Gazi UniversityJournal of Science
GU J Sci
28(4):683- 688 (2015)

Riemann Zeta Matrix Function

Levent KARGIN[1,♠], Veli KURT[2]

[1]Alanya Alaaddin Keykubat University Akseki Vocational School, Antalya Turkey
[2]Akdeniz University, Faculty of Science Department of Mathematics, Antalya, TURKEY

Received:28/05/2015 Accepted: 05/09/2015

ABSTRACT

In this study, obtaining the matrix analog of the Euler's reflection formula for the classical gamma function we expand the domain of the gamma matrix function and give a infinite product expansion of $sin\pi xP$. Furthermore we define Riemann zeta matrix function and evaluate some other matrix integrals. We prove a functional equation for Riemann zeta matrix function.
2010 Mathematics Subject Classification. 33C05, 33C25, 34A05, 15A60.

Key words: Gamma matrix function; Riemann Zeta matrix function; Matrix integrals.

1. INTRODUCTION

Matrix generalization of special functions has become important during last years. This importance is brought about by many facts. Their frequent appearance in physical problems, their ubiquitous use in statistics and probability theory and their applications in numerical analysis are just a few of these facts. Matrix polynomials, appeared in connection with matrix functions are introduced in ([6, 7, 8, 11, 12, 17, 19, 20]). Furthermore their important properties are studied in ([1, 2, 3, 4, 8, 15, 17]).

Throughout this paper for a matrix P in $\mathbb{C}^{r\times r}$, its spectrum $\sigma(P)$ denotes the set of all the eigenvalues of P, $\alpha(P)=max\{Re(z): z\epsilon\sigma(P)\}$ and $\beta(P)=min\{Re(z): z\epsilon\sigma(P)\}$. Besides for $t\geq 0$, it follows that

$$\|e^{tP}\|\leq e^{t\alpha(P)}\sum_{k=0}^{r-1}\frac{\left(\|P\|\sqrt{r}t\right)^k}{k!} \qquad (1)$$

where ‖P‖ denotes the 2-norm of P defined in [20].

If $f(z)$ and $g(z)$ are holomorphic functions in an open set Ω of the complex plane and P is a matrix in $\mathbb{C}^{r\times r}$ with $\sigma(P)\subseteq\Omega$, then from matrix functional calculus [9] it follows that $f(P)g(P)=g(P)f(P)$. Since the reciprocal Gamma function denoted by $\Gamma^{-1}(z)=1/\Gamma(z)$ is an entire function, $\Gamma^{-1}(P)$ is a well-defined matrix for any matrix P in $\mathbb{C}^{r\times r}$. Furthermore if P is a matrix in $\mathbb{C}^{r\times r}$ such that

$$P+nI \text{ is invertible for every integer } n\geq 0 \qquad (2)$$

♠Corresponding author, e-mail: lkargin@akdeniz.edu.tr

then $\Gamma(P)$ is invertible and

$$(P)_n = \Gamma(P+nI)\Gamma^{-1}(P), \quad n \geq 1 \qquad (3)$$

where $(P)_n$ is the matrix analogue expression of the factorial function is defined by [13]

$$(P)_n = P(P+I)(P+2I)\dots(P+(n-1)I), n \geq 1 \ (4)$$

where $(P)_0 = 1$.

If P is a matrix in $\mathbb{C}^{r\times r}$ such that $Re(z) > 0$ for all $z \in \sigma(P)$ (we say positive stable matrix for P), then the gamma matrix function $\Gamma(P)$ is defined in [12] as follows

$$\Gamma(P) = \int_0^\infty e^{-t}t^{P-I}dt,\ t^{P-I} = exp\big((P-I)lnt\big). \ (5)$$

Using (5) Jódar and Cortés in [12] have obtained a limit expression for the gamma matrix function

$$\Gamma\,(P) = \lim_{n\to\infty}(n-1)!\,(P)_n^{-1}n^P. \qquad (6)$$

In addition, if P and Q are commuting matrices in $\mathbb{C}^{r\times r}$ such that $P+nI, Q+nI$ and $P+Q+nI$ invertible for all integer $n \geq 0$ then

$$B(P,Q) = \Gamma(P)\Gamma(Q)\Gamma^{-1}(P)$$

where $B(P,Q)$ is the Beta matrix function given in [13]. Also Jódar and Sastre have proved asymptotic behaviour of Laguerre matrix polynomials in [14] by using

$$\mathrm{P}\Gamma(P) = e^{-\gamma P}\left[\prod_{n=1}^{\infty}\left(I+\frac{P}{n}\right)e^{-\frac{P}{n}}\right]^{-1} \qquad (7)$$

where P is a matrix in $\mathbb{C}^{r\times r}$ satisfying (2) and γ is the Euler-Mascheroni constant.

