Gazi University Journal of Science
GU J Sci
28(4):729-735 (2015)

Fixed Points Of Mappings On The Fuzzy Reflexive Spaces

Bayaz DARABY[1,♠], Zahra SOLIMANI[2]

[1]Department of Mathematics, University of Maragheh, P.O. Box: 55181-83111, Maragheh, Iran.
[2]Department of Mathematics, University of Maragheh, P.O. Box: 55181-83111, Maragheh, Iran.

Received: 18/05/2015 Accepted: 25/08/2015

ABSTRACT

In this paper, we first define the new notion of fuzzy uniform normal structure. Moreover, we prove that the spectrum of the category of fuzzy reflexive spaces is broader than the category of the spaces which have fuzzy normal structure. Also, we introduce the notions of *FNST, FNSTN* and we prove the theorems of the fixed points of some classes of mappings on the sets from the fuzzy reflexive space, which have some of the properties *FNST, FNSTN.*

Key Words: Fixed point, Fuzzy norm, Fuzzy normed spaces, Fuzzy reflexive spaces.

1. INTRODUCTION

Baillon and Schoneberg [6] showed that the space X have the Browder-Kirk-Gohde *(BKG)* property if any nonexpansive self-mapping of a nonempty closed convex and bounded subset E of X have a fixed point in E. Also, Browder and Gohde, independently, showed that uniformly convex Banach spaces have the property *BKG*. We know for some of the mappings on fuzzy Banach spaces that have the fuzzy normal structure, there are many results about the existence of the fixed point. W. A. Kirk showed that a class of Banach spaces, namely, reflexive Banach spaces with normal structure, have the property *BKG* or every reflexive Banach space with asymptotic normal structure has the property *BKG* and proved that if *K* is a subset of reflexive Banach space with normal structure, the nonexpansive mapping $T: K \longrightarrow K$ has a fixed point. The existence of the fixed point for one class of mappings on the sets which have the normal structure is considered in the paper [17]. Now we introduce the conditions *FNST* and *FNSTN* for subsets on the fuzzy reflexive spaces which extend the fuzzy normal structure and we obtain some new results in fixed point theory.

2. PRELIMINARIES

Let η be a fuzzy subset on ℝ, i.e. a mapping η: ℝ → [0, 1] with grade of membership η (t), for each real number *t* .

♠Corresponding author, e-mail: bdaraby@maragheh.ac.ir

In this paper we consider the concept of fuzzy real numbers (fuzzy intervals) in the sense of Xiao and Zhu which is defined below:

Definition 2.1.[18] A fuzzy subset η on ℝ is called a fuzzy real number (fuzzy intervals), whose α-level set is denoted by $[\eta]_\alpha$, i.e., $[\eta]_\alpha = \{t : \eta(t) \geq \alpha\}$, if it satisfies two axioms:

(N_1) There exists r' ∈ℝ such that η (r')=1.

(N_2) For all $0 < \alpha \leq 1$, there exists real number $-\infty < \eta_\alpha^- \leq \eta_\alpha^+ < +\infty$ such that $[\eta]_\alpha$ is equal to the closed interval $[\eta_\alpha^-, \eta_\alpha^+]$.

The set of all fuzzy real numbers (fuzzy intervals) is denoted by F(ℝ). If η ∈ F (ℝ) and η (t) = 0 whenever t < 0, then η is called a non-negative fuzzy real number and $F^*(\mathbb{R})$ denotes the set of all non-negative fuzzy real numbers. Real number $\eta_\alpha^- \geq 0$ for all $\eta \in F^*(\mathbb{R})$ and each α∈ (0, 1]. The number $\tilde{0}$ stands for the fuzzy real number as:

$$\tilde{0}(t) = \begin{cases} 1, & t = 0, \\ 0, & t \neq 0. \end{cases}$$

It is clear that $\tilde{0} \in F^*(\mathbb{R})$.

Fuzzy real number $\tilde{r} \in F(\mathbb{R})$ defined by

$$\tilde{r}(t) = \begin{cases} 1, & t = r, \\ 0, & t \neq r, \end{cases}$$

it follows that ℝ can be embedded in F (ℝ), that is if r ∈ (∞ , ∞), then $\tilde{r} \in F(\mathbb{R})$ satisfies $\tilde{r}(t) = \tilde{0}(t - r)$ and α-level of $\tilde{r}$ is given by $[\tilde{r}] = [r, r], \alpha \in (0,1]$.

Definition 2.2.[9] Let δ, γ ∈ F (ℝ) and $[\delta]_\alpha = [\delta_\alpha^-, \delta_\alpha^+], [\gamma]_\alpha = [\gamma_\alpha^-, \gamma_\alpha^+]$, for each α ϵ (0, 1]. Define a partial ordering by δ ≼ γ in F (ℝ) if and only if $\delta_\alpha^- \leq \gamma_\alpha^-$ and $\delta_\alpha^+ \leq \gamma_\alpha^+$, for each α∈ (0, 1]. The strict inequality in F (ℝ) is defined by δ ≺ γ if and only if $\delta_\alpha^- < \gamma_\alpha^-$ and $\delta_\alpha^+ < \gamma_\alpha^+$, for each α ∈ (0, 1].

