Gazi University Journal of Science
GU J Sci
28(4):715-728 (2015)

Taylor Polynomial Solutions of Second Order Linear Partial Differential Equations with Three Variables

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Received: 14/05/2014 *Accepted: 04/09/2015*

ABSTRACT

The purpose of this study is to give a Taylor polynomial approximation for the solution of second order linear partial differential equations with three variables and variable coefficients. For this purpose, Taylor matrix method for the approximate solution of second order linear partial differential equations with specified associated conditions in terms of Taylor polynomials about any point.

Keywords: *Taylor polynomial solutions, partial differential equations with three variables, approximate solutions of partial differential equations.*

1. INTRODUCTION

Numerical solutions of second order linear partial differential equations with two-dimensional are searched in [5], the different methods are searched for solving them in [1], [2], [4]. In this study, we consider that the second order linear partial differential equation

$$A(x,y,z)\frac{\partial^2 u}{\partial x^2}+B(x,y.z)\frac{\partial^2 u}{\partial y^2}+C(x,y.z)\frac{\partial^2 u}{\partial z^2}+D(x,y,z)\frac{\partial^2 u}{\partial x\partial y}+E(x,y.z)\frac{\partial^2 u}{\partial x\partial z}+F(x,y.z)\frac{\partial^2 u}{\partial y\partial z}+ G(x,y,z)\frac{\partial u}{\partial x}+H(x,y,z)\frac{\partial u}{\partial y}+I(x,y,z)\frac{\partial u}{\partial z}+J(x,y,z)u=K(x,y,z), \tag{1.1}$$

where $A(x,y,z),B(x,y.z),C(x,y.z),D(x,y,z),E(x,y.z),F(x,y.z),G(x,y,z),H(x,y,z)I(x,y,z),J(x,y,z),K(x,y,z),u(x,y,z),$ are functions having Taylor expressions on an interval $a\le x,y,z\le b$, under the given conditions,

which are

$$\sum_{i=0}^{1}\left[u^{(0,0,i)}(x,y,a)+u^{(0,0,i)}(x,y,b)+u^{(0,0,i)}(x,y,c_2)\right]=f(x,y), \tag{1.2}$$

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$$\sum_{i=0}^{1}\left[u^{(0,i,0)}(x,a,z)+u^{(0,i,0)}(x,b,z)+u^{(0,i,0)}(x,c_1,z)\right]=g(x,z),$$

$$\sum_{i=0}^{1}\left[u^{(i,0,0)}(a,y,z)+u^{(i,0,0)}(b,y,z)+u^{(i,0,0)}(c_0,y,z)\right]=h(y,z),$$

$$\begin{aligned}&u(a,a,a)+u(a,a,b)+u(a,a,c_2)+u(a,b,a)+u(a,b,b)+u(a,b,c_2)+\\&u(a,c_1,a)+u(a,c_1,b)+u(a,c_1,c_2)+u(b,a,a)+u(b,a,b)+u(b,a,c_2)+\\&u(b,b,a)+u(b,b,b)+u(b,b,c_2)+u(b,c_1,a)+u(b,c_1,b)+u(b,c_1,c_2)+\\&u(c_0,a,a)+u(c_0,a,b)+u(c_0,a,c_2)+u(c_0,b,a)+u(c_0,b,b)+u(c_0,b,c_2)+\\&u(c_0,c_1,a)+u(c_0,c_1,b)+u(c_0,c_1,c_2)=\lambda,\end{aligned}$$

where $a \le c_0, c_1, c_2 \le b$, where f is function of x, y, g is function of x, z, h is function of y, z, and λ is constant. The solution is expressed in the form

$$u(x,y,z)=\sum_{r=0}^{N}\sum_{s=0}^{N}\sum_{t=0}^{N}\frac{1}{r!.s!.t!}u^{(r,s,t)}(c_0,c_1,c_2)(x-c_0)^r(y-c_1)^s(z-c_2)^t,$$

$$a \le c_0, c_1, c_2 \le b \tag{1.3}$$

Which is a Taylor polynomial of degree N at $(x, y, z) = (c_0, c_1, c_2)$, where

$\frac{1}{r!.s!.t!}u^{(r,s,t)}(c_0,c_1,c_2)$, $r, s, t = 0,1,...,N$ are the coefficients to be determined.

2. FUNDAMENTAL RELATIONS

Now, we consider a function $u(x, y, z)$ of three variables. Let us assume that, in the range $a \le x, y, z \le b$, that the function $u(x, y, z)$ and the n th derivatives of $u(x, y, z)$ with respect to x can be expanded in Taylor series

$$u(x,y,z)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}a_{r,s,t}.(x-c_0)^r(y-c_1)^s(z-c_2)^t,\quad a \le c_0,c_1,c_2 \le b \text{ and}$$

$$u^{(n,0,0)}(x,y,z)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}a_{r,s,t}^{(n,0,0)}.(x-c_0)^r(y-c_1)^s(z-c_2)^t,\quad a \le c_0,c_1,c_2 \le b,$$ respectively, where $a_{r,s,t}^{(n,0,0)}$ and $a_{r,s,t}$ are Taylor coefficients; clearly $a_{r,s,t}^{(0,0,0)} = a_{r,s,t}$ and $u^{(0,0,0)}(x, y, z) = u\ (x, y, z)$.