In this paper, we focus on the generalization of two other properties of classical gamma function to gamma matrix function. These properties are

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$$

and

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{sin\pi z},$$

called Legendre's duplication formula and Euler's reflection formula, respectively. So we can evaluate the gamma values of a matrix P in $\mathbb{C}^{r\times r}$ such that $\sigma(P) = \{k: k \notin \mathrm{Z}\}$. Moreover we give a infinite product expansion of $\sin \pi x P$. The classical Riemann zeta function plays a pivotal role in analytic number theory and has applications in physics, probability theory and applied statistics. This fact together with its close relationship with the well-known classical gamma function motivates the definition of Riemann zeta matrix function $\zeta(P)$ as an infinite sum of matrix exponential. So, examining the properties of Riemann zeta matrix function will allow us to get information about infinite sum of matrix exponential. We evaluate some matrix integrals in terms Riemann zeta matrix function and gamma matrix function. We also give a functional equation for Riemann zeta matrix function.

2. ON GAMMA MATRIX FUNCTION

In this section we give duplication and reflection formula for gamma matrix function and a product formula for sin matrix function. Evaluation of sin matrix function for a matrix P in $\mathbb{C}^{r\times r}$ is given with an example. We discover an expression for $\Gamma(2A)$ in terms of $\Gamma(A)$ and $\Gamma\left(A+\frac{I}{2}\right)$ in the following theorem.

Theorem 1 Let P be a positive stable matrix in $\mathbb{C}^{r\times r}$. Then the dublication formula for gamma matrix function holds:

$$\sqrt{\pi}\,\Gamma(2A) = 2^{2A-I}\Gamma(A)\Gamma\left(A+\frac{I}{2}\right). \qquad (8)$$

Proof. Using (4) it is easy to show that

$$(P)_{2n} = 2^{2n}\left(\frac{P}{2}\right)_n\left(\frac{P+I}{2}\right)_n$$

Taking $P = 2A$ in the above equation and using (3) we obtain

$$\Gamma(2A)\Gamma^{-1}(A)\Gamma^{-1}\left(A+\frac{I}{2}\right) \qquad (9)$$

$$= 2^{-2n}\Gamma(2A+2nI)\Gamma^{-1}(A+nI)\Gamma^{-1}\left(A+\frac{I}{2}+nI\right).$$

We next insert in the right member of (9) appropriate factors to permit us to make use of the result in (6). By using (6), we rewrite (9) as

$$\Gamma(2A)\Gamma^{-1}(A)\Gamma^{-1}\left(A+\frac{I}{2}\right)$$

$$= 2^{2A}\lim_{n\to\infty}\frac{(2n-1)!\,n^{-\frac{I}{2}}}{2^{2n}(n-1)!\,(n-1)!}.$$

Setting $A=\frac{I}{2}$ it follows from (5) that the value of limit is $\frac{1}{2\sqrt{\pi}}$. This completes the proof.

To derive a reflection formula for gamma matrix function and the sine product formula, we must first give the following lemma.

Lemma 2 Let x be an arbitrary parameter. Define the matrix function $\varphi(xP)$, for a matrix P in $\mathbb{C}^{r\times r}$ such that $k\notin\sigma(xP)$ for $k\in Z$ to be

$$\varphi(xP)=\Gamma(xP)\Gamma(I-xP)\sin\pi xP \qquad (10)$$

then $\varphi(xP+I)=\varphi(xP)$.

Proof. By taking xP and $-xP$ in (3) for $n=1$, then we get

$$\Gamma\,(xP+I)=xP\Gamma(xP) \qquad (11)$$

$$\Gamma\,(I-xP)=-xP\Gamma(-xP) \qquad (12)$$

respectively. It is easy to show that $\sin\pi(xP+I)=\sin\pi xP$. Using (11) and rearranging (12) gives the proof.