Lemma 2.3. Let η ∈ F (ℝ). Then $\eta \succcurlyeq \tilde{0}$ if and only if $\eta \in F^*(\mathbb{R})$.

Proof. The proof follows immediately from Definition 2.2.

According to Mizumoto and Tanaka [13], fuzzy arithmetic operations ⊕, ⊖, ⊗ and ⊘ on F(ℝ) × F(ℝ) can be defined as:

$$(\eta \oplus \delta)(t) = \vee_{t=x+y} (\min (\eta(x), \delta(y)))$$
$$= \sup_{s\in\mathbb{R}} \{\eta(s) \wedge \delta(t - s)\}, t \in \mathbb{R}$$
$$(\eta \ominus \delta)(t) = \vee_{t=x-y} (\min (\eta(x), \delta(y)))$$
$$= \sup_{s\in\mathbb{R}} \{\eta(s) \wedge \delta(s - t)\}, t \in \mathbb{R}$$
$$(\eta \otimes \delta)(t) = \vee_{t=xy} (\min (\eta(x), \delta(y)))$$
$$= \sup_{s\in\mathbb{R}, s\neq 0} \{\eta(s) \wedge \delta(t / s)\}, t \in \mathbb{R}$$
$$(\eta \oslash \delta)(t) = \vee_{t=\frac{x}{y}} (\min (\eta(x), \delta(y)))$$
$$= \sup_{s\in\mathbb{R}}\{\eta(st) \wedge \delta(s)\}, t \in \mathbb{R}.$$

which are special cases of Zadeh's extension principle. The additive and multiplicative identities in F (ℝ) are $\tilde{0}$ and $\tilde{1}$, respectively. Let $\ominus\eta$ be defined as $\tilde{0} \ominus \eta$. It is clear that $\eta \ominus \delta = \eta \oplus (\ominus \delta)$.

Lemma 2.4.[15] Let γ, δ be fuzzy real numbers. Then

$\forall\alpha \in (0, 1]$, $[\gamma]_\alpha = [\delta]_\alpha \Leftrightarrow \forall t \in \mathbb{R}, \gamma(t) = \delta(t)$.

Definition 2.5. [9] The absolute value | η | of η ∈ F (ℝ) is defined by

$$|\eta|(t) = \begin{cases} \sup(\eta(t), \eta(-t)), & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Lemma 2.6. Let γ, δ ∈ F (ℝ) and $[\gamma]_\alpha = [\gamma_\alpha^-, \gamma_\alpha^+]$, $[\delta]_\alpha = [\delta_\alpha^-, \delta_\alpha^+]$. Then for all α ∈ (0, 1],

$$[\gamma \oplus \delta]_\alpha = [\gamma_\alpha^- + \delta_\alpha^-, \gamma_\alpha^+ + \delta_\alpha^+]$$
$$[\gamma \ominus \delta]_\alpha = [\gamma_\alpha^- - \delta_\alpha^+, \gamma_\alpha^+ - \delta_\alpha^-]$$
$$[\gamma \otimes \delta]_\alpha = [\gamma_\alpha^- \delta_\alpha^-, \gamma_\alpha^+ \delta_\alpha^+]$$
$$[1/\gamma]_\alpha = [\tfrac{1}{\gamma_\alpha^+}, \tfrac{1}{\gamma_\alpha^-}], \gamma_\alpha^- > 0$$

$[|\gamma|]_\alpha = [\max(0, \gamma_\alpha^-, -\gamma_\alpha^+), \max(|\gamma_\alpha^-|, |\gamma_\alpha^+|)]$

Proof. Lemma 2.1 [9].

Proposition 2.7. [2] If η_i, i=1, 2, be the fuzzy real numbers (fuzzy intervals) generated by the family of nested bounded closed intervals $\{[a_\alpha^s, b_\alpha^s]: 0 < \alpha \leq 1\}$, for s =-, + and for each α ∈ (0, 1], $a_\alpha^- \leq a_\alpha^+, b_\alpha^- \leq b_\alpha^+$, then $\eta_1 \preccurlyeq \eta_2$.

From [4] we know that if η is a fuzzy real number with $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$ and η^* is the fuzzy number (fuzzy interval) generated by the family of nested bounded closed intervals $[\eta_\alpha^-, \eta_\alpha^+]$, α ∈ (0, 1], then $\eta = \eta^*$.

A definition of fuzzy norm on a linear space was introduced by Felbin [7]. Bag and Samanta [4], changed slightly this definition to define a fuzzy Felbin's norm on a linear space as is given below.

