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Research Article

J-hyperideals and their expansions in a Krasner (m,n)-hyperring

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Abstract

Over the years, different types of hyperideals have been introduced in order to let us fully realize the structures of hyperrings in general. The aim of this research work is to define and characterize a new class of hyperideals in a Krasner (m,n)-hyperring that we call n-ary J-hyperideals. A proper hyperideal Q of a Krasner (m,n)-hyperring with the scalar identity 1_R is said to be an n-ary J-hyperideal if whenever $x_1^n \in R$ such that $g(x_1^n) \in Q$ and $x_i \notin J_{(m,n)}(R)$, then $g(x_1^{i-1}, 1_R, x_{i+1}^n) \in Q$. Also, we study the concept of n-ary δ-J-hyperideals as an expansion of n-ary J-hyperideals. Finally, we extend the notion of n-ary δ-J-hyperideals to (k,n)-absorbing δ-J-hyperideals. Let δ be a hyperideal expansion of a Krasner (m,n)-hyperring R and k be a positive integer. A proper hyperideal Q of R is called (k,n)-absorbing δ-J-hyperideal if for $x_1^{kn-k+1} \in R$, $g(x_1^{kn-k+1}) \in Q$ implies that $g(x_1^{(k-1)n-k+2}) \in J_{(m,n)}(R)$ or a g-product of $(k-1)n-k+2$ of $x_i^,$ s except $g(x_1^{(k-1)n-k+2})$ is in $\delta(Q)$.

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Keywords. n-ary J-hyperideal, n-ary δ-J-hyperideal, (k,n)-absorbing δ-J-hyperideal

1. Introduction

Hyperstructures represent a natural extension of classical algebraic structures and they were introduced by Marty. In 1934, he [21] defined the concept of a hypergroup as a generalization of groups during the 8^{th} Congress of the Scandinavian Mathematicians. Many papers and books have been written concerning hyperstructure theory. Some review of the theory of hyperstructures can be found in [7–9, 25, 28, 29]). The simplest algebraic hyperstructures which possess the properties of closure and associativity are said to be semihypergroups. n-ary semigroups and n-ary groups are algebras with one n-ary operation which is associative and invertible in a generalized sense. The notion of n-ary algebras goes back to Kasners lecture [15] at a scientific meeting in 1904. In 1928, Dorente wrote the first paper concerning the theory of n-ary groups [12]. Later on, Crombez and Timm [5, 6] defined the notion of the (m,n)-rings and their quotient structures. Mirvakili and Davvaz [20] defined (m,n)-hyperrings and obtained several results in this respect. In [10], they introduced a generalization of the notion of a hypergroup in the sense of Marty and a generalization of an n-ary group, which is called n-ary hypergroup. The n-ary structures

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have been studied in [2, 17–20, 26]. Mirvakili and Davvaz [23] defined (m, n)-hyperrings and obtained several results in this respect.

One important class of hyperrings was introduced by Krasner, where the addition is a hyperoperation, while the multiplication is an ordinary binary operation, which is called Krasner hyperring. In [22], a generalization of the Krasner hyperrings, which is a subclass of (m, n)-hyperrings, was defined by Mirvakili and Davvaz. It is called Krasner (m, n)-hyperring. Ameri and Norouzi in [1] introduced some important hyperideals such as Jacobson radical, n-ary prime and primary hyperideals, nilradical, and n-ary multiplicative subsets of Krasner (m, n)-hyperrings. Afterward, the notions of (k, n)-absorbing hyperideals and (k, n)-absorbing primary hyperideals were studied by Hila et. al. [14].

Norouzi et. al. proposed and analysed a new defnition for normal hyperideals in Krasner (m, n)-hyperrings, with respect to that one given in [22] and they showed that these hyperideals correspond to strongly regular relations [24]. In [26], Ostadhadi-Dehkordi and Davvaz dened the fundamental relation $\eta*$ on R as the smallest equivalence relation on R such that the quotient $[R : \eta*]$ is an (m, n)-ring. Asadi and Ameri introduced and studied direct limit of a direct system in the category of Krasner (m, n)-hyperrigs [4].

Dongsheng defined the notion of δ-primary ideals in a commutative ring where δ is a function that assigns to each ideal I an ideal $\delta(I)$ of the same ring [11]. Moreover, in [13] he and his colleague investigated 2-absorbing δ-primary ideals which unify 2-absorbing ideals and 2-absorbing primary ideals. Ozel Ay et al. generalized the notion of δ-primary on Krasner hyperrings [27]. The concept of δ-primary hyperideals in Krasner (m, n)-hyperrings, which unifies the prime and primary hyperideals under one frame, was defined in [3]. The notion of J-ideals as a generalization of n-ideals in ordinary rings was studied by Khashan and Bani-ata in [16].

Now in this paper, first we define the notion of n-ary J-hyperideals in a Krasner (m, n)-hyperring which is a generalization of J-ideals. We give several characterizations of n-ary J-hyperideals. Afterward, we study the concept of n-ary δ-J-hyperideals as an expansion of n-ary J-hyperideals. Several properties of them are provided. Moreover, we extend the notion of n-ary δ-J-hyperideals to (k, n)-absorbing δ-J-hyperideals.

2. Preliminaries

In this section we recall some definitions and results concerning n-ary hyperstructures which we need to develop our paper.
Let H be a nonempty set. Then the mapping $f : H^n \longrightarrow P^*(H)$, where $P^*(H)$ is the set of all the nonempty subsets of H, is called an n-ary hyperoperation and the algebraic system (H, f) is called an n-ary hypergroupoid. Suppose that $H_1, ..., H_n$ are non-empty subsets of H. We define $f(H_1^n) = f(H_1, ..., H_n) = \bigcup\{f(x_1^n) \mid x_i \in H_i, i = 1, ..., n\}$. The sequence $x_i, x_{i+1}, ..., x_j$ will be denoted by x_i^j and it is the empty symbol when $j < i$. Using this notation, $f(x_1, ..., x_i, y_{i+1}, ..., y_j, z_{j+1}, ..., z_n)$ will be written as $f(x_1^i, y_{i+1}^j, z_{j+1}^n)$. The expression will be written in the form $f(x_1^i, y^{(j-i)}, z_{j+1}^n)$ if $y_{i+1} = ... = y_j = y$. Assume that for all $1 \leq i < j \leq n$ and every $x_1, x_2, ..., x_{2n-1} \in H$,

$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$.

Then the n-ary hyperoperation f is associative. An n-ary hypergroupoid with the associative n-ary hyperoperation is said to be an n-ary semihypergroup.

An n-ary hypergroupoid (H, f) in which the equation $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$ has a solution $x_i \in H$ for every $a_1^{i-1}, a_{i+1}^n, b \in H$ and $i \in \{1, 2, \ldots, n\}$, is called an n-ary quasihypergroup, when (H, f) is an n-ary semihypergroup, (H, f) refers to an n-ary hypergroup.

If for all $\sigma \in \mathbb{S}_n$, the group of all permutations of $\{1, 2, 3, ..., n\}$, and for all $a_1^n \in H$ we have $f(a_1, ..., a_n) = f(a_{\sigma(1)}, ..., a_{\sigma(n)})$, then an n-ary hypergroupoid (H, f) is commutative. If $a_1^n \in H$, then the $(a_{\sigma(1)}, ..., a_{\sigma(n)})$ is denoted by $a_{\sigma(1)}^{\sigma(n)}$. t-ary hyperoperation

$f_{(l)}$ is given by $f_{(l)}(x_1^{l(n-1)+1}) = f(f(..., f(f(x_1^n), x_{n+1}^{2n-1}), ...), x_{(l-1)(n-1)+1}^{l(n-1)+1})$ if f is an n-ary hyperoperation and $t = l(n-1)+1$.

Definition 2.1. [22] Let (H, f) be an n-ary hypergroup and A be a non-empty subset of H. A is called an n-ary subhypergroup of (H, f), if $f(x_1^n) \subseteq A$ for $x_1^n \in A$, and the equation $b \in f(b_1^{i-1}, x_i, b_{i+1}^n)$ has a solution $x_i \in A$ for every $b_1^{i-1}, b_{i+1}^n, b \in A$ and $1 \leq i \leq n$. An element e in H is called a scalar neutral element if $x = f(e^{(i-1)}, x, e^{(n-i)})$, for every $1 \leq i \leq n$ and for every $x \in H$.

