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# Shift Filter of Quasi-ordered Residuated Systems

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**Abstract**
The concept of residuated relational systems ordered under a quasi-order relation was introduced in 2018 by S. Bonzio and I. Chajda as a structure $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, \preccurlyeq \rangle$, where $(A, \cdot)$ is a commutative semigroup with the identity 1 as the top element in this ordered monoid under a quasi-order $\preccurlyeq$. In 2020, the author introduced and analyzed the concepts of filters in this type of algebraic structures. In addition to the previous, the author continued to investigate some of the types of filters in quasi-ordered residuated systems such as, for example, implicative and comparative filters. In this article, as a continuation of previous author's research, the author introduced and analyzed the concepts of shift filters of quasi-ordered residuated systems and then compared it with other types of filters.

**Keywords:** Comparative filter, Filter, Implicative filter, Shift filter, Quasi-ordered residuated system
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## 1. Introduction

Let $(A, \cdot, 1)$ be a commutative semigroup with the identity 1. Suppose that on the carrier $A$ there exists another operation $\rightarrow$ and one relation $R$ that with multiplication in $A$ have a link $(x \cdot y, z) \in R \Longleftrightarrow (x, y \rightarrow z) \in R$ for each $x, y, z \in A$. A relational system designed in this way, when $R$ is a quasi-ordered relation on $A$, is in the focus of this paper.

The concept of residuated relational systems ordered under a quasi-order relation was introduced in 2018 by S. Bonzio and I. Chajda in [2]. Previously, this concept was discussed in [1]. This paper continues the investigations of quasi-ordered residuated systems and of their filters which were started in the author article [3]. In particular, the concept of shift filters of a quasi-ordered residuated system is introduced and analyzed. This type of filter is compared to the concept of filter and the concept of implicative (introduced in [4]) and comparative filters ([6]) in this algebraic system. It is shown (Theorem 3.2) that every comparative filter is a shift filter and vice versa does not have to be. In addition, it is shown (Theorem 3.3) that if the implicative filter $F$ satisfies the added condition

$$(\forall u, v \in A)((u \rightarrow v) \rightarrow v \in F \Longrightarrow (v \rightarrow u) \rightarrow u \in F))$$

then $F$ is a shift filter. The reverse, of course, does not have to be.

It should be said here that a quasi-ordered residuated system, in the general case, it does not have to be a commutative residauted lattice (see Example 2.8).

## 2. Preliminaries

### 2.1 Concept of quasi-ordered residuated systems

In article [2], S. Bonzio and I. Chajda introduced and analyzed the concept of residual relational systems.

**Definition 2.1** ([2], Definition 2.1)**.** *A* residuated relational system *is a structure* $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$*, where* $\langle A, \cdot, \rightarrow, 1\rangle$ *is an algebra of type* $\langle 2,2,0\rangle$ *and R is a binary relation on A and satisfying the following properties:*
(1) $(A, \cdot, 1)$ *is a commutative monoid;*
(2) $(\forall x \in A)((x,1) \in R)$*;*
(3) $(\forall x,y,z \in A)((x \cdot y, z) \in R \Longleftrightarrow (x, y \rightarrow z) \in R)$ .
*We will refer to the operation* $\cdot$ *as multiplication, to* $\rightarrow$ *as its residuum and to condition (3) as residuation.*

The basic properties for residuated relational systems are subsumed in the following:

**Theorem 2.2** ([2], Proposition 2.1)**.** *Let* $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$ *be a residuated relational system. Then*
(4) $(\forall x,y \in A)(x \rightarrow y = 1 \Longrightarrow (x,y) \in R)$*;*
(5) $(\forall x \in A)((x, 1 \rightarrow 1) \in R)$*;*
(6) $(\forall x \in A)((1, x \rightarrow 1) \in R)$*;*
(7) $(\forall x,y,z \in A)(x \rightarrow y = 1 \Longrightarrow (z \cdot x, y) \in R)$*;*
(8) $(\forall x,y \in A)((x, y \rightarrow 1) \in R)$.