Definition 2.8. [4] Let X be a linear space over ℝ. Suppose $\|.\|: X \longrightarrow F^*$ is a mapping satisfying:

(i) $\|x\| = \tilde{0}$ iff $x = 0$,

(ii) $\|rx\| = \tilde{r} \odot \|x\|$, $x \in X, r \in \mathbb{R}$,

(iii) for all $x, y \in X$, $\|x + y\| \preccurlyeq \|x\| \oplus \|y\|$, and

(A') : $x \neq 0 \Rightarrow \|x\|(t) = 0$, for all $t \leq 0$.

Then (X, ‖ . ‖) is called a fuzzy normed linear space and ‖ . ‖ is called a fuzzy norm on X.

In the rest of this paper was use this definition of fuzzy norm. We note that $\|x\|_\alpha^s$, $s = -, +$ are crisp norms on X where $[\|x\|]_\alpha = [\|x\|_\alpha^-, \|x\|_\alpha^+], 0 < \alpha \leq 1$.

Definition 2.9. [18] Let (X, ‖ . ‖) be a fuzzy normed linear space.

(i) A sequence $\{x_n\} \subset X$ is said to converge to $x \in$ X ($\lim_{n\to\infty} x_n = x$), if $\lim_{n\to\infty} \| x_n - x \|_\alpha^- = \lim_{n\to\infty} \| x_n - x \|_\alpha^+ = 0$, $\forall \alpha \in (0,1]$.

(ii) A sequence $\{x_n\} \subset X$ is said Cauchy, if $\lim_{n,m\to\infty} \| x_n - x_m \|_\alpha^- = \lim_{n,m\to\infty} \| x_n - x_m \|_\alpha^+ = 0, \forall \alpha \in (0,1]$.

Definition 2.10. [18] Let (X, ‖ . ‖) be a fuzzy normed linear space. A subset Ω of X is said to be complete, if every Cauchy sequence in Ω converges in Ω. The fuzzy normed space (X, ‖ . ‖) is said to be a fuzzy Banach space if it is complete.

Definition 2.11.[1] Let (X, ‖ . ‖) be a fuzzy normed linear space and α ∈ (0, 1]. The space X is said to be α-complete if it is complete w.r.t. $\|.\|_\alpha^-$ where $[\| x \|]_\alpha = [\| x \|_\alpha^-, \| x \|_\alpha^+]$.

Remark 2.12. [1] It is easily followed that if X is α-complete then it is also complete w.r.t. $\|.\|_\alpha^+$.

Definition 2.13. Let (X, ‖ . ‖) be fuzzy normed linear space. A subset S of X is said to be fuzzy bounded if there is a fuzzy real number $\eta \succcurlyeq \tilde{0}$ such that for all $x \in S$,

$$\| x \| \preccurlyeq \eta.$$

Definition 2.14. [3] Let $T: X \longrightarrow X$ be a mapping where (X, ‖ . ‖) is a fuzzy normed linear space.Then T is said to be fuzzy nonexpansive if for all $x, y \in$ X, $\| Tx - Ty \| \preccurlyeq \| x - y \|$.

Proposition 2.15. [3] Let (X, ‖ . ‖) be a fuzzy normed linear space. A mapping $T: X \longrightarrow X$ is fuzzy nonexpansive iff T is nonexpansive w.r.t. $\| . \|_\alpha^-$ and $\| . \|_\alpha^+$, for all α ∈ (0, 1].

Definition 2.16. A set $C \subset X$ is said to be fuzzy convex if $\lambda C + (1 - \lambda)C \subset C$ for $0 \le \lambda \le 1$.

Notice that Definition 2.16 is subtly different from definition of fuzzy convex set in [14] and is extention of it.

Theorem 2.17. [11] Let E be a nonempty, weakly compact, convex subset of a Banach space and E has normal structure. Then every nonexpansive mapping $T: E \longrightarrow E$ has a fixed point.

Remark 2.18. [5] Let (X, ‖ . ‖) be a fuzzy normed linear space. Then (X, $\| . \|_\alpha^-$) and (X, $\| . \|_\alpha^+$) are normed linear spaces for each α ∈ (0, 1]. We denote by ($X_\alpha^{*-}, \| . \|_\alpha^{*-}$) and by ($X_\alpha^{*+}, \| . \|_\alpha^{*+}$) the first conjugate space of (X, $\| . \|_\alpha^-$) and (X, $\| . \|_\alpha^+$) respectively; also we denote by X_α^{**-} and X_α^{**+} the second conjugate spaces of (X, $\| . \|_\alpha^-$) and (X, $\| . \|_\alpha^+$) respectively.

Remark 2.19. [5] $\varphi_{x,\alpha}^+(f) = f(x)$, for all $f \in X_\alpha^{*-} \subset X_\alpha^{*+}$. For fixed $x \epsilon X$, we define for each $\alpha \in (0,1]$, a functional $\varphi_{x,\alpha}^-$ on X_α^{*-} by $\varphi_{x,\alpha}^-(f) = f(x)$, for all $f \in X_\alpha^{*-}$.

