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g −Reciprocal Continuity in Probabilistic MetricSpaces

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ABSTRACT:

In this paper, we obtain a common fixed point theorem by employing the notion of g −reciprocal continuity in probabilistic metric space. We demonstrate that g −reciprocal continuity ensures the existence of common fixed point under strict contractive conditions, which otherwise do not ensure the existence of fixed points.

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1. INTRODUCTION

The theory of probabilistic metric spaces was introduced by Menger [1] in connection with some measurements in Physics. The first effort in this direction was made by Sehgal [3], who in his doctoral dissertation initiated the study of contraction mapping theorems in probabilistic metric spaces. Since then, Sehgal and Bharucha - Reid [6] obtained a generalization of Banach Contraction Principle on a complete Menger space which is an important step in the development of fixed point theorems in Mengar space. Over the years, the theory has found several important applications in the investigation of physical quantities in quantum particle physics and string theory as studied by El.Naschie [18, 20]. The area of probabilistic metric spaces is also of fundamental importance in probabilistic functional analysis.

Fixed point theory of strict contractive conditions constitutes a very important class of mappings and includes contraction mappings as their subclass. It may be observed that strict contractive conditions do not ensure the existence of common fixed points unless some strong condition is assumed either on the space or on the mappings. In such cases either

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the space is taken to be compact or some sequence of iterates is assumed to be Cauchy sequence. The studyof common fixed points of strict contractive conditions using noncompatibility was initiated by Pant [19]. The significance of this paper lies in the fact that we can obtain fixed point theorems for g-reciprocally continuous mappings in Probabilistic metric spaces.

Definition 1.1. [22]. A distribution function (on $[-\infty,+\infty]$) is a function F: $[-\infty, +\infty] \to [0, 1]$ which is left-continuous on R, non-decreasing and $F(-\infty) = 0$, $F(+\infty) = 1$. The Heaviside function H is a distribution function defined by,

$$H(t) = \begin{cases} 0, & if\ t \leq 0 \\ 1, & if\ t > 0. \end{cases}$$

Definition 1.2. [22]. A distance distribution function F: $[-\infty,+\infty] \to [0, 1]$ is distribution function with support contained in $[0,\infty]$.The family of all distance distribution functions will be denoted by Δ^+.We denote

$$D^+ = \left\{F: F \in \Delta^+, \lim_{x\to\infty} F(x) = 1\right\}.$$

Definition 1.3. [10]. A probabilistic metric space in the sense of Schweizer and Sklar is an ordered pair (X,F), where X is a nonempty set and $F : X \times X \to \Delta^+$, if and only if the following conditions are satisfied ($F(x,y) = F_{x,y}$for every $x,y \in X \times X$):

(i) for every $(x, y) \in X \times X$, $F_{x,y}(0) = 0$;

(ii) for every $(x,y) \in X \times X$, $F_{x,y} = F_{y,x}$;

(iii) $F_{x,y} = 1$, for every $t > 0 \Leftrightarrow x = y$;

(iv) for every $(x,y,z) \in X \times X \times X$ and for every $t_1, t_2 > 0$,

(v) $F_{x,y}(t_1) = 1$, $F_{y,z}(t_2) = 1, \Longrightarrow F_{x,z}(t_1+t_2) = 1$.

For each x and y in X and for each real number $t \geq 0, F_{x,y}(t)$ is to be thought of as the probability that the distance between x and y is less than t. Indeed, if (X,d) is a metric space, then the distribution function $F_{x,y}(t)$ defined by the relation $F_{x,y}(t) = H(t - d(x,y))$ induces a probabilistic metric space.

Definition 1.4.[10]. A t-norm is a function Δ: $[0,1]\times[0,1] \longrightarrow [0,1]$ satisfying the following conditions:

i. $T(a,1) = a$, $T(0,0) = 0$.

ii. $T(a,b) = T(b,a)$.

iii. $T(c,d) \geq T(a,b)$ for $c \geq a$, $d \geq b$.

iv. $T(T(a,b),c) = T(a,T(b,c))$ for all a, b, c in $[0,1]$.