Then the recurrence relation between the Taylor coefficients $a_{r,s,t}^{(n,0,0)}$ and $a_{r,s,t}^{(n+1,0,0)}$ of $u^{(n,0,0)}(x, y, z)$ and $u^{(n+1,0,0)}(x, y, z)$ is given by $a_{r,s,t}^{(n+1,0,0)} = (r+1)a_{r,s,t}^{(n,0,0)}$. (2.1)

Now, let us take $r = 0,1,..., N$ and assume $a_{r,s,t}^{(0,0,0)} = a_{r,s,t}^{(1,0,0)} = . \ . \ . = a_{r,s,t}^{(n,0,0)} = 0$ for $r > N$; then the Eq. (2.1) can be transformed into the matrix form,

$$A^{(n+1,0,0)} = MA^{(n,0,0)} \quad , \quad n = 0,1,2,... \tag{2.2}$$

$A^{(n,0,0)} = \left[a_{r,s,t}^{(n,0,0)}\right]$, $M = \left[m_{i,j}\right]$, $i, j = 0,1,..., N$, $a_{i,s,t}^{(n+1,0,0)} = \sum_{j=0}^{N} m_{i,j}.a_{j,s,t}^{(n,0,0)}$,where

$$A^{(n,0,0)}=\begin{bmatrix} & & & & & a_{0,0,N}^{(n,0,0)} & a_{0,1,N}^{(n,0,0)} & . & . & . & a_{0,N,N}^{(n,0,0)} \\ & & & & . & . & . & . & . & . & a_{1,N,N}^{(n,0,0)}. \\ & & & . & . & . & . & . & . & . & . \\ & & . & . & . & . & . & . & . & . & . \\ & a_{0,0,1}^{(n,0,0)} & a_{0,1,1}^{(n,0,0)} & . & . & . & a_{0,N,1}^{(n,0,0)} & . & . & . & . \\ \ldots a_{0,0,0}^{(n,0,0)} & a_{0,1,0}^{(n,0,0)} & . & . & . & a_{0,N,0}^{(n,0,0)} & a_{1,N,1}^{(n,0,0)} & . & . & . & a_{N,N,N}^{(n,0,0)}. \\ a_{1,0,0}^{(n,0,0)} & a_{1,1,0}^{(n,0,0)} & . & . & . & a_{1,N,0}^{(n,0,0)} & . & . & . & . & \\ . & . & . & . & . & . & . & . & . & & \\ . & . & . & . & . & . & . & . & & & \\ . & . & . & . & . & . & a_{N,N,1}^{(n,0,0)} & & & & \\ a_{N,0,0}^{(n,0,0)} & a_{N,1,0}^{(n,0,0)} & . & . & . & a_{N,N,0}^{(n,0,0)} & & & & & \end{bmatrix}_{(N+1).(N+1).(N+1)}$$

and

$$M=\begin{bmatrix} 0 & 1 & 0 & . & . & . & . & 0 \\ 0 & 0 & 2 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & 3 & 0 & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & N \\ 0 & 0 & . & . & . & . & . & 0 \end{bmatrix}.$$

For $n=0,1,\ldots,$ it follows from (2.2) that

$$\begin{aligned} &A^{(1,0,0)}=MA^{(0,0,0)},\\ &A^{(2,0,0)}=MA^{(1,0,0)}=M^{2}A^{(0,0,0)},\\ &A^{(3,0,0)}=MA^{(2,0,0)}=M^{3}A^{(0,0,0)},\\ &.\\ &.\\ &.\\ &A^{(n,0,0)}=MA^{(n-1,0,0)}=M^{n}A^{(0,0,0)},\\ &.\\ &.\\ &. \end{aligned} \tag{2.3}$$

where clearly $A^{(0,0,0)}=A$.

Let us assume, in the range $a\le x,y,z\le b$, that the n th derivatives of $u(x,y,z)$ with respect to y can be expanded in Taylor series

$$u^{(0,n,0)}(x,y,z)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}a_{r,s,t}^{(0,n,0)}.(x-c_0)^{r}(y-c_1)^{s}(z-c_2)^{t},\quad a\le c_0,c_1,c_2\le b,$$

respectively, where $a_{r,s,t}^{(0,n,0)}$ and $a_{r,s,t}$ are Taylor coefficients; clearly $a_{r,s,t}^{(0,0,0)}=a_{r,s,t}$ and $u^{(0,0,0)}(x,y,z)=u\ (x,y,z)$. Then the recurrence relation between the Taylor coefficients $a_{r,s,t}^{(0,n,0)}$ and $a_{r,s,t}^{(0,n+1,0)}$ of $u^{(0,n,0)}(x,y,z)$ and $u^{(0,n+1,0)}(x,y,z)$ is given by $a_{r,s,t}^{(0,n+1,0)}=(r+1)a_{r,s,t}^{(0,n,0)}$. (2.4)

Now, let us take $r=0,1,\ldots,N$ and assume $a_{r,s,t}^{(0,0,0)}=a_{r,s,t}^{(0,1,0)}=.\quad .\quad .=a_{r,s,t}^{(0,n,0)}=0$ for $r>N$; then the Eq. (2.4) can be transformed into the matrix form,

$$A_t^{(0,n+1,0)} = MA_t^{(0,n,0)} \quad , \quad n = 0,1,2,... \tag{2.5}$$

$$A_t^{(0,n,0)} = \left[a_{r,s,t}^{(0,n,0)}\right],\; M = \left[m_{i,j}\right],\; i,\, j = 0,1,...,N \;,\; a_{r,i,t}^{(0,n+1,0)} = \sum_{j=0}^{N} m_{i,j}\,.a_{r,j,t}^{(0,n,0)} \text{ ,where}$$

$$A_t^{(0,n,0} = \begin{bmatrix}
 & & & & a_{0,0,N}^{(0,n,0)} & a_{1,0,N}^{(0,n,0)} & \cdots & a_{N,0,N}^{(0,n,0)} \\
 & & & \cdot & \cdot & \cdot & \cdots & a_{N,1,N}^{(0,n,0)}\cdot \\
 & & & \cdot\;\cdot & \cdot & \cdot & \cdots & \cdot \\
 & & \cdot & \cdot\;\cdot & \cdot & \cdot & \cdots & \cdot \\
 & a_{0,0,1}^{(0,n,0)} & a_{1,0,1}^{(0,n,0)} & \cdot\;\cdot & \cdot & a_{N,0,1}^{(0,n,0)} & \cdots & \cdot \\
\ldots a_{0,0,0}^{(0,n,0)} & a_{1,0,0}^{(0,n,0)} & \cdot & \cdot\;\cdot & a_{N,0,0}^{(0,n,0)} & a_{N,1,1}^{(0,n,0)} & \cdots & a_{N,N,N}^{(0,n,0)}\cdot \\
a_{0,1,0}^{(0,n,0)} & a_{1,1,0}^{(0,n,0)} & \cdot & \cdot\;\cdot & a_{N,1,0}^{(0,n,0)} & \cdot & \cdots & \\
\cdot & \cdot & \cdot & \cdot\;\cdot & \cdot & \cdot & \cdot\;\cdot & \\
\cdot & \cdot & \cdot & \cdot\;\cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot\;\cdot & \cdot & a_{N,N,1}^{(0,n,0)} & & \\
a_{0,N,0}^{(0,n,0)} & a_{1,N,0}^{(0,n,0)} & \cdot & \cdot\;\cdot & a_{N,N,0}^{(0,n,0)} & & &
\end{bmatrix}_{(N+1).(N+1).(N+1)}$$

.