Remark 2.20. [5] For each $\alpha \in (0,1]$, we define a mapping $J_\alpha^- : X \longrightarrow X_\alpha^{**-}$ by $J_\alpha^-(x) = \varphi_{x,\alpha}^-$, for all $x \in X$ then J_α^- is isomorphically isometric between the space (X, $\| . \|_\alpha^-$) and the subspace $J_\alpha^-(X)$ of X_α^{**-} and $\| J_\alpha^- \|_\alpha^-$ =1, for all $\alpha \in (0,1]$.

Remark 2.21. [5] For each $\alpha \in (0,1]$, we define a mapping $J_\alpha^+ : X \longrightarrow X_\alpha^{**+}$ by $J_\alpha^+(x) = \varphi_{x,\alpha}^+$, for all $x \in X$ then J_α^+ is isomorphically isometric between the space (X, $\| . \|_\alpha^+$) and the subspace $J_\alpha^+(X)$ of X_α^{**+} and $\| J_\alpha^+ \|_\alpha^+$ =1, for all $\alpha \in (0,1]$.

Definition 2.22. [5] Let (X, ‖ . ‖) be a fuzzy normed linear space. ***X*** is said to be left fuzzy reflexive if there exists $\alpha_1 \in (0,1]$, such that J_α^- is onto, for all $\alpha \ge \alpha_1$, i.e. $J_\alpha^-(X) = X_\alpha^{**-}$ for all $\alpha \ge \alpha_1$.

Definition 2.23. [5] Let (X, ‖ . ‖) be a fuzzy normed linear space. X is said to be right fuzzy reflexive if there exists $\alpha_2 \in (0,1]$, such that J_α^+ is onto, for all $\alpha \le \alpha_2$, i.e. $J_\alpha^+(X) = X_\alpha^{**+}$ for all $\alpha \le \alpha_2$.

Definition 2.24. [5] Let (X, ‖ . ‖) be a fuzzy normed linear space. X is said to be fuzzy reflexive if it is both left and right fuzzy reflexive, i.e., there exists $\beta_1, \beta_2 \in (0,1]$ such that

$J_\alpha^-(X) = X_\alpha^{**-}$, for all $\alpha \ge \beta_1$,

and

$J_\alpha^+(X) = X_\alpha^{**+}$, for all $\alpha \le \beta_2$.

Since (X^{**}, ‖ . ‖) must always be a complete fuzzy normed linear space, no incomplete space can be fuzzy reflexive. However there exists fuzzy irreflexive complete fuzzy normed linear space.

3. EXTENTION OF FUZZY NORMAL STRUCTURE

Definition 3.1. Let (X, ‖ . ‖) be a fuzzy normed linear space. X is said to have fuzzy uniform normal structure if for nonempty convex closed and bounded subset C of X there exists $\tilde{N} < \tilde{1}$ (for N < 1) such that

$$\text{rad}(C) \oslash \text{diam}(C) \preccurlyeq \tilde{N}$$

i.e.,

$\{ \inf \{ \sup \| y-x \|, y \in C \}, x \in C \} \oslash \sup_{x,y \in C} \| y-x \| \preccurlyeq \tilde{N}$

thus since $[\tilde{N}]_\alpha = [N, N]$, for $\alpha \in (0,1]$, then

$$\frac{\{ \inf \{ \sup \| y-x \|_\alpha^-, y \in C \}, x \in C \}}{\sup_{x,y \in C} \| y-x \|_\alpha^-} \le N$$

and

$$\frac{\{ \inf \{ \sup \| y-x \|_\alpha^+, y \in C \}, x \in C \}}{\sup_{x,y \in C} \| y-x \|_\alpha^+} \le N.$$

Definition 3.2. Let (X, ‖ . ‖) be a fuzzy normed linear space. X is said to have fuzzy normal structure if there exists $\beta_1, \beta_2 \in (0,1]$ such that X has fuzzy uniform normal structure w.r.t. $\|\,.\,\|_\alpha^-$,for all $\alpha \geq \beta_1$ and w.r.t. $\|\,.\,\|_\alpha^+$ for all $\alpha \leq \beta_2$.

Now, we show that the spectrum of the category of fuzzy reflexive spaces is broader than the category of the spaces which have fuzzy normal structure.

Theorem 3.3. Let $(X, \|\,.\,\|)$ be α-complete fuzzy normed linear space for each $\alpha \in (0,1]$. If *X* has fuzzy normal structure, then it is fuzzy reflexive.

Proof. Since $(X, \|\,.\,\|)$ be α-complete for each $\alpha \in (0,1]$, then X is complete w.r.t. $\|\,.\,\|_\alpha^-$ and $\|\,.\,\|_\alpha^+$ for each $\alpha \in (0,1]$. Again X has fuzzy normal structure, thus by Definition 3.2, it follows that there exists $\beta_1, \beta_2 \in (0,1]$ such that X has fuzzy uniform normal structure w.r.t. $\|\,.\,\|_\alpha^-$ for all $\alpha \geq \beta_1$ and w.r.t. $\|\,.\,\|_\alpha^+$, for all $\alpha \leq \beta_2$. We know that if a Banach space has normal structure then it is reflexive. Thus *X* is reflexive w.r.t. $\|\,.\,\|_\alpha^-$ for all $\alpha \geq \beta_1$ and $\|\,.\,\|_\alpha^+$ for all $\alpha \leq \beta_2$. Hence X is fuzzy reflexive.