An element 0 of an n-ary semihypergroup (H, g) is called a zero element if for every $x_2^n \in H$ we have $g(0, x_2^n) = g(x_2, 0, x_3^n) = ... = g(x_2^n, 0) = 0$. If 0 and $0'$are two zero elements, then $0 = g(0', 0^{(n-1)}) = 0'$ and so the zero element is unique.

Definition 2.2. [17] An n-ary hypergroup (H, f) is called canonical if
(1) there exists a unique $e \in H$, such that for every $x \in H, f(x, e^{(n-1)}) = x$;
(2) for every $x \in H$ there exists a unique $x^{-1} \in H$, such that $e \in f(x, x^{-1}, e^{(n-2)})$;
(3) if $x \in f(x_1^n)$, then for all i, we have $x_i \in f(x, x^{-1}, ..., x_{i-1}^{-1}, x_{i+1}^{-1}, ..., x_n^{-1})$.

We say that e is the scalar identity of (H, f) and x^{-1} is the inverse of x. Notice that the inverse of e is e.

Definition 2.3. [22] A Krasner (m, n)-hyperring is an algebraic hyperstructure (R, f, g), or simply R, which satisfies the following conditions:
(1) (R, f) is a canonical m-ary hypergroup;
(2) (R, g) is a n-ary semigroup;
(3) the n-ary operation g is distributive with respect to the m-ary hyperoperation f , i.e., for all $a_1^{i-1}, a_{i+1}^n, x_1^m \in R$, and $1 \leq i \leq n$,
$g(a_1^{i-1}, f(x_1^m), a_{i+1}^n) = f(g(a_1^{i-1}, x_1, a_{i+1}^n), ..., g(a_1^{i-1}, x_m, a_{i+1}^n))$;
(4) 0 is a zero element (absorbing element) of the n-ary operation g, i.e., for every $x_2^n \in R$ we have
$g(0, x_2^n) = g(x_2, 0, x_3^n) = ... = g(x_2^n, 0) = 0$.

We assume throughout this paper that all Krasner (m, n)-hyperrings are commutative.

Let S is a non-empty subset of R. We say that S is a subhyperring of R if (S, f, g) is a Krasner (m, n)-hyperring. Let I be a non-empty subset of R. Then I is called a hyperideal of (R, f, g) if (I, f) is an m-ary subhypergroup of (R, f) and $g(x_1^{i-1}, I, x_{i+1}^n) \subseteq I$, for every $x_1^n \in R$ and $1 \leq i \leq n$.

Definition 2.4. [1] Let a is an element in a Krasner (m, n)-hyperring R. Then the hyperideal generated by a is denoted by $< a >$ and defined as follows:
$< a >= g(R, a, 1^{(n-2)}) = \{g(r, a, 1^{(n-2)}) \mid r \in R\}$

Definition 2.5. [1] A hyperideal M of a Krasner (m, n)-hyperring R is said to be maximal if for every hyperideal N of R, $M \subseteq N \subseteq R$ implies that $N = M$ or $N = R$.

The intersection of all maximal hyperideals of R is called the Jacobson radical of a Krasner (m, n)-hyperring R and it is denoted by $J_{(m,n)}(R)$. If R does not have any maximal hyperideal, we define $J_{(m,n)}(R) = R$.

Definition 2.6. [1] An element $x \in R$ is said to be invertible if there exists $y \in R$ with $1_R = g(x, y, 1_R^{(n-2)})$. Moreover, the subset U of R is invertible if and only if every element of U is invertible.

Definition 2.7. [1] A hyperideal $P \neq R$ of a Krasner (m, n)-hyperring R refers to a prime hyperideal if for hyperideals $P_1, ..., P_n$ of R, $g(P_1^n) \subseteq P$ implies that $P_i \subseteq P$ for some $1 \leq i \leq n$.

Lemma 2.8. *It was shown (Lemma 4.5 in [1]) that the hyperideal $P \neq R$ of a Krasner (m,n)-hyperring R is a prime hyperideal if for all $a_1^n \in R$, $g(a_1^n) \in P$ implies that $a_i \in P$ for some $1 \leq i \leq n$.*

Definition 2.9. [1] Let I be a hyperideal in a Krasner (m,n)-hyperring R with scalar identity. The radical (or nilradical) of I, denoted by $\sqrt{I}^{(m,n)}$ is the hyperideal $\bigcap P$, where the intersection is taken over all prime hyperideals P which contain I. If the set of all prime hyperideals containing I is empty, then $\sqrt{I}^{(m,n)}$ is defined to be R.

It was shown that if $a \in \sqrt{I}^{(m,n)}$ then there exists $t \in \mathbb{N}$ such that $g(a^{(t)}, 1_R^{(n-t)}) \in I$ for $t \leq n$, or $g_{(l)}(a^{(t)}) \in I$ for $t = l(n-1)+1$ [1].

Definition 2.10. [1] Let I be a proper hyperideal in a Krasner (m,n)-hyperring R with the scalar identity 1_R. Then I is called primary if $g(a_1^n) \in I$ and $a_i \notin I$ implies that $g(a_1^{i-1}, 1_R, x_{i+1}^n) \in \sqrt{I}^{(m,n)}$ for some $1 \leq i \leq n$.

If I is a primary hyperideal in a Krasner (m,n)-hyperring R with the scalar identity 1_R, then $\sqrt{I}^{(m,n)}$ is prime. (Theorem 4.28 in [1])

Definition 2.11. [1] Let S be a hyperideal of a Krasner (m,n)-hyperring (R,f,g). Then the set

$R/S = \{f(x_1^{i-1}, S, x_{i+1}^m) \mid x_1^{i-1}, x_{i+1}^m \in R\}$

endowed with m-ary hyperoperation f which for all $x_{11}^{1m}, ..., x_{m1}^{mm} \in R$

$f(f(x_{11}^{1(i-1)}, S, x_{1(i+1)}^{1m}), ..., f(x_{m1}^{m(i-1)}, S, x_{m(i+1)}^{mm}))$

$= f(f(x_{11}^{m1}), ..., f(x_{1(i-1)}^{m(i-1)}), S, f(x_{1(i+1)}^{m(i+1)}), ..., f(x_{1m}^{mm}))$

and with n-ary hyperoperation g which for all $x_{11}^{1m}, ..., x_{n1}^{nm} \in R$

$g(f(x_{11}^{1(i-1)}, S, x_{1(i+1)}^{1m}), ..., f(x_{n1}^{n(i-1)}, S, x_{n(i+1)}^{nm}))$

$= f(g(x_{11}^{n1}), ..., g(x_{1(i-1)}^{n(i-1)}), S, g(x_{1(i+1)}^{n(i+1)}), ..., f(x_{1m}^{nm}))$

construct a Krasner (m,n)-hyperring, and $(R/S, f, g)$ is called the quotient Krasner (m,n)-hyperring of R by S.

Definition 2.12. [22] Let (R_1, f_1, g_1) and (R_2, f_2, g_2) be two Krasner (m,n)-hyperrings. A mapping $h : R_1 \longrightarrow R_2$ is called a homomorphism if for all $a_1^m \in R_1$ and $b_1^n \in R_1$ we have

$h(f_1(a_1, ..., a_m)) = f_2(h(a_1), ..., h(a_m))$

$h(g_1(b_1, ..., b_n)) = g_2(h(b_1), ..., h(b_n))$.

Definition 2.13. [3] Let R be a Krasner (m,n)-hyperring. A function δ is called a hyperideal expansion of R if it assigns to each hyperideal I of R a hyperideal $\delta(I)$ of R with the following conditions:

(i) $I \subseteq \delta(I)$.

(ii) if $I \subseteq K$ for any hyperideals I, K of R, then $\delta(I) \subseteq \delta(K)$.

