be a pair of mappings such that

(i) $f(X)\subseteq g(X)$,
(ii) $g(X)$ or $f(X)$ is a complete subspace of X,
(iii) $\psi\left(\frac{1}{M(fx,fy,t)}-1\right)\le\alpha\left(\frac{1}{M(gx,gy,t)}-1\right)-\beta\left(\frac{1}{M(gx,gy,t)}-1\right)$,

whereψ and α are altering functions while $\beta:[0,\infty)\to[0,\infty)$ is continuous with $\beta(0)=0$, $\beta(t)>0$ (for $t>0$) and $\psi(t)-\alpha(t)+\beta(t)>0$ for all $t>0$.

Then the pair (f,g) has a coincidence point.

Proof. Letx_0 be any point in X. Then using condition (i), define a sequence

$$y_n=fx_n=gx_{n+1},n\ge 1. \tag{1}$$

Without loss of generality, we may assume $y_n\ne y_{n+1}$ for all $n\in\mathbb{N}$;otherwise f and g have a coincidence point and there is nothing to prove. In case $y_n\ne y_{n+1}$, firstly we assert that $M(y_n,y_{n+1},t)\ge M(y_{n-1},y_n,t)$. Let if possible, $M(y_n,y_{n+1},t)<M(y_{n-1},y_n,t)$, it implies $\left(\frac{1}{M(y_n,y_{n+1},t)}-1\right)>\left(\frac{1}{M(y_{n-1},y_n,t)}-1\right)$ as ψ is increasing, we have $\psi\left(\frac{1}{M(y_n,y_{n+1},t)}-1\right)>\psi\left(\frac{1}{M(y_{n-1},y_n,t)}-1\right)$ so that

$$\begin{aligned}\psi\left(\frac{1}{M(y_{n-1},y_n,t)}-1\right)&<\psi\left(\frac{1}{M(y_n,y_{n+1},t)}-1\right)\\&=\psi\left(\frac{1}{M(fx_n,fx_{n+1},t)}-1\right)\\&\le\alpha\left(\frac{1}{M(gx_n,gx_{n+1},t)}-1\right)\\&\quad-\beta\left(\frac{1}{M(gx_n,gx_{n+1},t)}-1\right)\\&=\alpha\left(\frac{1}{M(y_{n-1},y_n,t)}-1\right)\\&\quad-\beta\left(\frac{1}{M(y_{n-1},y_n,t)}-1\right)\end{aligned}$$

or,

$$\psi\left(\frac{1}{M(y_{n-1},y_n,t)}-1\right) \le \alpha\left(\frac{1}{M(y_{n-1},y_n,t)}-1\right) - \beta\left(\frac{1}{M(y_{n-1},y_n,t)}-1\right) \qquad (2)$$

yielding thereby $M(y_{n-1},y_n,t)=1$, which is a contradiction. Thus $M(y_n,y_{n+1},t) \ge M(y_{n-1},y_n,t)$ for all $n \in N$ and hence $M(y_{n-1},y_n,t)$ is an increasing sequence of positive real numbers in $(0,1]$.

Let $\gamma(t) = \lim_{n\to\infty} M(y_n,y_{n+1},t)$, then we show that $\gamma(t)=1$ for all $t>0$. If not, then there corresponds some $t>0$ such that $\gamma(t)<1$. Taking $n\to\infty$ in (2) , we get

$$\psi\left(\frac{1}{\gamma(t)}-1\right) \le \alpha\left(\frac{1}{\gamma(t)}-1\right) - \beta\left(\frac{1}{\gamma(t)}-1\right),$$

then we have $\gamma(t)=1$. Therefore $\lim_{n\to\infty} M(y_n,y_{n+1},t)=1$. Now, for each positive integer p

$$M(y_n,y_{n+p},t) \ge M\left(y_n,y_{n+1},\frac{t}{p}\right) * M\left(y_{n+1},y_{n+2},\frac{t}{p}\right) * \dots * M\left(y_{n+p-1},y_{n+p},\frac{t}{p}\right) = 1*1*\dots*=1.$$

Therefore, $\{y_n\}$ is a G-Cauchy sequence. Suppose that $g(X)$is a complete subspace of X, the subsequence $\{y_{2n+1}\}$ contained in $g(X)$ must get a limit z in $g(X)$. Let $\in g^{-1}(z)$, then $gu=z$. As $\{y_n\}$ is a G-Cauchy sequence containing a convergent subsequence $\{y_{2n+1}\}$, therefore the sequence $\{y_n\}$ also converges implying thereby the convergence of $\{y_{2n}\}$ being a subsequence of the convergent sequence $\{y_n\}$.