We begin with the following preliminaries:

Definition 1.5. [10]. A Menger probabilistic metric space (X,F,T) is an ordered triad, where T is a t - norm, and (X,F) is probabilistic metric space satisfying the following condition:

$F_{x,z}(t_1+t_2) \geq T(F_{x,y}(t_1), F_{y,z}(t_2))$for all x, y, z in X and $t_1, t_2 \geq 0$.

Definition 1.6. [10]. Let (X,F) be a probabilistic metric space. The $(\in,\lambda)-$ topology in (X,F) is generated by the family of neighborhoods $U = \left\{\left(U_v(\epsilon,\lambda)\right): (v,\epsilon,\lambda \in X \times R^+ \times (0,1))\right\}$, where $U_v(\in,\lambda) = \left\{u: u \in X, F_{u,v}(\in) > 1-\lambda\right\}$.

If a t - norm T is such that $supx < 1\ T(x,x) = 1$ then (X,F,T) is, with the$(\in,\lambda)$ topology, a metrizable topological space.

Definition 1.7. [10]. Let (X, F) be a probabilistic metric space. A sequence $\{x_n\}$ in (X,F) is said to converge a point $x \in X$ if for every $\in > 0$ and $\lambda > 0$, there exists a positive integer $N(\in,\lambda)$such that,

$F_{x_n,x}(\in) > 1-\lambda$for all $n \geq N(\in,\lambda)$.

Definition 1.8. [10]. Let (X, F) be a probabilistic metric space. A sequence $\{x_n\}$ in (X,F) is said to be a Cauchy sequence if for every $\in > 0$ and $\lambda > 0$, there exists a positive integer $N(\in,\lambda)$such that,

$F_{x_n,x_m}(\in) > 1-\lambda$for all $n,m \geq N(\in,\lambda)$.

Definition 1.9. [10]. A probabilistic metric space (X, F) with continuous t-norm is said to be complete if every cauchy sequence in X converge to a point in X.

Definition 1.10.[13]. Twoself maps f and g of a probabilistic metric space (X,F) are called compatible if $F_{fgx_n,gfx_n}(t) \to 1$ for all $t >$ 0 whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = u$.

Definition 1.11[16]. Twoself maps f and g of a probabilistic metric space (X,F) are called $g-$ compatible if $F_{fgx_n,gfx_n}(t) \to 1$ for all $t >$

0whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = u$

Definition 1.12[17]. Two selfmappings f and g of a probabilistic metric space (X, F) are called reciprocally continuous iff $fgx_n \to fu$ and $gfx_n \to gu$ whenever $\{x_n\}$ is a sequence such that $\lim_n fx_n = \lim_n gx_n = u$ for some u in X.

Definition 1.13 [23]. Two self mappings f and g of a probabilistic metric space (X, F) satisfy $(E.A)$ property if there exist a sequence $\{x_n\}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some u in X.

Definition 1.14[25]. Two selfmappings f and g of a probabilistic metric space (X, F) are called $g-$ reciprocally continuous iff $ffx_n \to fu$ and $gfx_n \to gu$ whenever $\{x_n\}$ is a sequence such that $\lim_n fx_n = \lim_n gx_n = u$ for some u in X.

Remark 1.1. It may be observed that if f and g are both continuous then they are obviously g-reciprocally continuous but the converse is not true. It may also be observed that g-reciprocal continuity is independent of the notion of reciprocal continuity.

Example 1.1. [19]. Let $X = [2, 20]$ and $F_{x,y}(t) = H(t - d(x, y))$,

where $d(x, y) = |x - y|$ .Define $f, g : X \to X$ by

$$f2 = 2 \text{ if } x = 2 \text{ or } x > 5,\ fx = 4 \text{ if } 2 < x \le 5,$$

$$g2 = 2, gx = 18 \text{ if } 2 < x \le 5,\ gx = \frac{(x+1)}{3} \text{ if } x > 5.$$

Then f and g are $g-$ reciprocally continuous but not reciprocally continuous. To see this let us consider the sequence $x_n = 5 + \frac{1}{n}$. Then

$$fx_n \to 2,\ gx_n \to 2, \lim_{n\to\infty} fgx_n \ne f2,$$

$$\lim_{n\to\infty} gfx_n = 2 = g2 \text{ and } \lim_{n\to\infty} ffx_n = 2 = f2.$$

Thus f and g are $g-$ reciprocally continuous but they are not reciprocally continuous.