For $n = 0,1,...,$ it follows from (2.5) that

$$\begin{aligned}
A_t^{(0,1,0)} &= MA_t^{(0,0,0)}, \\
A_t^{(0,2,0)} &= MA_t^{(0,1,0)} = M^2 A_t^{(0,0,0)}, \\
A_t^{(0,3,0)} &= MA_t^{(0,2,0)} = M^3 A_t^{(0,0,0)}, \\
&. \\
&. \\
&. \\
A_t^{(0,n,0)} &= MA_t^{(0,n-1,0)} = M^n A_t^{(0,0,0)}, \\
&. \\
&. \\
&.
\end{aligned} \tag{2.6}$$

where clearly $A_t^{(0,0,0)} = A_t = \left[A^{(0,0,0)}\right]^T$.

Let us assume, in the range $a \le x, y, z \le b$, that the n th derivatives of $u(x, y, z)$ with respect to z can be expanded in Taylor series
$u^{(0,0,n)}(x, y, z) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty} a_{r,s,t}^{(0,0,n)}.(x - c_0)^r (y - c_1)^s (z - c_2)^t$, $a \le c_0, c_1, c_2 \le b$, respectively, where $a_{r,s,t}^{(0,0,n)}$ and $a_{r,s,t}$ are Taylor coefficients; clearly $a_{r,s,t}^{(0,0,0)} = a_{r,s,t}$ and $u^{(0,0,0)}(x, y, z) = u\;(x, y, z)$.

Then the recurrence relation between the Taylor coefficients $a_{r,s,t}^{(0,0,n)}$ and $a_{r,s,t}^{(0,0,n+1)}$ of $u^{(0,0,n)}(x, y, z)$ and $u^{(0,0,n+1)}(x, y, z)$ is given by $a_{r,s,t}^{(0,0,n+1)} = (r+1)a_{r,s,t}^{(0,0,n)}$. (2.7)

Now, let us take $r = 0,1,\ldots,N$ and assume $a_{r,s,t}^{(0,0,0)} = a_{r,s,t}^{(0,0,1)} = . \; . \; . = a_{r,s,t}^{(0,0,n)} = 0$ for $r > N$; then the Eq. (2.7) can be transformed into the matrix form,

$$A^{(0,0,n+1)} = M A^{(0,0,n)} \quad , \quad n = 0,1,2,... \tag{2.8}$$

$$A^{(0,0,n)} = \left[a_{r,s,t}^{(0,0,n)}\right], \; M = \left[m_{i,j}\right], \; i, j = 0,1,...,N \; , \; a_{r,s,i}^{(0,0,n+1)} = \sum_{j=0}^{N} m_{i,j} . a_{r,s,j}^{(0,0,n)} \text{ , where}$$

$$A^{(0,0,n)} = \begin{bmatrix}
 & & & & & a_{0,0,N}^{(0,0,n)} & a_{0,1,N}^{(0,0,n)} & . \; . \; . & a_{0,N,N}^{(0,0,n)} \\
 & & & & . & . & . & . \; . \; . & a_{1,N,N}^{(0,n,0)} . \\
 & & & . & . & . & . & . \; . \; . & . \\
 & & . & . & . & . & . & . \; . \; . & . \\
 & a_{0,0,1}^{(0,0,n)} & a_{0,1,1}^{(0,0,n)} & . & . & . & a_{0,N,1}^{(0,0,n)} & . \; . \; . & . \\
 \ldots a_{0,0,0}^{(0,0,n)} & a_{0,1,0}^{(0,0,n)} & . & . & . & a_{0,N,0}^{(0,0,n)} & a_{1,N,1}^{(0,0,n)} & . \; . \; . & a_{N,N,N}^{(0,0,n)} . \\
 a_{1,0,0}^{(0,0,n)} & a_{1,1,0}^{(0,0,n)} & . & . & . & a_{1,N,0}^{(0,0,n)} & . & . \; . \; . & \\
 . & . & . & . & . & . & . & . \; . & \\
 . & . & . & . & . & . & . & . & \\
 . & . & . & . & . & . & a_{N,N,1}^{(0,0,n)} & & \\
 a_{N,0,0}^{(0,0,n)} & a_{N,1,0}^{(0,0,n)} & . & . & . & a_{N,N,0}^{(0,0,n)} & & &
\end{bmatrix}_{(N+1).(N+1).(N+1)}$$

For $n = 0,1,\ldots,$ it follows from (2.8) that

$$\begin{aligned}
A^{(0,0,1)} &= M A^{(0,0,0)} , \\
A^{(0,0,2)} &= M A^{(0,0,1)} \quad M^2 A^{(0,0,0)} , \\
A^{(0,0,3)} &= M A^{(0,0,2)} \quad M^3 A^{(0,0,0)} , \\
&. \\
&. \\
&. \\
A^{(0,0,n)} &= M A^{(0,0,n-1)} \quad M^n A^{(0,0,0)} , \\
&. \\
&. \\
&.
\end{aligned} \tag{2.9}$$

where clearly $A^{(0,0,0)} = A$. Furthermore, $A^{(n,n,n)}$ can be expressed as follows:

$$A^{(n,n,n)} = M^n \; M^n A \; \left(M^n\right)^T . \tag{2.10}$$

3. METHOD OF SOLUTION

Our purpose is to investigate the truncated Taylor series solution of Eq. (1.1), under the given condutions, in the series form