Example 2.14. Let R be a Krasner (m,n)-hyperring.

1. Define $\delta_0(I) = I$, for each hyperideal I of R. Then δ_0 is a hyperideal expansion of R.
2. Define $\delta_1(I) = \sqrt{I}^{(m,n)}$, for each hyperideal I of R. Then δ_1 is a hyperideal expansion of R.
3. Define $\delta_R(I) = R$, for each hyperideal I of R. Then δ_R is a hyperideal expansion of R.
4. Define $\delta_q(I/J) = \delta(I)/J$, for each hyperideal I of R containing hyperideal J and expansion function δ of R. Then δ_q is a hyperideal expansion of R/J.

Definition 2.15. [3] Let (R_1, f_1, g_1) and (R_2, f_2, g_2) be two Krasner (m,n)-hyperrings and $h : R_1 \longrightarrow R_2$ a hyperring homomorphism. Let δ and γ be hyperideal expansions of R_1

and R_2, respectively. Then h is said to be a $\delta\gamma$-homomorphism if $\delta(h^{-1}(I_2)) = h^{-1}(\gamma(I_2))$ for the hyperideal I_2 of R_2.

Note that $\gamma(h(I_1) = h(\delta(I_1)$ for $\delta\gamma$-epimorphism h and for hyperideal I_1 of R_1 with $Ker(h) \subseteq I_1$. For example, let (R_1, f_1, g_1) and (R_2, f_2, g_2) be two Krasner (m,n)-hyperrings. If δ_1 of R_1 and γ_1 of R_2 be the hyperideal expansions defined in Example 3.2, in ([3]), then each homomorphism $h : R_1 \longrightarrow R_2$ is a $\delta_1\gamma_1$-homomorphism.

3. n-ary J-hyperideals

Our aim in this section is to study the n-ary J-hyperideals in Krasner (m,n)-hyperrings. We begin with the following definition.

Definition 3.1. A proper hyperideal Q of a Krasner (m,n)-hyperring with the scalar identity 1_R is said to be n-ary J-hyperideal if whenever $x_1^n \in R$ with $g(x_1^n) \in Q$ and $x_i \notin J_{(m,n)}(R)$ implies that $g(x_1^{i-1}, 1_R, x_{i+1}^n) \in Q$.

Example 3.2. The set $A = \{0, 1, x\}$ with the following 3-ary hyperoeration f and 3-ary operation g is a Krasner $(3,3)$-hyperring such that f and g are commutative.

$$f(0,0,0) = 0, \quad f(0,0,1) = 1, \quad f(0,1,1) = 1, \quad f(1,1,1) = 1, \quad f(1,1,x) = A$$

$$f(0,1,x) = A, \quad f(0,0,x) = x, \quad f(0,x,x) = x, \quad f(1,x,x) = A, \quad f(x,x,x) = x$$

$$g(1,1,1) = 1, \quad g(1,1,x) = g(1,x,x) = g(x,x,x) = x$$

and for $x_1, x_2 \in R, g(0, x_1, x_2) = 0$. In the Krasner $(3,3)$-hyperring, hyperideals $\{0\}$ and $\{0, x\}$ are two n-ary J-hyperideals of A.

Example 3.3. The set $R = \{0, 1, \alpha, \beta\}$ with following 2-hyperoperation "$\oplus$" is a canonical 2-ary hypergroup.

$\oplus$	0	1	α	β
0	0	1	α	β
1	1	A	β	B
α	α	β	0	1
β	β	B	1	A

In which $A = \{0, 1\}$ and $B = \{\alpha, \beta\}$. Define a 4-ary operation g on R as follows:

$$g(a_1^n) = \begin{cases} \alpha & \text{if } a_1, a_2, a_3, a_4 \in B \\ 0 & \text{otherwise} \end{cases}$$

It follows that $(R, \oplus, g)$ is a Krasner (2,4)-hyperring. In the hyperring, $\{0\}$ is a 4-ary J-hyperideal.

Theorem 3.4. *Let Q be an n-ary J-hyperideal of a Krasner (m,n)-hyperring R. Then $Q \subseteq J_{(m,n)}(R)$.*

Proof. Let Q be an n-ary J-hyperideal of a Krasner (m,n)-hyperring R such that $Q \nsubseteq J_{(m,n)}(R)$. Suppose that $x \in Q$ but $x \notin J_{(m,n)}(R)$. Since Q is an n-ary J-hyperideal of R and $g(x, 1_R^{(n-1)}) \in Q$, then we have $g(1_R^{(n)}) \in Q$ which is a contradiction. Therefore, $Q \subseteq J_{(m,n)}(R)$. □

Next, we characterize the Krasner (m,n)-hyperring which every proper hyperideal is an n-ary J-hyperideal.

Theorem 3.5. *Let R be a Krasner (m,n)-hyperring. Then R is local if and only if every proper hyperideal of R is an n-ary J-hyperideal.*

Proof. $\Longrightarrow$ Let M be the only maximal hyperideal of R. Then $J_{(m,n)}(R) = M$. Suppose that Q is a proper hyperideal of R. Let $g(x_1^n) \in Q$ for $x_1^n \in R$ such that $x_i \notin M$. Therefore x_i is invertible. Then we have

$$\begin{aligned} g(x_i^{-1}, g(x_1^n), 1_R^{(n-2)}) &= g(g(x_i, x_i^{-1}, 1_R^{(n-2)}), g(x_1^{i-1}, 1_R, x_{i+1}^n), 1_R^{(n-2)}) \\ &= g(x_1^{i-1}, 1_R, x_{i+1}^n) \\ &\subseteq Q \end{aligned}$$

Hence, Q is a an n-ary J-hyperideal of R.

$\Longleftarrow$ Suppose that every proper hyperideal of R is an n-ary J-hyperideal. Assume that the hyperideal M of R is maximal. Let $x \in M$. By the hypothesis, the principal hyperideal $\langle x \rangle$ is an n-ary J-hyperideal of R. Since $g(x, 1_R^{(n-1)}) \in \langle x \rangle$, then we get $x \in J_{(m,n)}(R)$ or $g(1_R^{(n)}) \in \langle x \rangle$. Since the second case is a contradiction, then $x \in J_{(m,n)}(R)$ which implies $J_{(m,n)}(R) = M$. Consequently, R is a local Krasner (m, n)-hyperring. □

Theorem 3.6. *Let $\{Q_i\}_{i \in \Delta}$ be a nonempty set of n-ary J-hyperideals of a Krasner (m, n)-hyperring R. Then $\bigcap_{i \in \Delta} Q_i$ is an n-ary J-hyperideal of R.*

Proof. Since $0 \in Q_i$ for all $i \in \Delta$, then $\bigcap_{i \in \Delta} Q_i \neq \varnothing$. Let $g(x_1^n) \in \bigcap_{i \in \Delta} Q_i$ for some $x_1^n \in R$ such that $x_i \notin J_{(m,n)}(R)$. Then $g(x_1^n) \in Q_i$ for every $i \in \Delta$. Since Q_i is an n-ary J-hyperideal of R, we have $g(x_1^{i-1}, 1_R, x_{i-1}^n) \in Q_i$. Then $g(x_1^{i-1}, 1_R, x_{i-1}^n) \in \bigcap_{i \in \Delta} Q_i$. □

Theorem 3.7. *Let Q be a proper hyperideal of a Krasner (m, n)-hyperring R. Then the following statements are equivalent:*

(1) Q is an n-ary J-hyperideal of R.

(2) $Q = U_x$ where $U_x = \{y \in R \mid g(x, y, 1_R^{(n-2)}) \in Q\}$ for every $x \notin J_{(m,n)}(R)$.

(3) $g(I_1^n) \subseteq Q$ for some hyperideals I_1^n of R and $I_i \nsubseteq J_{(m,n)}(R)$ imply $g(I_1^{i-1}, 1_R, I_{i+1}^n) \subseteq Q$.