(8) can be written as

$$\Gamma\left(\frac{xP}{2}\right)\Gamma\left(\frac{xP+I}{2}\right)=2\sqrt{\pi}2^{-xP}\Gamma(xP).$$

Replacing xP with $I-xP$ and using (10) we get

$$\varphi\left(\frac{xP}{2}\right)\varphi\left(\frac{xP+I}{2}\right)=\pi\varphi(xP).$$

Using (11) and the infinite series expansion of $\sin\pi xP$, we get

$$\varphi(xP)$$

$$=\Gamma(xP+I)\Gamma(I-xP)\left(\pi-\frac{\pi^3(xP)^2}{3!}+\frac{\pi^5(xP)^4}{5!}-\cdots\right)$$

The right hand side of the equation equals πI when $x=0$. From there we see that $\varphi(\mathbf{0})=\pi I$, where $\mathbf{0}$ is zero matrix. Let $g(xP)$ be a periodic matrix function that is equal to second derivative of $\log\varphi(xP)$. It is periodic because $\log\varphi(xP)=\log(\Gamma(xP)\Gamma(I-xP)\sin\pi xP)$ is periodic. Since $g(xP)$ is periodic, then it satisfies the equation

$$g(xP)=\frac{1}{4}\left[g\left(\frac{xP}{2}\right)+g\left(\frac{xP+I}{2}\right)\right] \qquad (13)$$

Since $g(xP)$ is continuos on the interval $[0,1]$ it is bounded by a constant M, $\|g(xP)\|\le M$. From (13), we get

$$\|g(xP)\|\le\frac{1}{4}\left[\left\|g\left(\frac{xP}{2}\right)\right\|+\left\|g\left(\frac{xP+I}{2}\right)\right\|\right]\le\frac{M}{2}$$

From this we see that $g(xP)$ can actually be bounded by $\frac{M}{2}$. We can continue to repeat this process until the bound of $g(xP)$ goes to $\mathbf{0}$. Therefore $g(xP)=\mathbf{0}$, which means $\log\varphi(xP)$ is a linear function, because $g(xP)=\mathbf{0}$ is its second derivative. Since $\log\varphi(xP)$ is periodic, this implies that it is a constant, which also implies $\varphi(xP)$ is constant. We know $\varphi(0)=\pi I$ and therefore $\varphi(xP)$ must equal πI for all x.

Rearranging (10) and using the fact that $\varphi(0)=\pi I$, we obtain

$$\cong(xP)\cong(I-xP)=\pi[\sin\pi\mathrm{xP}]^{-1}. \qquad (14)$$

For $x=1$, we derive a relationship between the sine and gamma matrix function in the following theorem.

Theorem 3 Let P be a matrix in $\mathbb{C}^{r\times r}$ such that $k\notin\sigma(P)$ for $k\in Z$. Then we obtain a reflection formula for the gamma matrix function

$$\Gamma(P)\Gamma(I-P)=\pi[\sin\pi\mathrm{P}]^{-1} \qquad (15)$$

Let rewrite (15) as

$$\Gamma(P)=\pi[\Gamma(I-P)\sin\pi\mathrm{P}]^{-1}. \qquad (16)$$

Then (16) gives us important information for the gamma matrix function. Whereas the gamma matrix function is well-defined for the positive stable matrix in (5), we can evaluate the gamma values of the matrix P in $\mathbb{C}^{r\times r}$ such that $k\notin\sigma(P)$ for $k\in Z$.

Example 4 Let $A=\begin{bmatrix}-\frac{7}{2} & 0\\ 3 & -\frac{1}{2}\end{bmatrix}$ be a matrix in $\mathbb{C}^{2\times 2}$ with $\sigma(A)=\left\{-\frac{7}{2},-\frac{1}{2}\right\}$. Hence one gets

$$\Gamma(I-A)=\int_0^\infty e^{-t}t^{-A}dt$$

$$=\int_0^\infty\begin{bmatrix}e^{-t} & 0\\ 0 & e^{-t}\end{bmatrix}\begin{bmatrix}t^{\frac{7}{2}} & 0\\ t^{\frac{1}{2}}-t^{\frac{7}{2}} & t^{\frac{1}{2}}\end{bmatrix}dt$$

$$=\begin{bmatrix}\Gamma\left(\frac{9}{2}\right) & 0\\ \Gamma\left(\frac{3}{2}\right)-\Gamma\left(\frac{9}{2}\right) & \Gamma\left(\frac{3}{2}\right)\end{bmatrix}$$

$$=\begin{bmatrix}\frac{105}{16}\sqrt{\pi} & 0\\ -\frac{97}{16}\sqrt{\pi} & \frac{1}{16}\sqrt{\pi}\end{bmatrix}$$

and

$$\sin(\pi A) = \frac{exp(i\pi A) - exp(-i\pi A)}{2i} = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$$

Then from (16), we have

$$\Gamma(A) = \begin{bmatrix} \frac{16}{105}\sqrt{\pi} & 0 \\ -\frac{226}{105}\sqrt{\pi} & -2\sqrt{\pi} \end{bmatrix}$$

Note that the Weierstrass product formula given in (7) allows us to rewrite $\sin \pi x P$ as an infinite product expansion in the following corollary.