Definition 3.4. Let *M* be a nonempty closed fuzzy convex and fuzzy bounded subset of a fuzzy reflexive *X*. Suppose *S* be any closed and convex subset of *M* with more than one element. Let for $\alpha \in (0,1]$, diam (S)=[diam(S) $_\alpha^-$, diam(S) $_\alpha^+$]. For the set $M \subset X$, we say that it has the property *LFNST* if there is a mapping *T:M⟶ M*, so that for some $x_0 \in S$ there is $\alpha_1 \in (0,1]$, such that for $\alpha \geq \alpha_1$,

$$sup_{z\in S}\| x_0 - T^k z \|_\alpha^- < \text{diam}(S)_\alpha^-,$$

for some $k \in \mathbb{N}$, $T^k(S) \subset S$.

Definition 3.5. Let's assume that sets *M*, *S* satisfy conditions of the above definition. For the set $M \subset X$, we say that it has the property *RFNST* if there is a mapping *T:M ⟶ M*, so that for some $x_0 \in S$ there is $\alpha_2 \in (0,1]$, such that for $\alpha \leq \alpha_2$,

$$sup_{z\in S}\| x_0 - T^k z \|_\alpha^+ < \text{diam}(S)_\alpha^+,$$

for some $k \in \mathbb{N}$, $T^k(S) \subset S$.

Definition 3.6. Let's assume that sets *M*, *S* be as above. For the set $M \subset X$ we say that it has the property *FNST* if it is both *LFNST* and *RFNST*, i.e., there is a mapping *T:M⟶ M* so that for some $x_0 \in S$,

$$sup_{z\in S}\| x_0 - T^k z \| \prec \text{diam}(S),$$

for some $k \in \mathbb{N}$, $T^k(S) \subset S$.

Definition 3.7. Let's assume that sets *M*, *S* be as above. For the set $M \subset X$ we say that it has the property *LFNSTN* if there is a mapping *T:M ⟶ M*, so that for some $x_0 \in S$, there is $\alpha_1 \in (0,1]$, such that for $\alpha \geq \alpha_1$,

$$\sup_{n\in\mathbb{N}} \| x_0 - T^n x_0 \|_\alpha^- < \text{diam}(S)_\alpha^-,$$

and $Tx_0 \in S$.

Definition 3.8. Let's assume that sets *M*, *S* be as above. For the set $M \subset X$ we say that it has the property *RFNSTN* if there is a mapping *T:M ⟶ M*, so that for some $x_0 \in S$, there is $\alpha_2 \in (0,1]$, such that for $\alpha \leq \alpha_2$,

$$\sup_{n\in\mathbb{N}} \| x_0 - T^n x_0 \|_\alpha^+ < \text{diam}(S)_\alpha^+,$$

and $Tx_0 \in S$.

Definition 3.9. Let's assume that sets *M*, *S* be as above. For the set $M \subset X$ we say that it has the property *FNSTN* if it is both *LFNSTN* and *RFNSTN*, i.e., if there is a mapping *T:M ⟶ M*, so that for some $x_0 \in S$,

$$\sup_{n\in\mathbb{N}} \| x_0 - T^n x_0 \| \prec \text{diam}(S),$$

and $Tx_0 \in S$.

4. THE NEW RESULTS

Definition 4.1. Let $(X, \|\,.\,\|)$ be a fuzzy normed linear space. The fuzzy bounded set $K \subset X$ is said to be weakly compact if every open cover of *K* has a finite subcover.

Let *E* be a reflexive fuzzy space, i.e., right and left fuzzy reflexive,. We know on reflexive fuzzy spaces, a closed, convex and fuzzy bounded set is weakly compact.

Let us prove the existence of the fixed point for *T* that is defined on the sets from fuzzy reflexive space *X*, with the condition *FNST* or *FNSTN*.

Theorem 4.2. Let *K* be nonempty closed fuzzy convex and fuzzy bounded subset of the fuzzy reflexive space *X* and let *K* have the property *FNST*, where *T* is one of the mappings which is defined in the property *FNST*. If for every closed and convex subset $E \subset K$ it satisfy that $T(E) \subset E$, $k \in \mathbb{N}$ and for set *K* which has the property *FNST* is valid that

$$\| Tx - Ty \| \preccurlyeq sup_{z\in E}\| x - T^k z \|, \quad (1)$$

then the mapping *T* has a fixed point.