Recall that a *quasi-order relation* $' \preccurlyeq '$ on a set $A$ is a binary relation which is reflexive and transitive (Some authors use the term pre-order relation).

**Definition 2.3** ([2], Definition 3.1)**.** *A* quasi-ordered residuated system *is a residuated relational system* $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, \preccurlyeq\rangle$*, where* $\preccurlyeq$ *is a quasi-order relation in the monoid* $(A, \cdot)$

**Example 2.4.** *Let* $A = \{1, a, b, c, d\}$ *and operations '*$\cdot$*' and '*$\rightarrow$*' defined on A as follows:*

| $\cdot$ | 1 | a | b | c | d |
|---|---|---|---|---|---|
| 1 | 1 | a | b | c | d |
| s | a | a | d | c | d |
| b | b | d | b | d | d |
| c | c | c | d | c | d |
| d | d | d | d | d | d |

and

| $\rightarrow$ | 1 | a | b | c | d |
|---|---|---|---|---|---|
| 1 | 1 | a | b | c | d |
| a | 1 | 1 | b | c | d |
| b | 1 | a | a | c | c |
| c | 1 | 1 | b | 1 | b |
| d | 1 | 1 | 1 | 1 | 1 |

*Then* $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1\rangle$ *is a quasi-ordered residuated systems where the relation '*$\preccurlyeq$*' is defined as follows*

$$\preccurlyeq := \{(1,1),(a,1),(b,1),(c,1),(d,1),(b,b),(a,a),(c,a),(d,a),(d,b),(d,c)\}.$$

**Example 2.5.** *For a commutative monoid A, let* $\mathfrak{P}(A)$ *denote the powerset of A ordered by set inclusion and '*$\cdot$*' the usual multiplication of subsets of A. Then* $\langle \mathfrak{P}(A), \cdot, \rightarrow, A, \subseteq\rangle$ *is a quasi-ordered residuated system in which the residuum are given by*

$$(\forall X, Y \in \mathfrak{P}(A))(Y \rightarrow X := \{z \in A : Yz \subseteq X\}).$$

**Example 2.6.** *Let* $\mathbb{R}$ *be a field of real numbers. Define two binary operations '*$\cdot$*' and '*$\rightarrow$*' on* $A = [0,1] \subset \mathbb{R}$ *by*

$$(\forall x,y \in [0,1])(x \cdot y := max\{0, x+y-1\}) \text{ and } x \rightarrow y := min\{1, 1-x+y\}).$$

*Then, A is a commutative monoid with the identity* 1 *and* $\langle A, \cdot, \rightarrow, <, 1\rangle$ *is a quasi-ordered residuated system.*

**Example 2.7.** *Any commutative residuated lattice* $\langle A, \cdot, \rightarrow, 0, 1, \wedge, \vee, R\rangle$*, where R is a lattice quasi-order, is a quasi-ordered residuated system.*

The following example shows that a quasi-ordered residuated system $A$ does not have to be a lattice because:
- in the general case, $A$ does not have to have a common lower bound,
- $A$ doesn't have to be a lattice.

**Example 2.8.** *Let* $A = \langle -\infty, 1]\subset \mathbb{R}$ *(the real numbers field). If we define '*$\cdot$*' and '*$\rightarrow$*' as follows,* $(\forall y, v \in A)(u \cdot v := min\{u, v\})$ *and* $u \rightarrow v := 1$ *if* $u \leqslant v$ *and* $u \rightarrow v := v$ *if* $v < u$ *for all* $u, v \in A$*, then* $\mathfrak{A} := \langle A, \cdot, \rightarrow, 1, <\rangle$ *is a quasi-ordered residuated system.*

The following proposition shows the basic properties of quasi-ordered residuated systems.