Corollary 5 Let P be an invertible matrix in $\mathbb{C}^{r\times r}$. Then infinite product expansion of $\sin \pi x P$ holds:

$$\sin \pi x P = \pi x P \prod_{n=1}^{\infty}\left(I - \frac{(xP)^2}{n^2}\right)$$

3. RIEMANN ZETA MATRIX FUNCTİON

Let P be a matrix in $\mathbb{C}^{r\times r}$ such that $Re(z) > 1$ for all $z \in \sigma(P)$. Using inequality (1), it follows that

$$\sum_{n=1}^{\infty}\|n^{-P}\| \leq \sum_{k=0}^{r-1}\frac{\left(\|P\|\sqrt{r}\right)^k}{k!}\zeta^{(k)}(\beta(P)) < \infty,$$

where $\zeta^{(k)}(s)$ is the k times derivative of classical Riemann zeta function. Thus we can define Riemann zeta matrix function as

$$\zeta(P) = \sum_{n=1}^{\infty} n^{-P} = \sum_{n=1}^{\infty} exp(-Plnn). \qquad (17)$$

The following theorem is about integral representation of Riemann zeta matrix function.

Theorem 6 Let P be a matrix in $\mathbb{C}^{r\times r}$ such that $\mathrm{Re(z)} > 1$ for all $\mathrm{z} \in \sigma(\mathrm{P})$, then we get

$$\zeta(P)\Gamma(P) = \int_0^{\infty}\frac{x^{P-I}}{e^x - 1}dx$$

Proof. If we make the change of variable $t = nx$ for $\Gamma(P)$, where $n \geq 1$, to obtain

$$n^{-P}\Gamma(P) = \int_0^{\infty} e^{-nx}x^{P-I}dx,$$

and summing over all $n \geq 1$, we find

$$\zeta(P)\Gamma(P) = \sum_{n=1}^{\infty}\int_0^{\infty} e^{-nx}x^{P-I}dx. \qquad (18)$$

For each $n \geq 1$, by the inequality (1) and using $\ln x < x - 1$ for $x \in R^+$, we have

$$\|f_n(x,P)\| \leq \sum_{k=0}^{r-1}\frac{\left(\|P\|\sqrt{r}\right)^k}{k!}e^{-x}x^{k+\alpha(P)-1} = f(\mathrm{x},\mathrm{P})$$

where $f_n(x,P) = e^{-nx}x^{P-I}$. Furthermore it is clear that $f(\mathrm{x},\mathrm{P})$ is integrable in $[0,\infty]$. By the dominated convergence theorem, we can change the summation and integration in (18), obtaining

$$\zeta(P)\Gamma(P) = \int_0^{\infty}\sum_{n=1}^{\infty} e^{-nx}x^{P-I}\,dx.$$

If $x > 0$ we have $0 < |e^{-nx}| < 1$ and hence we have

$$\sum_{n=1}^{\infty} e^{-nx}x^{P-I} = \frac{x^{P-I}}{e^{x-1}}.$$

This completes the proof.

Proposition 7 Let P be a matrix in $\mathbb{C}^{\mathrm{r\times r}}$ such that $\mathrm{Re(z)} > 1$ for all $\mathrm{z} \in \sigma(\mathrm{P})$, and $\mathrm{Re(b)} > 0$ then we have

$$\int_0^{\infty}\frac{x^{P-I}}{sinhbx}dx = 2b^{-P}\,(I - 2^{-P})\,\Gamma(P)\zeta(P), \qquad (19)$$

$$\int_0^{\infty} x^{P-I}(1 - tanhbx)dx$$

$$= 2(2b)^{-P}(I - 2^{I-P})\Gamma(P)\zeta(P), \qquad (20)$$

$$\int_0^{\infty} x^{P-I}(cothbx - 1)dx = 2(2b)^{-P}\Gamma(P)\zeta(P). \qquad (21)$$