Example 1.2. [19]. Let $X = [2, 20]$ and $F_{x,y}(t) = H(t - d(x, y))$,where $d(x, y) = |x - y|$ .Define $f, g : X \to X$ by

$$f2 = 2,\ fx = 6 \text{ if } 2 < x \le 5,\ fx = \frac{(x+5)}{5} \text{ if } x > 5,$$

$$gx = 2 \text{ if } x = 2 \text{ or } x > 5,\ gx = \frac{(x+4)}{3} \text{ if } 2 < x \le 5.$$

Then f and g are reciprocally continuous but not $g-$ reciprocally continuous. To see this let us consider the sequence $x_n = 5 + \frac{1}{n}$. Then

$$fx_n \to 2,\ gx_n \to 2, \lim_{n\to\infty} fgx_n = 2 = f2,$$

$$\lim_{n\to\infty} gfx_n = 2 = g2 \text{ and}$$

$\lim_{n\to\infty} ffx_n = 6 \ne f2.$ Thus f and g are reciprocally continuous but they are not $g-$ reciprocally continuous.

Remark 1.2. Both the Examples areclearly showing that reciprocal continuity and $g-$ reciprocally continuous reciprocal continuity are independent of each other.

2.MAIN RESULT

Theorem 2.1. Let f and g be g- reciprocally continuous self mappings of probabilistic metric space (X, F) satisfying,

(1) $F_{fx,fy}(t) > F_{gx,gy}(t)$

whenever right hand side is not equal to one. Suppose f and g satisfy property $(E.A)$. If f and g are g- compatible then f and g have a unique common fixed point.

Proof. Since f and g satisfy property $(E.A)$there exists a sequence x_n in X such that $fx_n \to u$ and $gx_n \to u$ for some $u$in X. Suppose that f and

g are g - compatible. Then $F_{fgx_n, gfx_n}(t) \to 1$, g - reciprocal continuity of f and g implies that $ffx_n \to fu$ and $gfx_n \to gu$. The last two limits together imply $fu = gu$. Since g - compatibility implies commutativity at coincidence points, i.e., $fgu = gfu$ and, hence $ffu = fgu = gfu = ggu$. If $fu \neq ffu$ then by using(1), we get $F_{fu,ffu}(t) > F_{gu,gfu}(t) = F_{fu,ffu}(t)$. a contradiction. Hence $fu = ffu = gfu$ and fu is a common fixed point of fand g.

The next example illustrates the above theorem.

Example 2.1. Let $X = [2, 20]$ and and $F_{x,y}(t) = H(t - d(x,y))$, where $d(x, y) = |x - y|$. Define $f, g : X \to X$ as follows

$$fx = 2 \ if \ x = 2 \ or \ x > 5, \quad fx = 6 \ if \ 2 < x \leq 5,$$

$$g2 = 2, gx = 12 \ if \ 2 < x \leq 5, \ gx = (x+1)/3 \ if \ x > 5.$$

Then f and g satisfy all the conditions of Theorem 2.*1* and have a unique common fixed point at $x = 2$. It can be verified in this example that f and g satisfy the contraction condition (i). Furthermore, f and g are $g-$ reciprocally continuous $g-$ compatible mappings. It is also obvious that f and g are not reciprocally continuous. Here f and g are not reciprocally continuous mappings.

Remark: In the above result we have not assumed strong conditions, e.g., completeness of the space, containment of the ranges of the mappings, closedness of the range of any one of the involved mappings and continuity of any mapping. In this paper we have proved a result using generalized strict contractive condition.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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