$u(x,y,z)=\sum_{r=0}^{N}\sum_{s=0}^{N}\sum_{t=0}^{N} a_{r,s,t}.(x-c_0)^r (y-c_1)^s (z-c_2)^t$ or in the matrix form $u(x,y,z)= \; X \; \overset{Z}{A} \; Y$. To obtain the solution of Eq. (1.1) in the form of expression (1.3) we first reduce Eq. (1.1) to a differential equation whose coefficients are polynomials. For this purpose, we assume that the functions

$A(x,y,z), B(x,y.z), C(x,y.z), D(x,y,z), E(x,y.z), F(x,y.z), G(x,y,z), H(x,y,z), I(x,y,z), J(x,y,z), K(x,y,z)$ can be expressed in the form

$$A(x,y,z)=\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N} a_{i,j,k}(x-c_0)^i (y-c_1)^j (z-c_2)^k \; , B(x,y,z)=\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N} b_{i,j,k}(x-c_0)^i (y-c_1)^j (z-c_2)^k \; ,$$

$$C(x,y,z)=\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N} c_{i,j,k}(x-c_0)^i (y-c_1)^j (z-c_2)^k \; , D(x,y,z)=\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N} d_{i,j,k}(x-c_0)^i (y-c_1)^j (z-c_2)^k \; ,$$

$$E(x,y,z)=\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N} e_{i,j,k}(x-c_0)^i (y-c_1)^j (z-c_2)^k \; , F(x,y,z)=\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N} f_{i,j,k}(x-c_0)^i (y-c_1)^j (z-c_2)^k \; ,$$

$$G(x,y,z)=\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N} g_{i,j,k}(x-c_0)^i (y-c_1)^j (z-c_2)^k \; , H(x,y,z)=\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N} h_{i,j,k}(x-c_0)^i (y-c_1)^j (z-c_2)^k \; ,$$

$$I(x,y,z)=\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N} \iota_{i,j,k}(x-c_0)^i (y-c_1)^j (z-c_2)^k \; , J(x,y,z)=\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N} j_{i,j,k}(x-c_0)^i (y-c_1)^j (z-c_2)^k \; ,$$

$$K(x,y,z)=\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N} k_{i,j,k}(x-c_0)^i (y-c_1)^j (z-c_2)^k \tag{3.1}$$

which are Taylor polynomials at $(x,y,z)=(c_0,c_1,c_2)$. By using the expressions (3.1) in Eq., we get

$$\begin{aligned}
\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N}\Bigg[& a_{i,j,k}(x-c_0)^i (y-c_1)^j (z-c_2)^k \frac{\partial^2 u}{\partial x^2} + b_{i,j,k}(x-c_0)^i (y-c_1)^j (z-c_2)^k \frac{\partial^2 u}{\partial y^2} \\
& + c_{i,j,k}(x-c_0)^i (y-c_1)^j (z-c_2)^k \frac{\partial^2 u}{\partial z^2} + d_{i,j,k}(x-c_0)^i (y-c_1)^j (z-c_2)^k \frac{\partial^2 u}{\partial x \partial y} \\
& + e_{i,j,k}(x-c_0)^i (y-c_1)^j (z-c_2)^k \frac{\partial^2 u}{\partial x \partial z} + f_{i,j,k}(x-c_0)^i (y-c_1)^j (z-c_2)^k \frac{\partial^2 u}{\partial y \partial z} \\
& + g_{i,j,k}(x-c_0)^i (y-c_1)^j (z-c_2)^k \frac{\partial u}{\partial x} + h_{i,j,k}(x-c_0)^i (y-c_1)^j (z-c_2)^k \frac{\partial u}{\partial y} \\
& + \iota_{i,j,k}(x-c_0)^i (y-c_1)^j (z-c_2)^k \frac{\partial u}{\partial z} + j_{i,j,k}(x-c_0)^i (y-c_1)^j (z-c_2)^k u \Bigg] \\
& = \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N} k_{i,j,k}(x-c_0)^i (y-c_1)^j (z-c_2)^k \; .
\end{aligned} \tag{3.2}$$

The matrix representation of Taylor expansions $(x-c_0)^i (x-c_1)^j (x-c_2)^k u^{r,s,t}$ $\left(r=0,1,2;\, s=0,1,2;\, t=0,1,2;\, i=0,1,\ldots,N;\, j=0,1,\ldots,N\right)$, are obtained by means of the formula

$$(x-c_0)^i u^{(r,0,0)}(x,y,z) = \; XC_i M^r \; \overset{Z}{A} \; Y, \qquad (x-c_0)^i u^{(0,s,0)}(x,y,z) = \; XC_i \; \overset{Z}{A} \; \left(M^s\right)^T Y,$$

$$\left(y-c_1\right)^j u^{(r,0,0)}(x,y,z) = XM^r \overset{Z}{A} \left(C_j\right)^T Y, \qquad \left(y-c_1\right)^j u^{(0,s,0)}(x,y,z) = X \overset{Z}{A} \left(C_i M^s\right)^T Y,$$

$$\left(z-c_2\right)^k u^{(r,0,0)}(x,y,z) = XM^r \overset{ZC_k}{A} Y, \qquad \left(z-c_2\right)^k u^{(0,s,0)}(x,y,z) = X \overset{ZC_k}{A} \left(M^s\right)^T Y,$$

$$\left(x-c_0\right)^i u^{(0,0,t)}(x,y,z) = XC_i M^t \overset{ZM^t}{A} Y, \qquad \left(y-c_1\right)^j u^{(0,0,t)}(x,y,z) = X \overset{ZM^t}{A} \left(C_j\right)^T Y,$$

$$\left(z-c_2\right)^k u^{(0,0,t)}(x,y,z) = X \overset{ZC_k M^t}{A} Y,$$

$$\left(x-c_0\right)^i \left(y-c_1\right)^j \left(z-c_2\right)^k u^{(r,s,t)}(x,y,z) = XC_i M^r \overset{ZC_k M^t}{A} \left(C_j M^s\right)^T Y, \tag{3.3}$$

where $M^0 = I$ (unit matrix),

$$X = \left[1 \;\; \left(x-c_0\right) \;\; \left(x-c_0\right)^2 \;\; \ldots \;\; \left(x-c_0\right)^N\right],$$

$$Y = \left[1 \;\; \left(y-c_1\right) \;\; \left(y-c_1\right)^2 \;\; \ldots \;\; \left(y-c_1\right)^N\right]^T,$$

$$Z = \left[1 \;\; \left(z-c_2\right) \;\; \left(z-c_2\right)^2 \;\; \ldots \;\; \left(x-c_2\right)^N\right],$$