Proof. Let *G* be the set of all nonempty closed and convex subsets *E* of set *K* for which $T(E) \subset E$. The set *G* is nonempty, because $K \in G$. Set inclusion defines a relation of ordering on G. From the fuzzy norming and mapping of the space *X*, it follows that the space *X* is complete, so every chain in *G*, which is consists of nonempty closed fuzzy convex and fuzzy bounded set of *G*, by reflexivity of *X*, has nonempty intersection. By Zorn's Lemma there is the minimal element *S* of the set *G*.

If *S* consists only of one element, on the basis of supposition that $T(S) \subset S$, this element is also the fixed point of the mapping *T*.

If *S* has more than one point with the property *FNST*, it satisfy that

$$sup_{z\in S}\| x_0 - T^k z \| \prec diam\,(S),$$

for certain $k \in \mathbb{N}$ and $x_0 \in S$.

If in the inequality (1) we put that $x = x_0$, we have that

$$\| \ Tx_0 - Ty \ \| \leqslant sup_{z\in S} \| \ x_0 - T^k \ z \ \|,$$

so that all Ty, $y \in S$ are in the ball with the center in Tx_0 and radius $sup_{z\in S} \| \ x_0 - T^k \ z \ \| = r$, i.e., $T(S) \subset B(Tx_0, r)$ and that is also $T^k(S) \subset B(Tx_0, r)$.

Since $T(S) \subset S$ it implies that $T^k(S) \subset S$ so that $T^k(S) \subset B(Tx_0, r) \cap S$ and on the basis of minimality of the set S, it satisfy $B(Tx_0, r) \cap S = S$, so that $S \subset B(Tx_0, r)$. From the relation $S \subset B(Tx_0, r)$ it implies that

$$\| \ Tx_0 - y \ \| \leqslant sup_{z\in S} \| \ x_0 - T^k \ z \ \|, \qquad (2)$$

for all $y \in S$.

Let us form the set

$$S' = \{ v \in S : sup_{z\in S} \| \ v - z \ \| \leqslant sup_{z\in S} \| \ x_0 - T^k \ z \ \| \}.$$

On the basis of definition of the set S' and the relation (2) we conclude that the set S' is fuzzy bounded and closed, with regard to $Tx_0 \in S'$, then S' also nonempty set.

Let us prove that for all $v \in S'$, it satisfy that $Tv \in S'$. Since S is a nonempty, limited, closed and convex set with more than one element and it is a minimal element of the family G and is valid that $TS \subset S$, then is

$$S = \overline{CoTS}.$$

If $z \in S$, then z can be calculated as convex combination of the elements from TS, i.e.

$$z = \sum_{i=1}^{n} \alpha_i \ Tz_i, \ \sum_{i=1}^{n} \alpha_i = 1, \alpha_i \geq 0, z_i \in S.$$

Now

$$\| Tv - z \|_{\alpha}^{-} = \| Tv - \sum_{i=1}^{n} \alpha_i Tz_i \|_{\alpha}^{-} \leq \sum_{i=1}^{n} \alpha_i \ \| T v - Tz_i \|_{\alpha}^{-}$$

$$\leq \sum_{i=1}^{n} \alpha_i \, sup_{z\in S} \| \ v - T^k z \|_{\alpha}^{-}$$

$$\leq \sum_{i=1}^{n} \alpha_i \, sup_{z\in S} \| \ v - z \|_{\alpha}^{-}$$

$$\leq sup_{z\in S} \| \ x_0 - T^k z \ \|_{\alpha}^{-} \sum_{i=1}^{n} \alpha_i$$

$$= sup_{z\in S} \| \ x_0 - T^k z \|_{\alpha}^{-},$$

and

$$\| Tv - z \|_{\alpha}^{+} = \| Tv - \sum_{i=1}^{n} \alpha_i Tz_i \|_{\alpha}^{+} \leq \sum_{i=1}^{n} \alpha_i \ \| T v - Tz_i \|_{\alpha}^{+}$$

$$\leq \sum_{i=1}^{n} \alpha_i \, sup_{z\in S} \| \ v - T^k z \|_{\alpha}^{+}$$

$$\leq \sum_{i=1}^{n} \alpha_i \, sup_{z\in S} \| \ v - z \|_{\alpha}^{+}$$

$$\leq sup_{z\in S} \| \ x_0 - T^k z \ \|_{\alpha}^{+} \sum_{i=1}^{n} \alpha_i$$

$$= sup_{z\in S} \| \ x_0 - T^k z \|_{\alpha}^{+},$$

i.e., we have

$$\| \ Tv - z \ \| \leqslant sup_{z\in S} \| \ x_0 - T^k z \ \|,$$

so that

$$TS' \subset S'.$$

Let us give the sequence $\{ \alpha_n \} \subset S'$, for all $n \in \mathbb{N}$ and let $\alpha_n \longrightarrow \alpha \in S$ when $n \longrightarrow \infty$.

Now

$$sup_{z\in S} \| \ \alpha - z \ \| \leqslant sup_{z\in S} (\| \ \alpha - \alpha_n \ \| + \| \ \alpha_n - z \ \|)$$

$$= \| \ \alpha - \alpha_n \ \| + sup_{z\in S} \| \ \alpha_n - z \ \|$$

$$\leqslant \| \ \alpha - \alpha_n \ \| + r.$$

When $n \longrightarrow \infty$ we get that

$$sup_{z\in S} \| \ \alpha - z \ \| \leqslant r,$$

then the set S' is closed.