Proof. It is clear that a proof is needed only for the case $b = 1$ (then use the substitution $x \to bx$). For (19), by using geometric series we write

$$\frac{1}{sinhx} = \frac{2e^{-x}}{1 - e^{-2x}} = 2\sum_{n=0}^{\infty} e^{-(2n+1)x}.$$

Multiplying this by x^{P-I} and integrating from zero to infinity we obtain

$$\int_0^\infty \frac{x^{P-I}}{sinhbx}dx = 2\Gamma(P)\sum_{n=0}^{\infty}(2n+1)^{-P}$$
$$= 2\Gamma(P)\{\zeta(P) - 2^{-P}\zeta(P)\}.$$

This gives (19). The equations (20) and (21) can be proved similarly.

Differentiating (20) and (21) for the variable b we obtain two more representations.

Corollary 8 For a matrix P in $\mathbb{C}^{r\times r}$ such that $\mathrm{Re}(z) > -1$ for all $z \in \sigma(P)$ and $\mathrm{Re}(b) > 0$ we obtain the following matrix integrals.

$$\int_0^\infty \frac{x^{P-I}}{cosh^2bx}dx$$
$$= 4(2b)^{-(P+I)}(I - 2^{I-P})\,\Gamma(P+I)\zeta(P), \quad (22)$$

$$\int_0^\infty \frac{x^{P-I}}{sinh^2bx}dx = 4(2b)^{-(P+I)}\,\Gamma(P+I)\zeta(P). \quad (23)$$

An important item for this section is the integral (cosine transform of hyperbolic cosecant)

$$\int_0^\infty \frac{t\cos xt}{sinh\frac{\pi}{2}t}dt = \frac{1}{cosh^2x}, \quad (24)$$

which will be used to prove the matrix analog of the functional equation for the Riemann zeta function. An elementary introduction based on classical analysis can be found in chapter 3 of [21]. Also the following lemma is important to prove the matrix analog of the functional equation for the Riemann zeta function.

Lemma 9 For a matrix P in $\mathbb{C}^{r\times r}$ such that $\mathrm{Re}(z) > -1$ for all $z \in \sigma(P)$ and $t > 0$ we get

$$\int_0^\infty x^{P-I}\cos xt\,dx = -\Gamma\,(P+I)t^{-(P+I)}\sin\frac{\pi P}{2}. \quad (25)$$

Proof. Using (5) we get

$$\int_0^\infty x^{P-I}\cos xt\,dx = \frac{1}{2}\int_0^\infty x^{P-I}\left(e^{ixt} + e^{-ixt}\right)dx$$
$$= \frac{1}{2}\Gamma(P+I)t^{-(P+I)}i\left[e^{\frac{\pi}{2}iP} - e^{-\frac{\pi}{2}iP}\right]$$
$$= -\Gamma(P+I)t^{-(P+I)}\sin\frac{\pi P}{2}.$$

Now we can give the following theorem.

Theorem 10 If P is a matrix in $\mathbb{C}^{r\times r}$ such that $1 \notin \sigma(P)$, then we obtain a functional equation for Riemann zeta matrix function

$$\zeta(I-P)\Gamma(I-P)\sin\frac{\pi P}{2} = \pi\zeta(P)(2\pi)^{-P}. \quad (26)$$

Proof. We shall evaluate the integral in (22) in a different way. Using (24) and Fubini Tonelli theorem we can write

$$\int_0^\infty \frac{x^{P-I}}{cosh^2bx}dx = \int_0^\infty x^P\left[\int_0^\infty \frac{t\cos xt}{sinh\frac{\pi}{2}t}dt\right]dx$$
$$= \int_0^\infty\left[\int_0^\infty x^P\cos xt\,dx\right]\frac{tdt}{sinh\frac{\pi}{2}t}.$$

By using (25) and (19), we get

$$\int_0^\infty \frac{x^P}{cosh^2x}dx = -2\,(I - 2^{P-I})\left(\frac{\pi}{2}\right)^{P-I}\sin\frac{\pi P}{2}$$
$$\times\,\Gamma\,(P+I)\Gamma(I-P)\zeta(I-P)$$

Comparing the above equation to (22) we obtain (26).