The matrix form $C_p = \left[c_{i,j}\right]$ is defined in [3]. Also we assume that the matrix form of the equation $K(x,y,z)$ can be defined as follows

$$K(x,y,z) = X \overset{Z}{K} Y, \tag{3.4}$$

where $K = \left[k_{i,j,k}\right]$ $i, j, k = 0,1,\ldots,N.$

Substituting the expressions (3.3) and (3.4) into the Eq. (3.2), and simplifying the result, we have the matrix equation

$$\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N}\Bigg[a_{i,j,k}\; C_i M^2 \overset{C_k}{A} \left(C_j\right)^T + b_{i,j,k}\; C_i \overset{C_k}{A} \left(C_j M^2\right)^T$$

$$+ c_{i,j,k}\; C_i \overset{C_k M^2}{A} \left(C_j\right)^T + d_{i,j,k}\; C_i M \overset{C_k}{A} \left(C_j M\right)^T$$

$$+ e_{i,j,k}\; C_i M \overset{C_k M}{A} \left(C_j\right)^T + f_{i,j,k}\; C_i \overset{C_k M}{A} \left(C_j M\right)^T$$

$$+ g_{i,j,k}\; C_i M \overset{C_k}{A} \left(C_j\right)^T + h_{i,j,k}\; C_i \overset{C_k}{A} \left(C_j M\right)^T$$

$$+\,i_{i,j,k}\; C_i \overset{C_k M}{A} \left(C_j\right)^T + j_{i,j,k}\; C_i \overset{C_k}{A} \left(C_j\right)^T \Bigg] = K\,, \tag{3.5}$$

which correspondes to a system $(N+1).(N+1).(N+1)$ algebraic equations fort he unknown Taylor coefficients $a_{r,s,t}\,; r,s,t = 0,1,\ldots,N$. Breafly, we can assume that Eq. (3.5) is given in the form

$$\sum_{\sigma=1}^{10} W_\sigma A Y_\sigma^{Z_\sigma} = K\,, \tag{3.6}$$

where $W_\sigma = \left[w_{i,j}\right], Y_\sigma = \left[y_{i,j}\right], Z_\sigma = \left[z_{i,j}\right], \sigma = 1,2,\ldots,10$. Matrix equation (3.6) can be reduce to new matrix equation by making use of

$$x_{i(N+1)^2+v(N+1)+k,\, j(N+1)^2+u(N+1)+l} = w_{i,j}.y_{u,v}.z_{k,l}\,;\; i = 0,1,\ldots,N,\; j = 0,1,\ldots,N,$$

$$k = 0,1,\ldots,N,\; l = 0,1,\ldots,N,\; u = 0,1,\ldots,N,\; v = 0,1,\ldots,N.$$

Then, the new matrix equation (the fundamental matrix equation) is

$$\sum_{\sigma=1}^{10} X_\sigma \bar{A} = \bar{G}\,, \tag{3.7}$$

where $X_\sigma = \left[x_{z,q}\right], z,q = 0,1,\ldots,(N+1)(N+1)(N+1), \sigma = 1,2,\ldots,10$ and

$$\bar{A} = \big[a_{0,0,0}\;\; a_{0,0,1}\;\cdots\; a_{0,0,N}\;\; a_{0,1,0}\;\; a_{0,1,1}\;\cdots\; a_{0,1,N}\;\cdots\; a_{0,N,0}\;\; a_{0,N,1}\;\cdot\; a_{0,N,N}$$
$$a_{1,0,0}\;\; a_{1,0,1}\;\cdots\; a_{1,N,N}\;\cdots\; a_{N,N,N-1}\;\; a_{N,N,N}\big]^T,$$

$$\bar{G} = \big[g_{0,0,0}\;\; g_{0,0,1}\;\cdots\; g_{0,0,N}\;\; g_{0,1,0}\;\; g_{0,1,1}\;\cdots\; g_{0,1,N}\;\cdots\; g_{0,N,0}\;\; g_{0,N,1}\;\cdot\; g_{0,N,N}$$
$$g_{1,0,0}\;\; g_{1,0,1}\;\cdots\; g_{1,N,N}\;\cdots\; g_{N,N,N-1}\;\; g_{N,N,N}\big]^T.$$

4. MATRIX FORMS OF CONDITIONS

Let us consider the matrices forms of given condition

$$u^{(0,0,0)}(x,y,a) + u^{(0,0,0)}(x,y,b) + u^{(0,0,0)}(x,y,c_2) + u^{(0,0,1)}(x,y,a) + u^{(0,0,1)}(x,y,b) + u^{(0,0,1)}(x,y,c_2) = f(x,y), \tag{4.1}$$

$$u^{(0,0,0)}(x,a,z) + u^{(0,0,0)}(x,b,z) + u^{(0,0,0)}(x,c_1,z) + u^{(0,1,0)}(x,a,z) + u^{(0,1,0)}(x,b,z) + u^{(0,1,0)}(x,c_1,z) = g(x,z), \tag{4.2}$$

$$u^{(0,0,0)}(a,y,z) + u^{(0,0,0)}(b,y,z) + u^{(0,0,0)}(c_0,y,z) + u^{(1,0,0)}(a,y,z) + u^{(1,0,0)}(b,y,z) + u^{(1,0,0)}(c_0,y,z) = h(y,z), \tag{4.3}$$

$$\begin{aligned}&u(a,a,a)+u(a,a,b)+u(a,a,c_2)+u(a,b,a)+u(a,b,b)+u(a,b,c_2)+\\&u(a,c_1,a)+u(a,c_1,b)+u(a,c_1,c_2)+u(b,a,a)+u(b,a,b)+u(b,a,c_2)+\\&u(b,b,a)+u(b,b,b)+u(b,b,c_2)+u(b,c_1,a)+u(b,c_1,b)+u(b,c_1,c_2)+\\&u(c_0,a,a)+u(c_0,a,b)+u(c_0,a,c_2)+u(c_0,b,a)+u(c_0,b,b)+u(c_0,b,c_2)+\\&u(c_0,c_1,a)+u(c_0,c_1,b)+u(c_0,c_1,c_2)=\lambda.\end{aligned} \tag{4.4}$$