Proof. (1) $\Longrightarrow$ (2) Let Q be an n-ary J-hyperideal of R. We have $Q \subseteq U_x$ for every $x \in R$. Suppose that $y \in U_x$ such that $x \notin J_{(m,n)}(R)$. This means $g(x, y, 1_R^{(n-2)}) \in Q$. Since Q is an n-ary J-hyperideal of R and $x \notin J_{(m,n)}(R)$, then $y = g(y, 1_R^{(n-2)}) \in Q$. Hence, we get $Q = U_x$.

(2) $\Longrightarrow$ (3) Let $g(I_1^n) \subseteq Q$ for some hyperideals I_1^n of R such that $I_i \nsubseteq J_{(m,n)}(R)$. Take $x_i \in I_i$ such that $x_i \notin J_{(m,n)}(R)$. Hence, $g(I_1^{i-1}, x_i, I_{I+1}^n) \subseteq Q$ which means $g(I_1^{i-1}, 1_R, I_{i+1}^n) \subseteq U_{x_i}$. Since $Q = U_{x_i}$ for every $x_i \notin J(R)$, then $g(I_1^{i-1}, 1_R, I_{i+1}^n) \subseteq Q$.

(3) $\Longrightarrow$ (1) Let us consider $g(x_1^n) \in Q$ for some $x_1^n \in R$ with $x_i \notin J_{(m,n)}(R)$. We have $g(\langle x_1 \rangle, ..., \langle x_n \rangle) = g(\langle g(x_1^n) \rangle, 1_R^{(n-1)}) \subseteq Q$ but $\langle x_i \rangle \nsubseteq J_{(m,n)}(R)$. Then we get $g(\langle x_1 \rangle, ..., \langle x_{i-1} \rangle, 1_R, \langle x_{i+1} \rangle, ..., \langle x_n \rangle) = g(\langle g(x_1^{i-1}, 1_R, x_{i+1}^n) \rangle, 1^{(n-1)}) \in Q$ which implies $g(x_1^{i-1}, 1_R, x_{i+1}^n) \in Q$. Therefore, Q is an n-ary J-hyperideal of R. □

Theorem 3.8. *Let Q be a proper hyperideal of a Krasner (m, n)-hyperring R. Then Q is an n-ary J-hyperideal of R if and only if $U_x \subseteq J_{(m,n)}(R)$ with $U_x = \{y \in R \mid g(x, y, 1_R^{(n-2)}) \in Q\}$ for every $x \notin Q$.*

Proof. $\Longrightarrow$ Let $y \in U_x$ such that $x \notin Q$. So, $g(x, y, 1^{(n-2)}) \in Q$. Then we have $y \in J(R)$ as Q is an n-ary J-hyperideal of R and $x = g(x, 1^{(n-2)}) \notin Q$.

$\Longleftarrow$ Let $g(x_1^n) \in Q$ for some $x_1^n \in R$ such that $x_i \notin J_{(m,n)}(R)$. If $g(x_1^{i-1}, 1_R, x_{i+1}^n) \notin Q$, then $x_i \in U_{g(x_1^{i-1}, 1_R, x_{i+1}^n)} \subseteq J_{(m,n)}(R)$ which is a contradiction. Then we conclude that $g(x_1^{i-1}, 1_R, x_{i+1}^n) \in Q$. Thus, Q is an n-ary J-hyperideal of R. □

Theorem 3.9. *Let Q be a hyperideal of a Krasner (m,n)-hyperring R and S be a nonempty subset of R such that $S \nsubseteq Q$. If Q is an n-ary J-hyperideal of R, then $U_S = \{x \in R \mid g(x,S,1_R^{(n-2)}) \in Q\}$ is an n-ary J-hyperideal of R.*

Proof. Let $U_S = R$. Then $1_R \in U_S$ which implies $S \subseteq Q$ a contradiction. Hence, U_S is a proper hyperideal of R. Suppose that $g(x_1^n) \in U_S$ for some $x_1^n \in R$ such that $x_i \notin J_{(m,n)}(R)$. This means $g(g(x_1^n),S,1_R^{(n-2)}) = \bigcup_{s\in S} g(g(x_1^n),s,1^{(n-2)}) \subseteq Q$ which implies for each $s \in S$, $g(g(x_1^n),s,1_R^{(n-2)}) = g(g(x_1^{i-1},s,x_{i+1}^n),x_i,1_R^{(n-2)}) \in Q$. Then $g(x_1^{i-1},s,x_{i+1}^n) = g(g(x_1^{i-1},s,x_{i+1}^n),1^{(n-1)}) \in Q$ for all $s \in S$ as Q is an n-ary J-hyperideal of R. This means $g(x_1^{i-1},1_R,x_{i+1}^n) \in U_S$. Thus, U_S is an n-ary J-hyperideal of R. □

Theorem 3.10. *Let Q be an n-ary J-hyperideal of a Krasner (m,n)-hyperring R such that there is no n-ary J-hyperideal which contains Q properly. Then Q is an n-ary prime hyperideal of R.*

Proof. Assume that Q is an n-ary J-hyperideal of a Krasner (m,n)-hyperring R such that there is no n-ary J-hyperideal which contains Q properly. Suppose that $g(x_1^n) \in Q$ for some $x_1^n \in R$ such that for every $1 \leq i \leq n-1$, $x_i \notin Q$. Put $S = \{x_1,...,x_{n-1}\}$. By Theorem 3.9, $(Q:S)$ is an n-ary J-hyperideal of R. Since $Q \subseteq U_S = \{x \in R \mid g(x,S,1^{(n-2)}) \in Q\}$, we conclude that $x_n \in (Q:S) = Q$, by the hypothesis. Thus, Q is a prime hyperideal. □

In Theorem 3.10, if $Q = J_{(m,n)}(R)$, then the inverse of the theorem is true.

Theorem 3.11. *Let $J_{(m,n)}(R)$ be an n-ary prime of R. Then $J_{(m,n)}(R)$ is an n-ary J-hyperideal of R such that there is no J-hyperideal which contains $J_{(m,n)}(R)$ properly.*

Proof. Put $Q = J_{(m,n)}(R)$ such that $J_{(m,n)}(R)$ is an n-ary prime of R. Let $g(x_1^n) \in Q$ for some $x_1^n \in R$ with $x_i \notin J(R)$. Since Q is an n-ary prime hyperideal of R, then there exists $1 \leq j \leq i-1$ or $i+1 \leq j \leq n$ such that $x_j \in Q = J_{(m,n)}(R)$ which means the hyperideal $J(R)$ of R is an n-ary J-hyperideal. By Theorem 3.4, there is no J-prime hyperideal which contains Q properly. □

4. n-ary δ-J-hyperideals

In this section, we define and study the concept of n-ary δ-J-hyperideals as an expansion of n-ary J-hyperideals.

Definition 4.1. Let δ be a hyperideal expansion of a Krasner (m,n)-hyperring R. A proper hyperideal Q of R is called n-ary δ-J-hyperideal if for $x_1^n \in R$, $g(x_1^n) \in Q$ implies that $x_i \in J_{(m,n)}(R)$ or $g(x_1^{i-1},1_R,x_{i+1}^n) \in \delta(Q)$.

Example 4.2. The hyperideal $I = \{0\mathbb{Z}_{12}^\star, 4\mathbb{Z}_{12}^\star\}$ in $\mathbb{Z}_{12}/\mathbb{Z}_{12}^\star$ of Example 4.1 in [14] is a δ_1-J-hyperideal.

Theorem 4.3. *Let Q be a proper hyperideal of a Krasner (m,n)-hyperring R. If $\delta(Q)$ is an n-ary J-hyperideal of R, then Q is an n-ary δ-J-hyperideal of R.*

Proof. Suppose that $\delta(Q)$ is an n-ary J-hyperideal of R. Let $g(x_1^n) \in Q$ for some $x_1^n \in R$ such that $x_i \notin J_{(m,n)}(R)$. Since $\delta(Q)$ is an n-ary J-hyperideal of R and $Q \subseteq \delta(Q)$, then $g(x_1^{i-1},1_R,x_{i+1}^n) \in \delta(Q)$ which implies Q is a δ-J-hyperideal of R. □

The next Theorem shows that the inverse of Theorem 4.3 is true if $\delta = \delta_1$.