It is good to notice that (26) is a well defined matrix function for a matrix P in $\mathbb{C}^{r\times r}$ such that $1 \notin \sigma(P)$ whereas (17) is well defined matrix function for a matrix P in $\mathbb{C}^{r\times r}$ such that $\mathrm{Re}(z) > 1$ for all $z \in \sigma(P)$. Let P be a matrix in $\mathbb{C}^{r\times r}$ such that $\sigma(P) = \{-2k : k \in Z^+\}$. Then using $\sin\frac{\pi P}{2} = 0$ in

$$\zeta(P) = \left(\frac{2\pi}{\pi}\right)^P\zeta(I-P)\Gamma(I-P)\sin\frac{\pi P}{2}$$

we get $\zeta(P) = \mathbf{0}$. Moreover let $P = \mathbf{0} \in \mathbb{C}^{r\times r}$. Then one can obtain

$$exp(-P\ln n) = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} = I$$

From (17), we get

$$\zeta(P) = \sum_{n=1}^{\infty} n^{-P} = \begin{bmatrix} \zeta(0) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \zeta(0) \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{1}{2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -\frac{1}{2} \end{bmatrix}$$

So we have $\zeta(\mathbf{0}) = -\frac{I}{2}$.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES

[1] R. Aktaş, B. Çekim, and R. Şahin, The Matrix Version for the Multivariable Humbert Polynomials, Miskolc Mathematical Notes 13 (2) (2012) 197-208.

[2] R. Aktaş, B. Çekim and A. Çevik, Extended Jacobi matrix polynomials, Utilitas Mathematica, 92, 47-64, 2013.

[3] A. Altın and B. Çekim, Some properties associated with Hermite matrix polynomials, Utilitas Mathematica, 88, 171-181, 2012.

[4] A. Altın and B. Çekim, Some miscellaneous properties for Gegenbauer matrix polynomials, Utilitas Mathematica, 92, 377-387, 2013.

[5] S. Varma, B. Çekim, and F. Taşdelen, On Konhauser matrix polynomials, Ars Combinatoria 100 (2011) 193-204.

[6] E. Defez and L. Jódar, Chebyshev matrix polynomials and second order matrix differential equations, Utilitas Math. 61 (2002) 107-123.

[7] E. Defez and L. Jódar, Jacobi Matrix Differential Equation, Polynomial Solutions, and Their Properties, Computers and Math. with Appl., 48 (2004) 789-803.

[8] E. Defez, A Rodrigues-type formula for Gegenbauer matrix polynomials, Appl. Math. Lett. 26 (2013) 899–903.

[9] N. Dunford and J. Schwartz, Linear Operators, Vol. I, Interscience, New York, 1963.

[10] L. Jódar, R. Company and E. Navarro, Laguerre matrix polynomials and systems of second order differential equations, Appl. Numer. Math. 15 (1994) 53-63.

[11] L. Jódar and R. Company, Hermite matrix polynomials and second order matrix differential equations, J. Approx. Theory Appl. 12 (2) (1996) 20-30.

[12] L. Jódar, J.C. Cortés, Some properties of gamma and beta functions, Appl. Math. Lett. 11 (1) (1998) 89-93.

[13] L. Jódar and J.C. Cortés, On the hypergeometric matrix function, J. Comput. Appl. Maths., 99 (1998) 205-217.

[14] L. Jódar and J. Sastre: The growth of Laguerre matrix polynomials on bounded intervals. Appl. Math. Lett. 13 (2000) 21-26.

[15] L. Kargin and V. Kurt, Some relations on Hermite matrix polynomials. Mathematical and Computational Applications, Mathematical and Computational Applications, 18 (3) (2013) 323-329.

[16] L. Kargin and V. Kurt, Modified Laguerre Matrix Polynomials. Filomat, 10 (28) (2014) 2069-2076.

[17] L. Kargin and V. Kurt, On Generalized Humbert Matrix Polynomials. Miskolc Mathematical Notes, 15 (2) (2014) 509-524.

[18] L. Kargin and V. Kurt, Chebyshev-type Matrix Polynomials and Integral Transforms. Hacettepe Journal of Mathematics and Statistics, 44 (2) (2015) 341 – 350.

[19] L.M. Upadhyaya and A. Shehata; On Legendre matrix polynomials and its applications, Inter. Tran. Mathematical Sci. Computer 4 (2) (2011) 291-310.

[20] G. Golub and C.F. Van Loan, Matrix Computations, Johns Hopkins Univ. Press, Baltimore, 1995.

[21] D.V. Widder, An Introduction to Transform Theory, Academic Press, New York, 1971.