Now, we try to obtain the corresponding matrices form for the given condition as follows:

$$\begin{aligned}&X^{(0)}(c_0)=[1\ \ 0\ \ 0\ \ \ldots\ \ 0],X^{(0)}(a)=[1\ \ h_0\ \ h_0^2\ \ \ldots\ \ h_0^N],\\&X^{(0)}(b)=[1\ \ k_0\ \ k_0^2\ \ \ldots\ \ k_0^N],X^{(1)}(c_0)=[0\ \ 1\ \ 0\ \ \ldots\ \ 0],\\&X^{(1)}(a)=[0\ \ 1\ \ h_0\ \ \ldots\ \ h_0^{N-1}],X^{(1)}(b)=[0\ \ 1\ \ k_0\ \ \ldots\ \ k_0^{N-1}],\end{aligned} \tag{4.5}$$

where $h_0 = a - c_0$ and $k_0 = b - c_0$,

$$\begin{aligned}&Y^{(0)}(c_1)=[1\ \ 0\ \ 0\ \ \ldots\ \ 0]^T,Y^{(0)}(a)=[1\ \ h_1\ \ h_1^2\ \ \ldots\ \ h_1^N]^T,\\&Y^{(0)}(b)=[1\ \ k_1\ \ k_1^2\ \ \ldots\ \ k_1^N]^T,Y^{(1)}(c_1)=[0\ \ 1\ \ 0\ \ \ldots\ \ 0]^T,\\&Y^{(1)}(a)=[0\ \ 1\ \ h_1\ \ \ldots\ \ h_1^{N-1}]^T,Y^{(1)}(b)=[0\ \ 1\ \ k_1\ \ \ldots\ \ k_1^{N-1}]^T,\end{aligned} \tag{4.6}$$

where $h_1 = a - c_1$ and $k_1 = b - c_1$,

$$\begin{aligned}&Z^{(0)}(c_2)=[1\ \ 0\ \ 0\ \ \ldots\ \ 0],Z^{(0)}(a)=[1\ \ h_2\ \ h_2^2\ \ \ldots\ \ h_2^N],\\&Z^{(0)}(b)=[1\ \ k_2\ \ k_2^2\ \ \ldots\ \ k_2^N],Z^{(1)}(c_2)=[0\ \ 1\ \ 0\ \ \ldots\ \ 0],\\&Z^{(1)}(a)=[0\ \ 1\ \ h_2\ \ \ldots\ \ h_2^{N-1}],Z^{(1)}(b)=[0\ \ 1\ \ k_2\ \ \ldots\ \ k_2^{N-1}],\end{aligned} \tag{4.7}$$

where $h_2 = a - c_2$ and $k_2 = b - c_2$. We assume that the function $f(x,y)$, $g(x,z)$ and $h(y,z)$ can be expanded as

$f(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f_{r,s}.(x-c_0)^r(y-c_1)^s$, $g(x,z) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} g_{r,s}.(x-c_0)^r(z-c_2)^s$, and

$h(y,z) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} h_{r,s}.(y-c_1)^r(z-c_2)^s$ or in the matrix forms

$$f(x,y) = X\ \ f\ \ Y,\ g(x,z) = X\ \ g\ \ Z^T,\ \text{and}\ h(y,z) = Z\ \ h\ \ Y \tag{4.8}$$

where

$$f=\begin{bmatrix}f_{0,0}&f_{0,1}&\cdots&f_{0,n}\\f_{1,0}&f_{1,1}&\cdots&f_{1,N}\\.&.&\cdots&.\\.&.&\cdots&.\\f_{N,0}&f_{N,1}&\cdots&f_{N,N}\end{bmatrix},\ g=\begin{bmatrix}g_{0,0}&g_{0,1}&\cdots&g_{0,n}\\g_{1,0}&g_{1,1}&\cdots&g_{1,N}\\.&.&\cdots&.\\.&.&\cdots&.\\g_{N,0}&g_{N,1}&\cdots&g_{N,N}\end{bmatrix},\ h=\begin{bmatrix}h_{0,0}&h_{0,1}&\cdots&h_{0,n}\\h_{1,0}&h_{1,1}&\cdots&h_{1,N}\\.&.&\cdots&.\\.&.&\cdots&.\\h_{N,0}&h_{N,1}&\cdots&h_{N,N}\end{bmatrix}.$$

We obtain the matrices forms

$$u(x,y,\mu) = X \quad \overset{Z^{(0)}(\mu)}{A} \quad Y, \quad \mu = a,b,c_2, \quad u^{(0,0,1)}(x,y,\mu) = X \quad \overset{Z^{(1)}(\mu)}{A} \quad Y, \quad \mu = a,b,c_2,$$

$$u(x,\eta,z) = X \quad \overset{Z}{A} \quad Y^{(0)}(\eta), \quad \eta = a,b,c_1, \quad u^{(0,1,0)}(x,\eta,z) = X \quad \overset{Z}{A} \quad Y^{(1)}(\eta), \quad \eta = a,b,c_1,$$

(4.9)

$$u(\varepsilon,y,z) = X^{(0)}(\varepsilon) \quad \overset{Z}{A} \quad Y, \quad \varepsilon = a,b,c_0, \quad u^{(1,0,0)}(\varepsilon,y,z) = X^{(1)}(\varepsilon) \quad \overset{Z}{A} \quad Y, \quad \varepsilon = a,b,c_0,$$

$$u(\varepsilon,\eta,\mu) = X^{(0)}(\varepsilon) \quad \overset{Z^{(0)}(\mu)}{A} \quad Y^{(0)}(\eta), \quad \varepsilon = a,b,c_0,\ \eta = \mathrm{a,b,c}_1, \quad \mu = a,b,c_2. \tag{4.10}$$