Theorem 4.4. *Let Q be a proper hyperideal of a Krasner (m,n)-hyperring R. If Q is an n-ary δ_1-J-hyperideal of R, then $\sqrt{Q}^{(m,n)}$ is an n-ary J-hyperideal of R.*

Proof. Let for $x_1^n \in R$, $g(x_1^n) \in \sqrt{Q}^{(m,n)}$ such that $x_i \notin J_{(m,n)}(R)$. By $g(x_1^n) \in \sqrt{Q}^{(m,n)}$, it follows that there exists $t \in \mathbb{N}$ such that if $t \leq n$, then $g(g(x_1^n)^{(t)}, 1_R^{(n-t)}) \in Q$. Hence by associativity we get

$$g(x_i^{(t)}, g(x_i^{i-1}, 1_R, x_{i+1}^n)^{(t)}, 1_R^{(n-2t)})$$
$$= g(x_i^{(t)}, g(x_i^{i-1}, 1_R, x_{i+1}^n)^{(t)}, g(1_R^{(n)}), 1_R^{(n-2t-1)})$$
$$= g(g(x_i^{(t)}, 1_R^{(n-t)}), g(x_i^{i-1}, 1_R, x_{i+1}^n)^{(t)}, 1_R^{(n-t-1)})$$
$$\subseteq Q.$$

Since Q is an n-ary δ_1-J-hyperideal of R, then

$$g(x_i^{(t)}, 1^{(n-t)}) \in J_{(m,n)}(R)$$

or $g(g(x_i^{i-1}, 1_R, x_{i+1}^n)^{(t)}, 1_R^{(n-t)}) \in \delta_1(Q) = \sqrt{Q}^{(m,n)}$.

If $g(x_i^{(t)}, 1^{(n-t)}) \in J_{(m,n)}(R)$, then $x_i \in \sqrt{J_{(m,n)}(R)}^{(m,n)} = J_{(m,n)}(R)$ which is a contradiction. Then we have

$$g(g(x_i^{i-1}, 1_R, x_{i+1}^n)^{(t)}, 1_R^{(n-t)}) \in \sqrt{Q}^{(m,n)}$$

which means $g(x_i^{i-1}, 1_R, x_{i+1}^n) \in \sqrt{Q}^{(m,n)}$. Thus we conclude that $\sqrt{Q}^{(m,n)}$ is an n-ary J-hyperideal of R. If $t = l(n-1)+1$, then by using a similar argument, one can easily complete the proof. □

Theorem 4.5. *Let Q be a proper hyperideal of a Krasner (m,n)-hyperring R and let δ and γ be two hyperideal expansions of R. If $\delta(Q)$ is an n-ary γ-J-hyperideal of R, then Q is an n-ary $\gamma \circ \delta$-J-hyperideal of R.*

Proof. Suppose that $\delta(Q)$ is an n-ary γ-J-hyperideal of R. Let $g(x_1^n) \in Q$ for some $x_1^n \in R$ such that $x_i \notin J_{(m,n)}(R)$. We get $g(x_1^n) \in \delta(Q)$ as $Q \subseteq \delta(Q)$. Since $\delta(Q)$ is an n-ary γ-J-hyperideal of R and $x_i \notin J_{(m,n)}(R)$, then $g(x_1^{i-1}, 1_R, x_{i+1}^n) \in \gamma(\delta(Q))$ which means $g(x_1^{i-1}, 1_R, x_{i+1}^n) \in \gamma \circ \delta(Q)$. Thus, Q is an n-ary $\gamma \circ \delta$-J-hyperideal of R. □

Theorem 4.6. *Let Q_1, Q_2 and Q_3 be three proper hyperideals of a Krasner (m,n)-hyperring R such that $Q_1 \subseteq Q_2 \subseteq Q_3$. If Q_3 is an n-ary δ-J-hyperideal of R and $\delta(Q_1) = \delta(Q_3)$, then Q_2 is an n-ary δ-J-hyperideal of R.*

Proof. Let $g(x_1^n) \in Q_2$ for some $x_1^n \in R$ such that $x_i \notin J_{(m,n)}(R)$. Since $Q_2 \subseteq Q_3$ and Q_3 is an n-ary δ-J-hyperideal of R, then we have $g(x_1^{i-1}, 1_R, x_{i+1}^n) \in \delta(Q_3)$. Then we conclude that $g(x_1^{i-1}, 1_R, x_{i+1}^n) \in \delta(Q_1)$, by the hypothesis. Since $Q_1 \subseteq Q_2$, then $\delta(Q_1) \subseteq \delta(Q_2)$. This implies that $g(x_1^{i-1}, 1_R, x_{i+1}^n) \in \delta(Q_2)$ as needed. □

Theorem 4.7. *Let Q be a δ-J-hyperideal of a Krasner (m,n)-hyperring R such that $\sqrt{\delta(Q)}^{(m,n)} \subseteq \delta(\sqrt{Q}^{(m,n)})$. Then $\sqrt{Q}^{(m,n)}$ is a δ-J-hyperideal of R.*

Proof. Let for $x_1^n \in R$, $g(x_1^n) \in \sqrt{Q}^{(m,n)}$ such that $x_i \notin J_{(m,n)}(R)$. By $g(x_1^n) \in \sqrt{Q}^{(m,n)}$, it follows that there exists $t \in \mathbb{N}$ such that if $t \leq n$, then $g(g(x_1^n)^{(t)}, 1_R^{(n-t)}) \in Q$. Hence by associativity we get

$$g(x_i^{(t)}, g(x_i^{i-1}, 1_R, x_{i+1}^n)^{(t)}, 1_R^{(n-2t)})$$
$$= g(x_i^{(t)}, g(x_i^{i-1}, 1_R, x_{i+1}^n)^{(t)}, g(1_R^{(n)}), 1_R^{(n-2t-1)})$$
$$= g(g(x_i^{(t)}, 1_R^{(n-t)}), g(x_i^{i-1}, 1_R, x_{i+1}^n)^{(t)}, 1_R^{(n-t-1)})$$
$$\subseteq Q.$$

Since Q is an n-ary δ-J-hyperideal of R, then

$$g(x_i^{(t)}, 1^{(n-t)}) \in J_{(m,n)}(R) \text{ or } \quad g(g(x_i^{i-1}, 1_R, x_{i+1}^n)^{(t)}, 1_R^{(n-t)}) \in \delta(I).$$

If $g(x_i^{(t)}, 1^{(n-t)}) \in J_{(m,n)}(R)$, then $x_i \in \sqrt{J_{(m,n)}(R)}^{(m,n)} = J_{(m,n)}(R)$ which is a contradiction. Then we have

$g(g(x_i^{i-1}, 1_R, x_{i+1}^n)^{(t)}, 1_R^{(n-t)}) \in \delta(Q)$
which means $g(x_i^{i-1}, 1_R, x_{i+1}^n) \in \sqrt{\delta(Q)}^{(m,n)}$. Hence, by the assumption, we obtain $g(x_i^{i-1}, 1_R, x_{i+1}^n) \in \delta(\sqrt{Q}^{(m,n)})$.
Thus we conclude that $\sqrt{Q}^{(m,n)}$ is a δ-J-hyperideal of R. If $t = l(n-1)+1$, then by using a similar argument, one can easily complete the proof. □

We say that δ has the property of intersection preserving if it satisfies $\delta(I \cap J) = \delta(I) \cap \delta(J)$, for all hyperideals I, J of R. For example, the hyperideal expansion δ_1 of a Krasner (m,n)-hyperring R has the property of intersection preserving.