Let v and u be two points from S'. For $\lambda \in [0,1]$ we have that

$$\| \ \lambda v + (1-\lambda)u - z \ \|_{\alpha}^{-} = \| \ \lambda v + (1-\lambda)u - \lambda z + \lambda z - z \ \|_{\alpha}^{-}$$

$$\leq \lambda \, sup_{z\in S} \| \ x_0 - T^k z \ \|_{\alpha}^{-} + (1-\lambda) sup_{z\in S} \| \ x_0 - T^k z \ \|_{\alpha}^{-}$$

$$= r,$$

and

$$\| \ \lambda v + (1-\lambda)u - z \ \|_{\alpha}^{+} = \| \ \lambda v + (1-\lambda)u - \lambda z + \lambda z - z \ \|_{\alpha}^{+}$$

$$\leq \lambda \, sup_{z\in S} \| \ x_0 - T^k z \ \|_{\alpha}^{+} + (1-\lambda) sup_{z\in S} \| \ x_0 - T^k z \ \|_{\alpha}^{+}$$

$$= r,$$

i.e., we have $\| \ \lambda v + (1-\lambda)u - z \ \| \leqslant r$, so the set S' is convex.

Let for $\alpha \in (0,1]$, $[diam\,(S')]_{\alpha} = [\,diam\,(S')_{\alpha}^{-}, diam\,(S')_{\alpha}^{+}]$ and similarly for $diam\,(S)$.

For v, $w \in S'$ we have that

$$diam\,(S') = sup_{v,w\in S'} \| \ v - w \ \| \leqslant sup_{v\in S', z\in S} \| \ v - z \ \|$$

$$\leqslant sup_{z\in S} \| \ x_0 - T^k z \ \| \, diam\,(S).$$

Now S' is a nonempty closed convex fuzzy bounded subset of K for which $T(S') \subset S'$, S' belongs to the family G and it satisfy that $S' \subset S$ and $S' \neq S$, which this contradicts the minimality of the set S, so the set S has only one point and it is fixed point of mapping T. By this the proof of Theorem 4.2 is completed. □

Theorem 4.3. Let us introduce the mapping $T: K \longrightarrow K$ where K is a nonempty closed convex fuzzy bounded subset of the fuzzy reflexive space X and let K have the property *FNTN*, where *T* is one of the mappings that is defined in the property *FNSTN*.

If for any closed and convex subset $E \subset K$ with more than one element it satisfy that $T(E) \subset E$ and

$$\| Tx - Ty \| \preccurlyeq sup_{k \in \mathbb{N}} \| x - T^k x \| , \quad (3)$$

for all $x, y \in E$, then the mapping *T* has the fixed point.

Proof. In the same way as in Theorem 4.2, we come to the set *S*. If the set *S* has one element, with regard to $TS \subset S$, it is also the fixed point of mapping *T*.

Let us presume that the set *S* has more than one element. On the basis of the property *FNSTN* and the condition (3) for $x = x_0$, we get that

$$\| Tx_0 - Ty \| \preccurlyeq sup_{k \in \mathbb{N}} \| x_0 - T^k x_0 \| ,$$

for all $y \in S$.

By the similar reasoning, as in the Theorem 4.2, we come to the relation

$$\| Tx_0 - y \| \preccurlyeq sup_{k \in \mathbb{N}} \| x_0 - T^k x_0 \| ,$$

for all $y \in S$.

Let us form the set

$$S'' = \left\{ v \in S : sup_{z \in S} \| v - z \| \preccurlyeq sup_{k \in \mathbb{N}} \| x_0 - T^k z \| \right\}.$$

The set S'' is nonempty. If $v \in S''$ let us prove that $Tv \in S''$. Since the set *S* is a convex combination of elements from *T(S)* for every $z_i \in S$ and $k \in \mathbb{N}$ the inequalities are valid. Since

$$\| Tv - z \|_\alpha^- = \| Tv - \sum_{i=1}^n \alpha_i \, Tz_i \|_\alpha^-$$
$$\leq \sum_{i=1}^n \alpha_i \| Tv - Tz_i \|_\alpha^-$$
$$\leq \sum_{i=1}^n \alpha_i \, sup_{k \in \mathbb{N}} \| v - T^k v \|_\alpha^- \leq sup_{z \in S} \| v - z \|_\alpha^-$$
$$\leq sup_{k \in \mathbb{N}} \| x_0 - T^k x_0 \|_\alpha^-,$$

and

$$\| Tv - z \|_\alpha^+ = \| Tv - \sum_{i=1}^n \alpha_i \, Tz_i \|_\alpha^+$$
$$\leq \sum_{i=1}^n \alpha_i \| Tv - Tz_i \|_\alpha^+$$
$$\leq \sum_{i=1}^n \alpha_i \, sup_{k \in \mathbb{N}} \| v - T^k v \|_\alpha^+ \leq sup_{z \in S} \| v - z \|_\alpha^+$$
$$\leq sup_{k \in \mathbb{N}} \| x_0 - T^k x_0 \|_\alpha^+,$$

i.e., we have

$$\| Tv - z \| \preccurlyeq sup_{k \in \mathbb{N}} \| x_0 - T^k x_0 \|,$$

thus $T(S'') \subset S''$.