Substituting these matrices forms into conditions (4.1), (4.2), (4.3), and (4.4), then simplifying, we get the fundamental matrix equations of conditions as follows

$$\overset{U}{A} = f, \quad A \quad V = g, \quad T \quad A = h, \quad Q_1 \quad \overset{Q_3}{A} \quad Q_2 = \lambda, \quad \text{where}$$

$$U = Z^{(0)}(\mu) + Z^{(1)}(\mu), \quad \mu = a,b,c_2, \quad V = Y^{(0)}(\eta) + Y^{(1)}(\eta), \quad \eta = a,b,c_1,$$

$$T = X^{(0)}(\varepsilon) + X^{(1)}(\varepsilon), \quad \varepsilon = a,b,c_0, \quad Q_1 = X^{(0)}(\varepsilon), \quad \varepsilon = a,b,c_0,$$

$$Q_2 = Y^{(0)}(\eta), \quad \eta = a,b,c_1, \quad Q_3 = Z^{(0)}(\mu), \quad \mu = a,b,c_2.$$

5. FORMER METHOD FOR THE SOLUTION

We can assume that the Eq. (3.7) is in the form

$$\overline{X}\,\overline{A} = \overline{G}, \quad \text{where} \quad \overline{X} = \sum_{\sigma=1}^{10} X_\sigma. \tag{5.1}$$

Then the augmented matrix of (5.1) becomes $\left[\overline{X};\overline{G}\right]$ or

$$\begin{bmatrix} \overline{x_{0,0}} & \overline{x_{0,1}} & \cdots & \overline{x_{0,N[(N+1)(N+2)+1]}} & ; & g_{0,0,0} \\ \overline{x_{1,0}} & \overline{x_{1,1}} & \cdots & \overline{x_{1,N[(N+1)(N+2)+1]}} & ; & g_{0,0,1} \\ . & . & \cdots & . & ; & . \\ . & . & \cdots & . & ; & . \\ . & . & \cdots & . & ; & . \\ \overline{x_{N[(N+1)(N+2)+1],0}} & \overline{x_{N[(N+1)(N+2)+1],1}} & \cdots & \overline{x_{N[(N+1)(N+2)+1],N[(N+1)(N+2)+1]}} & ; & g_{N,N,N} \end{bmatrix}. \tag{5.2}$$

If we take the matrix forms of the conditions as $\overline{U}\,\overline{A} = \overline{f}$, $\overline{V}\,\overline{A} = \overline{g}$, $\overline{T}\,\overline{A} = \overline{h}$, $\overline{Q}\,\overline{A} = \lambda$, respectively, the augmented matrices of them become $\left[\overline{U};\overline{f}\right]$, $\left[\overline{V};\overline{g}\right]$, $\left[\overline{T};\overline{h}\right]$, $\left[\overline{Q};\lambda\right]$ or more clearly

$$\begin{bmatrix} u_{0,0} & u_{0,1} & \cdots & u_{0,N[(N+1)(N+2)+1]} & ; & f_{0,0} \\ u_{1,0} & u_{1,1} & \cdots & u_{1,N[(N+1)(N+2)+1]} & ; & f_{0,1} \\ . & . & \cdots & . & ; & . \\ . & . & \cdots & . & ; & . \\ . & . & \cdots & . & ; & . \\ u_{N[(N+1)(N+2)+1],0} & u_{N[(N+1)(N+2)+1],1} & \cdots & u_{N[(N+1)(N+2)+1],N[(N+1)(N+2)+1]} & ; & f_{N,N} \end{bmatrix}, \tag{5.3}$$

$$\begin{bmatrix} v_{0,0} & v_{0,1} & \cdots & v_{0,N[(N+1)(N+2)+1]} & ; & g_{0,0} \\ v_{1,0} & v_{1,1} & \cdots & v_{1,N[(N+1)(N+2)+1]} & ; & g_{0,1} \\ . & . & \cdots & . & ; & . \\ . & . & \cdots & . & ; & . \\ . & . & \cdots & . & ; & . \\ v_{N[(N+1)(N+2)+1],0} & v_{N[(N+1)(N+2)+1],1} & \cdots & v_{N[(N+1)(N+2)+1],N[(N+1)(N+2)+1]} & ; & g_{N,N} \end{bmatrix}, \tag{5.4}$$

$$\begin{bmatrix} t_{0,0} & t_{0,1} & \cdots & t_{0,N[(N+1)(N+2)+1]} & ; & h_{0,0} \\ t_{1,0} & t_{1,1} & \cdots & t_{1,N[(N+1)(N+2)+1]} & ; & h_{0,1} \\ . & . & \cdots & . & ; & . \\ . & . & \cdots & . & ; & . \\ . & . & \cdots & . & ; & . \\ t_{N[(N+1)(N+2)+1],0} & t_{N[(N+1)(N+2)+1],1} & \cdots & t_{N[(N+1)(N+2)+1],N[(N+1)(N+2)+1]} & ; & h_{N,N} \end{bmatrix} \tag{5.5}$$

and

$$\begin{bmatrix} q_{0,0} & q_{0,1} & \cdots & q_{0,N[(N+1)(N+2)+1]} & ; & \lambda \end{bmatrix}. \tag{5.6}$$

Consequently, by replacing (5.3), (5.4), (5.5) and (5.6) by the last $3(N+1)(N+1)+1$ rows of (5.2). we have the new augmented matrix