Theorem 4.8. *Suppose that Q_1^n are n-ary δ-J-hyperideal of a Krasner (m,n)-hyperring R and the hyperideal expansion δ of R has the property of intersection preserving. Then $Q = \bigcap_{i=1}^n Q_i$ is an n-ary δ-J-hyperideal of R.*

Proof. Let $g(x_1^n) \in Q$ for some $x_1^n \in R$ such that $g(x_1^{i-1}, 1_R, x_{i+1}^n) \notin \delta(Q)$. Since the hyperideal expansion δ of R has the property of intersection preserving, then there exists $1 \leq j \leq n$ such that $g(x_1^{i-1}, 1_R, x_{i+1}^n) \notin \delta(Q_j)$. Thus, we get $x_i \in J_{(m,n)}(R)$ as Q_j ia an n-ary δ-J-hyperideal of R. Consequently, $Q = \bigcap_{i=1}^n Q_i$ is an n-ary δ-J-hyperideal of R. □

Theorem 4.9. *Let Q be a proper hyperideal of a Krasner (m,n)-hyperring R. Then the following are equivalent:*

(1) Q is an n-ary δ-J-hyperideal of R.

(2) If Q_1^{n-1} are some hyperideals of R and $x \in R$ such that $g(Q_1^{n-1}, x) \subseteq Q$, then $x \in J_{(m,n)}(R)$ or $g(Q_1^{n-1}, 1_R) \subseteq \delta(Q)$.

(3) If Q_1^n are some hyperideals of R and $g(Q_1^n) \subseteq Q$, then either $Q_i \subseteq J_{(m,n)}(R)$ or $g(Q_1^{i-1}, 1_R, Q_{i+1}^n) \subseteq \delta(Q)$.

Proof. (1) $\Longrightarrow$ (2) Let Q be an n-ary δ-J-hyperideal of R. Assume that $g(Q_1^{n-1}, x) \subseteq Q$ for some hyperideals Q_1^{n-1} of R such that $x \notin J_{(m,n)}(R)$. Therefore, for each $q_i \in Q_i$ with $1 \leq i \leq n-1$ we have $g(q_1^{n-1}, x) \in Q$. Since Q is an n-ary δ-J-hyperideal of R and $x \notin J_{(m,n)}(R)$, then $g(q_1^{n-1}, 1_R) \in \delta(Q)$ which means $g(Q_1^{n-1}, 1_R) \subseteq Q$.

(2) $\Longrightarrow$ (3) Let $g(Q_1^n) \subseteq Q$ for some hyperideals Q_1^n of R such that $Q_i \nsubseteq J_{(m,n)}(R)$. Take $x \in Q_i$ but $x \notin J_{(m,n)}(R)$. Since $g(Q_1^{i-1}, x, Q_{i+1}^n) \subseteq Q$ and $x \notin J_{(m,n)}$, then $g(Q_1^{i-1}, 1_R, Q_{i+1}^n) \subseteq \delta(Q)$, by the hypothesis.

(3) $\Longrightarrow$ (1) Let $g(x_1^n) \in Q$ for some $x_1^n \in R$ and $x_i \notin J_{(m,n)}(R)$. Therefore we get $g(\langle x_1 \rangle, ..., \langle x_n \rangle) \subseteq Q$ but $\langle x_i \rangle \nsubseteq J_{(m,n)}$. Thus we have
$g(\langle x_1 \rangle, ..., \langle x_{i-1} \rangle, 1_R, \langle x_{i+1} \rangle, ..., \langle x_n \rangle) \subseteq \delta(Q)$.
This means $g(x_1^{i-1}, 1_R, x_{i+1}^n) \in \delta(Q)$. □

Theorem 4.10. *Let Q be a proper hyperideal of a Krasner (m,n)-hyperring R. Then the following are equivalent:*

(1) Q is an n-ary δ-J-hyperideal of R.

(2) $Q \subseteq J_{(m,n)}(R)$ and if $g(x_1^n) \in Q$ for some $x_1^n \in R$, then either x_i is in the intersection of all maximal hyperideals of R containing Q or $g(x_1^{i-1}, 1_R, x_{i+1}^n) \in \delta(Q)$.

Proof. (1) $\Longrightarrow$ (2) Suppose that Q is an n-ary δ-J-hyperideal of R. Let $Q \nsubseteq J_{(m,n)}(R)$. Take $x \in Q$ such that $x \notin J_{(m,n)}(R)$. Since $g(x, 1^{(n-1)}) \in Q$, then $g(1^{(n)}) \in Q$, a contradiction. Hence, $Q \subseteq J_{(m,n)}(R)$. Since $J_{(m,n)}(R)$ is in the intersection of all maximal hyperideals of R containing Q, then the second assertion follows.

(2) $\Longrightarrow$ (1) Let $g(x_1^n) \in Q$ for some $x_1^n \in R$ such that $x_i \notin J_{(m,n)}(R)$. The intersection of all maximal hyperideals of R containing Q is in $J_{(m,n)}(R)$ as $Q \subseteq J_{(m,n)}(R)$. This

means x_i is not in the intersection of all maximal hyperideals of R containing Q. Then $g(x_1^{i-1}, 1_R, x_{i+1}^n) \in \delta(Q)$. Consequently, Q is an n-ary δ-J-hyperideal of R. □

Theorem 4.11. *Let R be a Krasner (m,n)-hyperring. Then the following are equivalent:*

(1) R is a local Krasner (m,n)-hyperring such that $J_{(m,n)}(R)$ is the only maximal hyperideal of R.

(2) Every proper principal hyperideal is an n-ary δ-J-hyperideal of R.

(3) Every proper hyperideal is an n-ary δ-J-hyperideal of R.

Proof. (1) $\Longrightarrow$ (2) Let R be a local Krasner (m,n)-hyperring such that $J_{(m,n)}(R)$ is the only maximal hyperideal of R. Consider the principal hyperideal $Q = \langle a \rangle$ for some element a of R which is not invertible. Suppose that $g(x_1^n) \in Q$ such that $x_i \notin J_{(m,n)}(R)$. This means x_i is invertible member of R. Then we have

$g(g(x_1^n), x_i^{-1}, 1_R^{(n-2)}) = g(g(x_1^{i-1}, 1_R, x_{i+1}^n), x_i, x_i^{-1}, 1_R^{(n-3)}) = g(x_1^{i-1}, 1_R, x_{i+1}^n) \subseteq Q \subseteq \delta(Q)$.

Thus, Q is an n-ary δ-J-hyperideal of R.

(2) $\Longrightarrow$ (3) Suppose that P is a proper hyperideal of R and $g(x_1^n) \in P$ for some $x_1^n \in R$ such that $x_i \notin J_{(m,n)}(R)$. Consider the principal hyperideal $Q = \langle g(x_1^n) \rangle$. By the assumption, Q is an n-ary δ-J-hyperideal of R. Since $g(x_1^n) \in Q$ and $x_i \notin J_{(m,n)}(R)$, then $g(x_1^{i-1}, 1_R, x_{i+1}^n) \in \delta(Q)$. Since $Q \subseteq P$, then $\delta(Q) \subseteq \delta(P)$ and so $g(x_1^{i-1}, 1_R, x_{i+1}^n) \in \delta(P)$. This completes the proof.

(3) $\Longrightarrow$ (1) Suppose that M is a maximal hyperideal of R. By the hypothesis, M is an n-ary δ-J-hyperideal of R. By theorem 4.10, we have $M \subseteq J_{(m,n)}(R)$. Since $J_{(m,n)}(R) \subseteq M$, then $M = J_{(m,n)}(R)$. This implies that $J_{(m,n)}(R)$ is the only maximal hyperideal of R. Thus, R is a local Krasner (m,n)-hyperring. □

Recall that a proper hyperideal I of R is said to be δ-primary if for all $x_1^n \in R$, $g(x_1^n) \in I$ implies that $x_i \in I$ or $g(x_1^{i-1}, 1_R, x_{i+1}^n) \in \delta(I)$ for some $1 \leq i \leq n$ [3].

Theorem 4.12. *Let Q be a δ-primary hyperideal of R. Then Q is an n-ary δ-J-hyperideal of R if and only if $Q \subseteq J_{(m,n)}(R)$.*

Proof. $\Longrightarrow$ It follows by Theorem 4.10.