Now we assert that u is a coincidence point of $f$and g. Using (iii), we have

$$\psi\left(\frac{1}{M(fx_n,fu,t)}-1\right) \le \alpha\left(\frac{1}{M(gx_n,gu,t)}-1\right) - \beta\left(\frac{1}{M(gx_n,gu,t)}-1\right).$$

On making $n\to\infty$, we get $z=fu=gu$ which shows that u is a coincidence point of f and g. The proof is similar when $f(X)$ is a complete subspace of X. This concludes the proof.

The following example illustrates Theorem 3.1.

Example 3.1 Let $X=[0,10)$ equipped with $a*b=ab$ and $M(x,y,t)=\frac{t}{t+|x-y|}$ for all $x,y\in X$and $t>0$. Define the mappings $f,g\colon X\to X$ by $f(x)=\frac{x}{4}$ and $g(x)=5-\frac{x}{2}$ for all $x\in X$. Define $\psi,\alpha,\beta\colon[0,\infty)\to[0,\infty)$as $\psi(t)=3t$, $\alpha(t)=4t$ and $\beta(t)=2t$, then we notice that $\psi(t)-\alpha(t)+\beta(t)=t>0$. One can see that the condition (iii) can be easily verified.

$$M(fx,fy,t)=\frac{t}{t+\frac{|x-y|}{4}}, \left[\frac{1}{M(fx,fy,t)}-1\right]=\frac{|x-y|}{4t}$$

$$M(gx,gy,t)=\frac{t}{t+\frac{|x-y|}{2}}, \left[\frac{1}{M(gx,gy,t)}-1\right] = \frac{|x-y|}{2t}$$

$$\alpha\left[\frac{1}{M(gx,gy,t)}-1\right]-\beta\left[\frac{1}{M(gx,gy,t)}-1\right] = \alpha\left(\frac{|x-y|}{2t}\right)-\beta\left(\frac{|x-y|}{2t}\right) = \frac{|x-y|}{t} \ge \frac{3}{4}\frac{|x-y|}{t} = \psi\left[\frac{1}{M(fx,fy,t)}-1\right].$$

Hence, the mappings f and g has a point of coincidence $x=\frac{20}{3}$ which is not a common fixed point. Therefore the necessity of weak compatibility is required to ensure the existence of common fixed point.

Theorem 3.2 Let $(X,M,*)$ be a fuzzy metric space and $f,g\colon X\to X$ be two mappings. Suppose that the conditions (i)-(iii) of Theorem 3.1 are satisfied. Then f and g have a common fixed point in X provided that the pair (f,g) is weakly compatible.

Proof. By Theorem 3.1, the mappings f and g have a point of coincidence u in X such that $fu=gu=z$. Since the pair (f,g) is weakly compatible, we have $fgu=gfu$ and hence $fz=gz$. Now we assert that $fz=z$. If not, then using (iii), we have

$$\psi\left[\frac{1}{M(fx_n,fz,t)}-1\right] \le \alpha\left[\frac{1}{M(gx_n,gz,t)}-1\right] - \beta\left[\frac{1}{M(gx_n,gz,t)}-1\right].$$

Taking limit as $n\to\infty$, we get

$$\psi\left[\frac{1}{M(z,fz,t)}-1\right] \le \alpha\left[\frac{1}{M(z,fz,t)}-1\right] - \beta\left[\frac{1}{M(z,fz,t)}-1\right],$$

then we get $z=fz=gz$hence the result follows.