$$\begin{bmatrix} x_{0,0} & x_{0,1} & \cdots & x_{0,N[(N+1)(N+2)+1]} & ; & g_{0,0,0} \\ x_{1,0} & x_{1,1} & \cdots & x_{1,N[(N+1)(N+2)+1]} & ; & g_{0,0,1} \\ . & . & \cdots & . & ; & . \\ . & . & \cdots & . & ; & . \\ . & . & \cdots & . & ; & . \\ x_{(N+1)^2(N-2)-2,0} & x_{(N+1)^2(N-2)-2,1} & \cdots & x_{(N+1)^2(N-2)-2,N[(N+1)(N+2)+1]} & ; & g_{N,N-3,N-1} \\ u_{0,0} & u_{0,1} & \cdots & u_{0,N[(N+1)(N+2)+1]} & ; & f_{0,0} \\ . & . & \cdots & . & ; & . \\ . & . & \cdots & . & ; & . \\ . & . & \cdots & . & ; & . \\ u_{N[(N+1)(N+2)+1],0} & u_{N[(N+1)(N+2)+1],1} & \cdots & u_{N[(N+1)(N+2)+1],N[(N+1)(N+2)+1]} & ; & f_{N,N} \\ v_{0,0} & v_{0,1} & \cdots & v_{0,N[(N+1)(N+2)+1]} & ; & g_{0,0} \\ . & . & \cdots & . & ; & . \\ . & . & \cdots & . & ; & . \\ . & . & . & . & ; & . \\ v_{N[(N+1)(N+2)+1],0} & v_{N[(N+1)(N+2)+1],1} & \cdots & v_{N[(N+1)(N+2)+1],N[(N+1)(N+2)+1]} & ; & g_{N,N} \\ t_{0,0} & t_{0,1} & \cdots & v_{0,N[(N+1)(N+2)+1]} & ; & h_{0,0} \\ . & . & . & . & ; & . \\ . & . & . & . & ; & . \\ . & . & . & . & ; & . \\ t_{N[(N+1)(N+2)+1],0} & t_{N[(N+1)(N+2)+1],1} & \cdots & t_{N[(N+1)(N+2)+1],N[(N+1)(N+2)+1]} & ; & h_{N,N} \\ q_{0,0} & q_{0,1} & \cdots & q_{0,N[(N+1)(N+2)+1]} & ; & \lambda \end{bmatrix}$$

.

From the solution of this system we can find matrix $\overline{A}$ or matrix A.

6. ILLUSTRATIONS

The Taylor matrix method applied in this study is useful in finding approximate solutions of second order linear partial differential equations with three variables in terms of Taylor polynomials. We illustrate it by the following examples.

Example 1. We now consider the problem

$$u_{zz} = u_{yy} + u_{xx} + 8;$$
$$u(x,y,0) + u_z(x,y,0) = (x+2y)(x+2y+6),$$
$$u_y(x,0,z) = 4(x+3z),$$

(6.1)

and seek the solution in the form

$$u(x,y,z) = \sum_{r=0}^{2}\sum_{s=0}^{2}\sum_{t=0}^{2} a_{r,s,t}.(x-c_0)^r (y-c_1)^s (z-c_2)^t, \quad (c_0,c_1,c_2) = (0,0,0). \tag{6.2}$$

Then we obtain the matrix equation

$$-M^2 A \quad -A\left(M^2\right)^T + \quad \overset{M^2}{A} \quad = 8R \tag{6.3}$$

where

$$R = \begin{bmatrix} & & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & & \end{bmatrix}$$

and the condition matrices are

$$\overset{U}{A} = \begin{bmatrix} 0 & 12 & 4 \\ 6 & 4 & 0 \\ 1 & 0 & 0 \end{bmatrix}; \; U = Z^{(0)}(0) + Z^{(1)}(0), \qquad AY^{(1)}(0) = \begin{bmatrix} 0 & 12 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{6.4}$$

By replacing the new matrix form of (6.4) in the new matrix form of (6.3), we have the matrix equation under given conditions as follows

$$
\begin{bmatrix}
0&0&2&0&0&0&-2&0&0&0&0&0&0&0&0&0&0&0&-2&0&0&0&0&0&0&0&0\\
0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-2&0&0&0&0&0\\
0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-2&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-2&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0
\end{bmatrix}
.
\begin{bmatrix}
a_{000}\\ a_{001}\\ a_{002}\\ a_{010}\\ a_{011}\\ a_{012}\\ a_{020}\\ a_{021}\\ a_{022}\\ a_{100}\\ a_{101}\\ a_{102}\\ a_{110}\\ a_{111}\\ a_{112}\\ a_{120}\\ a_{121}\\ a_{122}\\ a_{200}\\ a_{201}\\ a_{202}\\ a_{210}\\ a_{211}\\ a_{212}\\ a_{220}\\ a_{221}\\ a_{222}
\end{bmatrix}
=
\begin{bmatrix}
8\\0\\0\\0\\0\\0\\0\\0\\0\\0\\12\\4\\6\\4\\0\\1\\0\\0\\0\\12\\0\\4\\0\\0\\0\\0\\0
\end{bmatrix}
$$

Hence, we can solve this matrix from with aiding computer.We obtain Taylor coffecients from the solution of this system as follows

$a_{0,0,2} = 9, a_{0,1,1} = 12, a_{0,2,0} = a_{1,1,0} = 4, a_{1,0,1} = 6, a_{2,0,0} = 1,$

the others coefficients equal zero and thereby the solution of problem (6.1) becomes $u(x,y,z) = x^2 + 4y^2 + 9z^2 + 4xy + 6xz + 12yz$, which is exact solution.

7.CONCLUSIONS

Analytic solutions of the second order linear partial differential equations with three variables and with variable coefficients are usually difficult. The Taylor matrix method can be proposed for approximate solutions of this equations.

In this study, the usefulness of the Taylor matrix method presented for the approximate solution of the second order linear partial differential equations with three variables is discussed. A considerable advantage of the method is that the solution is expressed as a truncated Taylor series and thereby a Taylor polynomial about any point-wise $(x,y,z) = (c_0, c_1, c_2)$. Furthermore, after calculation of the series coefficients, the solution $u(x, y, z)$ of the equations can be easily evaluated for arbitrary values of (x, y, z) at low computation effort. If the functions $A(x, y, z), B(x, y.z), C(x, y.z), D(x, y, z), E(x, y.z), F(x, y.z), G(x, y, z),$ $H(x, y, z), I(x, y, z), J(x, y, z), K(x, y, z),$ in the equation, can be expended to the Taylor series, then there exists the solution $u(x, y, z)$; otherwise, the method can not be used.

To get the best approximating solution of the equation, we must take more terms from the Taylor expansion of functions; that is, the truncation limit N must be chosen sufficiently large. Briefly, for computational efficiency, an estimate for N is important. If N is chosen too large, more work than necessary will have been done; for N too small, the computation will have to be repeated. Therefore N must be chosen sufficiently large. In cases, it may be required computer aid. The method can be also extend to the solution of the higher order linear partial differential equations as the solution of the schrodinger equation depend on time in three dimension.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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