$\Longleftarrow$ Let $Q \subseteq J_{(m,n)}(R)$. Suppose that $g(x_1^n) \in Q$ for some $x_1^n \in R$ such that $x_i \notin J_{(m,n)}(R)$. Then $x_i \notin Q$. Thus, we get $g(x_1^{i-1}, 1_R, x_{i+1}^n) \in \delta(Q)$ as Q is a δ-primary hyperideal of R. This implies that Q is an n-ary δ-J-hyperideal of R. □

Theorem 4.13. *Let Q be a maximal hyperideal of R. Then Q is an n-ary δ-J-hyperideal of R if and only if $Q = J_{(m,n)}(R)$.*

Proof. $\Longrightarrow$ Let Q be an n-ary δ-J-hyperideal of R. Since Q is a maximal hyperideal of R, then $J_{(m,n)}(R) \subseteq Q$. Let $g(x_1^n) \in Q$ for some $x_1^n \in R$ such that $x_i \notin Q$. Then $x_i \notin J_{(m,n)}(R)$. Since Q is an n-ary δ-J-hyperideal of R and $x_i \notin J_{(m,n)}(R)$, we conclude that $g(x_1^{i-1}, 1_R, x_{i+1}^n) \in \delta(Q)$. This implies that Q is a δ-primary hyperideal of R. Therefore, we get $Q \subseteq J_{(m,n)}(R)$, by Theorem 4.12. Hence $Q = J_{(m,n)}(R)$.

$\Longleftarrow$ It is obvious. □

Theorem 4.14. *Let (R_1, f_1, g_1) and (R_2, f_2, g_2) be two Krasner (m,n)-hyperrings and $h : R_1 \longrightarrow R_2$ be a $\delta\gamma$-homomorphism such that δ and γ be hyperideal expansions of R_1 and R_1, respectively. Then the following statements hold :*

(1) *If h is a monomorphism and I_2 is an n-ary γ-J_2-hyperideal of R_2, then $h^{-1}(I_2)$ is an n-ary δ-J_1-hyperideal of R_1.*

(2) *Let h be an epimorphism and I_1 be a hyperideal of R such that $Ker(h) \subseteq I_1$. If I_1 is an n-ary δ-J_1-hyperideal of R_1, then $h(I_1)$ is an n-ary γ-J_2-hyperideal of R_2.*

Proof. (1) Let for $x_1^n \in R_1$, $g_1(x_1^n) \in h^{-1}(I_2)$. Then we get $h(g_1(x_1^n)) = g_2(h(x_1^n)) \in I_2$. Since I_2 is a γ-J_2-hyperideal of R_2, it implies that either $h(x_i) \in J_{(m,n)}(R_2)$ which follows $x_i \in J_{(m,n)}(R_1)$ as h is a monomorphism, or

$$g_2(h(x_1), ..., h(x_{i-1}), 1_{R_2}, h(x_{i+1}), ..., h(x_n))$$
$$= h(g_1(x_1^{i-1}, 1_{R_1}, x_{i+1}^n))$$
$$\in \gamma(I_2),$$

which follows $g_1(x_1^{i-1}, 1_{R_1}, x_{i+1}^n) \in h^{-1}(\gamma(I_2)) = \delta(h^{-1}(I_2)$. Thus $h^{-1}(I_2)$ is a δ-J_1-hyperideal of R_1.

(2) Let for $y_1^n \in R_2$, $g_2(y_1^n) \in h(I_1)$ such that $y_i \notin J_{(m,n)}(R_2)$. Since h is an epimorphism, then there exist $x_1^n \in R_1$ such that $h(x_1) = y_1, ..., h(x_n) = y_n$. Hence

$$h(g_1(x_1^n)) = g_2(h(x_1), ..., h(x_n)) = g_2(y_1^n) \in h(I_1).$$

Since $Ker(h) \subseteq I_1$, then we get $g_1(x_1^n) \in I_1$. Since $y_i \notin J_{(m,n)}(R_2)$, then $x_i \notin J_{(m,n)}(R_1)$. Since I_1 is a δ-J_1-hyperideal of R_1 and $x_i \notin J_{(m,n)}(R_1)$, it implies that $g_1(x_1^{i-1}, 1_{R_1}, x_{i+1}^n) \in \delta(I_1)$ which implies

$$\begin{aligned} h(g_1(x_1^{i-1}, 1_{R_1}, x_{i+1}^n)) &= g_2(h(x_1), ..., h(x_{i-1}), 1_{R_2}, h(x_{i+1}), ..., h(x_n)) \\ &= g_2(y_1^{i-1}, 1_{R_2}, y_{i+1}^n) \\ &\in h(\delta(I_1)) \\ &= \gamma(h(I_1)) \end{aligned}$$

Thus $h(I_1)$ is a γ-J_2-hyperideal of R_2. □

Corollary 4.15. *Let Q and J be hyperideals of a Krasner (m,n)-hyperring R such that $I \subseteq Q$. If Q is an n-ary δ-J-hyperideal of R, then Q/I is an n-ary δ_q -J-hyperideal of R/I.*

Proof. Consider the map $\pi : R \longrightarrow R/I$, defined by $r \longrightarrow f(r, I, 0^{(m-2)})$. The map is a homomorphism of Krasner (m,n)-hyperrings, by Theorem 3.2 in [1]. Now, by using Theorem 4.14 (2), the claim can be proved. □

5. (k,n)-absorbing δ-J-hyperideals

In this section, we extend the notion of n-ary δ-J-hyperideals to (k,n)-absorbing δ-J-hyperideals.

Definition 5.1. Let δ be a hyperideal expansion of a Krasner (m,n)-hyperring R and k be a positive integer. A proper hyperideal Q of R is called (k,n)-absorbing δ-J-hyperideal if for $x_1^{kn-k+1} \in R$, $g(x_1^{kn-k+1}) \in Q$ implies that $g(x_1^{(k-1)n-k+2}) \in J_{(m,n)}(R)$ or a g-product of $(k-1)n-k+2$ of $x_i^,$ s except $g(x_1^{(k-1)n-k+2})$ is in $\delta(Q)$.

Example 5.2. Suppose that $H = [0,1]$ and define a 2-ary hyperoperation "$\boxplus$" on H as follows:

$$a \boxplus b = \begin{cases} \{max\{a,b\}\}, & if \;\; a \neq b \\ [0,a] & if \;\; a = b \end{cases}$$

Let "$\cdot$" is the usual multiplication on real numbers. In the Krasner $(2,3)$-hyperring H, the hyperideal $T = [0, 0.5]$ is a $(2,2)$-absorbing δ_1-J-hyperideal of R.

Theorem 5.3. *If Q is δ-J-hyperideal of a Krasner (m,n)-hyperring R, then Q is $(2,n)$-absorbing δ-J-hyperideal.*

Proof. Let for $x_1^{2n-1} \in R$, $g(x_1^{2n-1}) \in Q$. Since Q is a δ-J-hyperideal of R, then $g(x_1^n) \in J_{(m,n)}(R)$ or $g(x_{n+1}^{2n-1}) \in \delta(Q)$. This implies that for $1 \leq i \leq n$, $g(x_i, x_{n+1}^{2n-1}) \in \delta(Q)$, since $x_1^n \in R$ and $\delta(Q)$ is a hyperideal of R. Consequently, hyperideal Q is $(2,n)$-absorbing δ-primary. □

Theorem 5.4. *If Q is (k,n)-absorbing δ-J-hyperideal of a Krasner (m,n)-hyperring R, then Q is (s,n)-absorbing δ-J-hyperideal for $s > n$.*

Proof. Let for $x_1^{(k+1)n-(k+1)+1} \in R$, $g(x_1^{(k+1)n-(k+1)+1}) \in Q$. Put $g(x_1^{n+2}) = x$. Since hyperideal Q of R is (k,n)-absorbing δ-J-hyperideal, $g(x, x_{n+3}^{(k+1)n-(k+1)+1}) \in J_{(m,n)}(R)$ or a g-product of $kn-k+1$ of the $x_i^,$s except $g(x, x_{n+3}^{(k+1)n-(k+1)+1})$ is in $\delta(I)$. This implies that for $1 \leq i \leq n+2$, $g(x_i, x_{n+3}^{(k+1)n-(k+1)+1}) \in \delta(I)$ which means hyperideal Q is $(k+1,n)$-absorbing δ-J-hyperideal. Thus hyperideal Q of R is (s,n)-absorbing δ-J-hyperideal for $s > n$. □