Example 3.2 Let $X=[-1,2)$ equipped with $a*b=ab$ and $M(x,y,t)=\frac{t}{t+|x-y|}$ for all $x,y\in X$and $t>0$. Define the mappings $f,g\colon X\to X$ as follows

$$f(x)=\begin{cases}0, \text{if } x\le 0;\\ \frac{1}{2}, \text{if } x>0.\end{cases} \qquad g(x)=\begin{cases}0, \text{if } x\le 0;\\ 1, \text{if } x>0.\end{cases}$$

Define $\psi,\alpha,\beta\colon[0,\infty)\to[0,\infty)$as $\psi(t)=t$, $\alpha(t)=2t$ and $\beta(t)=\frac{3t}{2}$, then we can see that $\psi(t)-\alpha(t)+\beta(t)=\frac{t}{2}>0$. One can easily verified condition (iii). Discuss the following subcases.

Case 1: when $x,y\le 0$ or $x,y>0$ then we have $M(fx,fy,t)=1=M(gx,gy,t)$and condition (iii) is trivial.

Case 2: If $x\le 0, y>0$then $M(fx,fy,t)=\frac{t}{t+\frac{1}{2}}\Rightarrow \left(\frac{1}{M(fx,fy,t)}-1\right)=\frac{1}{2t}$,$M(gx,gy,t)=\frac{t}{t+1}\Rightarrow$

$\left(\frac{1}{M(gx,gy,t)}-1\right)=\frac{1}{t}$ and hence $\psi\left(\frac{1}{2t}\right)-\alpha\left(\frac{1}{t}\right)+\beta\left(\frac{1}{t}\right)$ that is $\frac{1}{2t}\leq\frac{2}{t}-\frac{3}{2t}\Rightarrow\frac{1}{2t}\leq\frac{1}{2t}$ which is true.

Case 3: If $x>0, y\leq 0$, then case is similar to previous one and other subcases are also true. Hence all the conditions of Theorem 3.2 are satisfied and 0 is a common fixed point of the mappings f and g.

Now we utilized the notion of pair-wise commuting due to Imdad et al. [9].

Definition 3.1 Two families of self mappings$\{f_i\}_{i=1}^{m}$ and $\{g_k\}_{k=1}^{n}$ are said to be pair-wise commuting if

(i) $f_i f_j = f_j f_i$ for all $i,j\in\{1,2,\ldots,m\}$,
(ii) $g_k g_l = g_l g_k$ for all $k,l\in\{1,2,\ldots,n\}$,
(iii) $f_i g_k = g_k f_i$ for all $i\in\{1,2,\ldots,m\}$ and $j\in\{1,2,\ldots,n\}$.

Our next result is defined for two finite families of self mappings.

Corollary 3.1 Let $\{f_i\}_{i=1}^{m}$ and $\{g_k\}_{k=1}^{n}$be two finite families of self mappings of a fuzzy metric space $(X,M*)$ with $f=f_1f_2\ldots f_m$ and $g=g_1g_2\ldots g_n$ satisfying conditions (i)-(iii) of Theorem 3.1. Then the pair (f,g) has a point of coincidence.

Moreover $\{f_i\}_{i=1}^{m}$ and $\{g_k\}_{k=1}^{n}$ have a common fixed point if the pair $(\{f_i\},\{g_k\})$ commutes pair-wise where $i=\{1,2,\ldots,m\}$and $k=\{1,2,\ldots,n\}$.

Proof. The proof of this theorem can be completed on the lines of a theorem of Imdad et al. [9].

Corollary 3.2 Let f,g be two self mappings of a fuzzy metric space $(X,M,*)$. Suppose that

(i) $f^m(X)\subseteq g^n(X)$,
(iv) $g^n(X)$ or $f^m(X)$ is a complete subspace of X,
(v) $\psi\left(\frac{1}{M(f^mx,f^my,t)}-1\right)\leq\alpha\left(\frac{1}{M(g^nx,g^ny,t)}-1\right)-\beta\left(\frac{1}{M(g^nx,g^ny,t)}-1\right)$,

wherem,n are fixed positive integers andψ,α are altering distance functions and $\beta:[0,\infty)\to[0,\infty)$ is continuous with $\beta(t)>0$ for $t>0$ and $\beta(0)=0$ and $\psi(t)-\alpha(t)+\beta(t)>0$ for all $t>0$.

Then f and g have a unique common fixed point provided$fg=gf$.

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CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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