It is simple to prove that the set S'' is closed and convex. On the basis of the definition of the set S'' for all $v, w \in S''$, we have that

$$diam\,(S'') = sup_{v,w \in S''} \| v - w \| \preccurlyeq sup_{v \in S'', z \in S} \| v - z \|$$
$$\preccurlyeq sup_{k \in \mathbb{N}} \| x_0 - T^k z \| \prec diam\,(S).$$

Now S'' is a nonempty closed convex fuzzy bounded subset of *S* and it satisfy that $S'' \neq S$, which is impossible because of the minimality of the set *S*. This completes the proof of Theorem 4.3. □

5. CONCLUSION

One of the most important topics of fixed point theory is fixed point results in fuzzy analysis, because of that, we see in [15] the spectrum of the class of fuzzy normed spaces is broader than the class of normed spaces. By means of introduction of the notions of *FNST, FNSTN* we prove the theorems of the fixed points of some classes of mappings on the sets from the fuzzy reflexive space, which have some of the properties *FNST, FNSTN.*

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES

[1] Bag, T., “Some fundamental theorems in Felbin’s type fuzzy normed linear spaces”, International Journal of Mathematics and Scientific Computing, 1 (2): 44-49, (2011).

[2] Bag, T. and Samanta, S. K., “Fixed point theorems in Felbin’s type fuzzy normed linear spaces” , J. Fuzzy Math. , 16 (1): 243-260, (2008).

[3] Bag, T. and Samanta, S. K., “Fixed point theorems in fuzzy normed linear spaces”, Information Sciences, 176: 2910-2931, (2006).

[4] Bag, T. and Samanta, S. K., “fuzzy bounded linear operators in Felbin’s type fuzzy normed linear spaces”, Fuzzy Sets and Systems 159: 685-707, (2008).

[5] Bag, T. and Samanta, S K., “Fuzzy reflexivity of Felbin’s type fuzzy normed linear spaces and fixed point theorems in such spaces” , Iranian Journal of Fuzzy Systems, 8 (5): 103-115, (2011).

[6] Baillon, J. B. and Schoneberg, R.,“ Asymptotic normal structure and fixed points of nonexpansive mappings”, Proc. Am. Math. Soc., 81: 257-264, (1981).

[7] Felbin, C.,"Finite dimensional fuzzy normed linear space", Fuzzy Sets and Systems, 48: 239-248, (1992).

[8] Hasankhani, A., Nazari, A. and Saheli, M.," Some properties of fuzzy Hilbert spaces and norm of operators", Iranian Journal of Fuzzy Systems, 7 (3): 129-157, (2010).

[9] Kaleva, O. and Seikkala, S., "On fuzzy metric spaces", Fuzzy Sets and Systems, 12: 215-229, (1984).

[10] Kiany, F. and Amini-Harandi, A., " Fixed point and endpoint theorems for set-valued fuzzy contraction maps in fuzzy metric spaces", Fixed Point Theory and Applications, 2011 (94): 1-10, (2011).

[11] Kirk, W. A., "A fixed point theorem for mappings which do not increase distance", Amer. Math. Monthly, 72: 1004-1006, (1965).

[12] Mijajlovic, B., "Fixed points of mapping on the normed and reflexive spaces", Kragujevac J. Math. 29: 113-120 .(2006) ,

[13] Mizumoto, M. and Tanaka, J., "Some properties of fuzzy numbers in: M. M. Gupta et al. Editors", Advances in Fuzzy Set Theory and Applications, North-Holland, New-York: 153-164, (1979).

[14] Sadeqi, I. and Solaty kia, F., "Some fixed point theorems in fuzzy reflexive Banach spaces", Chaos, Solitons and Fractals, 41: 2606-2612 (2009).

[15] Sadeqi, I., Moradlou, F. and Salehi, M., "On approximate cauchy equation in Felbin's type fuzzy normed linear spaces", Iranian Journal of Fuzzy Systems, 10 (3): 51-63, (2013).

[16] Smulian, V.," On the principle of inclusion in the space of the type (B)", Math. Sb. (N. S.), 5: 237-238, (1939).

[17] Vajzovic, F. and Stefanovic, M., "Two fixed point theorems", Radovi Akad. Nauka Bosnei Herceg., 24: 63-71, (1985).

[18] Xiao, J. and Zhu, X., "On linearly topological structure and property of fuzzy normed linear space", Fuzzy Sets and Systems, 125: 153-161, (2002).