Theorem 5.5. *If Q is a (k,n)-absorbing J-hyperideal of a Krasner (m,n)-hyperring R, then $\sqrt{Q}^{(m,n)}$ is a (k,n)-absorbing δ-J-hyperideal.*

Proof. Let for $x_1^{kn-k+1} \in R$, $g(x_1^{kn-k+1}) \in \sqrt{Q}^{(m,n)}$. We suppose that none of the g-products of $(k-1)n-k+2$ of the $x_i^,$s other than $g(x_1^{(k-1)n-k+2})$ are in $\delta(\sqrt{Q}^{(m,n)})$. Since $g(x_1^{kn-k+1}) \in \sqrt{Q}^{(m,n)}$ then for some $t \in \mathbb{N}$ we have for $t \leq n$, $g(g(x_1^{kn-k+1})^{(t)}, 1_R^{(n-t)}) \in Q$ or for $t > n$ with $t = l(n-1)+1$, $g_{(l)}(g(x_1^{kn-k+1})^{(t)}) \in Q$. In the former case, since all g-products of the $x_i^,$s other than $g(x_1^{(k-1)n-k+2})$ are not in $\delta(\sqrt{Q}^{(m,n)})$, then they are not in Q. Since Q is a (k,n)-absorbing J-hyperideal of R, then we have $g(g(x_1^{(k-1)n-k+2)})^{(l)}, 1_R^{(n-t)}) \in J_{(m,n)}(R)$ which means $g(x_1^{(k-1)n-k+2}) \in \sqrt{J_{(m,n)}(R)}^{(m,n)} = J_{(m,n)}(R)$. By using similar argument for the second case, the claim is completed. □

Theorem 5.6. *If $\delta(Q)$ is $(2,n)$-absorbing J-hyperideal for hyperideal Q of a Krasner (m,n)-hyperring R, then Q is a $(3,n)$-absorbing δ-J-hyperideal of R.*

Proof. Let for $x_1^{3n-2} \in R$, $g(x_1^{3n-2}) \in Q$ but $g(x_1^{2n-1}) \notin J_{(m,n)}(R)$. By $g(x_1^{3n-2}) \in Q$ it follows that $g(g(x_1, x_{2n}^{3n-2}), x_2^{2n-1}) \in Q \subseteq \delta(Q)$. Since $\delta(Q)$ is a $(2,n)$-absorbing J-hyperideal and $g(x_2^{2n-1}) \notin J_{(m,n)}(R)$, then we have $g(x_1^n, x_{2n}^{3n-2}) \in \delta(Q)$ or $g(x_1, x_{n+1}^{2n-1}, x_{2n}^{3n-2}) \in \delta(Q)$. Hence Q is a $(3,n)$-absorbing δ-J-hyperideal of R. □

Theorem 5.7. *If $\delta(Q)$ is a $(k+1,n)$-absorbing δ-J-hyperideal for the hyperideal Q of a Krasner (m,n)-hyperring R, then Q is $(k+1,n)$-absorbing δ-J-hyperideal.*

Proof. Let for $x_1^{(k+1)n-(k+1)+1} \in R$, $g(x_1^{(k+1)n-(k+1)+1}) \in Q$ but $g(x_1^{kn-k+1}) \notin J_{(m,n)}(R)$. Then we get $g(x_1^{(k+1)n-(k+1)+1}) = g(x_1^{kn-k}, g(x_{kn-k+1}^{(k+1)n-(k+1)+1})) \in Q \subseteq \delta(Q)$. Since hyperideal $\delta(I)$ is $(k+1,n)$-absorbing δ-J-hyperideal and $g(x_1^{kn-k}) \notin J_{(m,n)}(R)$, we get for $1 \leq i \leq n$, $g(x_1^{i-1}, x_{i+1}^{kn-k}, g(x_{kn-k+1}^{(k+1)n-(k+1)+1})) \in \delta(I)$. Consequently, hyperideal I is a $(k+1,n)$-absorbing δ-J-hyperideal. □

Theorem 5.8. *Let (R_1, f_1, g_1) and (R_2, f_2, g_2) be two Krasner (m,n)-hyperrings and $h : R_1 \longrightarrow R_2$ be a $\delta\gamma$-homomorphism such that δ and γ are two hyperideal expansions of Krasner (m,n)-hyperring R_1 and R_2, respectively. Then the following statements hold :*

(1) *Let h be a monomorphism. If Q_2 is a (k,n)-absorbing γ-J_2-hyperideal of R_2, then $h^{-1}(Q_2)$ is a (k,n)-absorbing δ-J_1-hyperideal of R_1.*

(2) *If h is an epimorphism and Q_1 is a (k,n)-absorbing δ-J_1-hyperideal of R_1 such that $Ker(h) \subseteq Q_1$, then $h(Q_1)$ is a γ-J_2-hyperideal of R_2.*

Proof. (1) Let for $x_1^{kn-k+1} \in R_1$, $g_1(x_1^{kn-k+1}) \in h^{-1}(Q_2)$. It means $h(g_1(x_1^{kn-k+1})) = g_2(h(x_1), ..., h(x_{kn-k+1})) \in Q_2$. Since Q_2 is a (k,n)-absorbing γ-J_2-hyperideal of R_2, we get $g_2(h(x_1), ..., h(x_{(k-1)n-k+2})) = h(g_1(x_1^{(k-1)n-k+2}) \in J_{(m,n)}(R_2)$. This implies that $g_1(x_1^{(k-1)n-k+2}) \in J_{(m,n)}(R_1)$, as h is a monomorphism, or

$$g_2(h(x_1), ..., h(x_{i-1}), h(x_{i+1}), ..., h(x_{kn-k+1)}) = h(g_1(x_1^{i-1}, x_{i+1}^{kn-k+1})) \in \gamma(Q_2)$$

which means $g_1(x_1^{i-1}, x_{i+1}^{kn-k+1})) \in h^{-1}(\gamma(Q_2)$ for $1 \leq i \leq n$. Since h is a $\delta\gamma$-homomorphism then $g_1(x_1^{i-1}, x_{i+1}^{kn-k+1})) \in \delta(h^{-1}(Q_2))$ for $1 \leq i \leq n$. Therefore we conclude that $h^{-1}(Q_2)$

is a (k,n)-absorbing δ-J_1-hyperideal of R_1. (2) Let for $y_1^{kn-k+1} \in R_2$, $g_2(y_1^{kn-k+1}) \in h(Q_1)$ such that $g_2(y_1^{(k-1)n-k+2}) \notin J_{(m,n)}(R_2)$. Then there are $x_1^{(k-1)n-k+2} \in R_1$ such that $h(x_i) = y_i$ for $1 \leq i \leq (k-1)n-k+2$. Hence, $h(g_1(x_1^{kn-k+1}) = g_2(h(x_1),...,h(x_{kn-k+1}) \in h(Q_1)$. Since Q_1 containing $Ker(h)$ then $g_1(x_1^{kn-k+1}) \in Q_1$. Since Q_1 is a (k,n)-absorbing δ-J_1-hyperideal of R_1 and $g_1(x_1^{(k-1)n-k+2}) \notin J_{(m,n)}(R_1)$, then $g_1(x_1^{i-1}, x_{i+1}^{kn-k+1}) \in \delta(Q_1)$ which means

$$\begin{aligned} h(g_1(x_1^{i-1}, x_{i+1}^{kn-k+1})) &= g_2(h(x_1),...,h(x_{i-1}),h(x_{i+1}),...,h(x_{kn-k+1})) \\ &= g_2(y_1^{i-1}, y_{i+1}^{kn-k+1}) \in h(\delta(Q_1)) \end{aligned}$$

for $1 \leq i \leq (k-1)n-k+2$. Since h is a $\delta\gamma$-epimorphism then we have $g_2(y_1^{i-1}, y_{i+1}^{kn-k+1}) \in \gamma(h((Q_1))$. Consequently, $h(Q_1)$ is a γ-J_2-hyperideal of R_2. □

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