**Proposition 2.9** ([2], Proposition 3.1)**.** *Let A be a quasi-ordered residuated system. Then*
(9) $(\forall x,y,z \in A)(x \preccurlyeq y \Longrightarrow (x \cdot z \preccurlyeq y \cdot z \wedge z \cdot x \preccurlyeq z \cdot y))$*;*
(10) $(\forall x,y,z \in A)(x \preccurlyeq y \Longrightarrow (y \rightarrow z \preccurlyeq x \rightarrow z \wedge z \rightarrow x \preccurlyeq z \rightarrow y))$*;*
(11) $(\forall x,y \in A)(x \cdot y \preccurlyeq x \wedge x \cdot y \preccurlyeq y)$.

Estimating that this topic is interesting ([1]-[3]), it is certain that there is interest in the development of the concept of some substructures such as some types of filters [4]-[7] in these systems.

## 2.2 Concepts of filters

In the article [3], in order to determine the concept of filters of quasi-ordered residuated systems, the relationships between the following conditions are analyzed:

(F0) $1 \in F$;

(F1) $(\forall u, v \in A)((u \cdot v \in F \implies (u \in F \wedge v \in F))$;

(F2) $(\forall u, v \in A)((u \in F \wedge u \preccurlyeq v) \implies v \in F)$;

(F3) $(\forall u, v \in A)((u \in F \wedge u \rightarrow v \in F) \implies v \in F)$.

It is shown ([3], Proposition 3.2) that $(F2) \Longrightarrow (F1)$. In addition, it is shown ([3], Proposition 3.4) that for every nonempty subset $F$ of system $\mathfrak{A}$ is valid $(F2) \Longrightarrow (F0)$.

Based on our previous analysis of the interrelationship between conditions (F1), (F2) and (F3) in a quasi-ordered residual system, we introduced the concept of filters in the following definition.

**Definition 2.10** ([3], Definition 3.1)**.** *For a subset $F$ of a quasi-ordered residuated system $\mathfrak{A}$, we say that it is a* filter *of $\mathfrak{A}$ if it satisfies conditions (F2) and (F3).*

**Example 2.11.** *Let $\mathfrak{A}$ be as in Example 2.8. All filters in this quasi-ordered residuated system are of the form $\langle -\infty, 1]$, where $x < 1$.*

**Lemma 2.12** ([4], Lemma 3.1)**.** *Let a subset $F$ of a quasi-ordered residuated system $\mathfrak{A}$ satisfy the condition* (F2)*. Then the following holds*

(12) $(\forall u \in A)(u \in F \iff 1 \rightarrow u \in F)$.

**Lemma 2.13** ([4], Lemma 3.4)**.** *Let a subset $F$ of a quasi-ordered residuated system $\mathfrak{A}$ satisfy the condition* (F2)*. Then the following holds*

(13) $(\forall u, v, z \in A)(u \rightarrow (v \rightarrow z) \in F \iff v \rightarrow (u \rightarrow z) \in F)$.

**Lemma 2.14** ([6])**.** *Let $\mathfrak{A}$ be a quasi-ordered residuated system. Then*

(14) $(\forall u, v, z \in A)(u \rightarrow v \preccurlyeq (v \rightarrow z) \rightarrow (u \rightarrow z))$ *and*

(15) $(\forall u, v, z \in A)(v \rightarrow z \preccurlyeq (u \rightarrow v) \rightarrow (u \rightarrow z))$.

Terms covering some of the requirements used herein to identify various types of filters in the observed algebraic structure are mostly taken from papers on UP-algebras.

**Definition 2.15** ([4], Definition 3.1)**.** *For a non-empty subset $F$ of a quasi-ordered residuated system $\mathfrak{A}$, we say that the* implicative filter *of $\mathfrak{A}$ if (F2) and the following condition*

*(IF)* $(\forall u, v, z \in A)((u \rightarrow (v \rightarrow z) \in F \wedge u \rightarrow v \in F) \implies u \rightarrow z \in F)$

*are valid.*

It is known that every implicative filter of a quasi-ordered residuated system $\mathfrak{A}$ is a filter of $\mathfrak{A}$ ([4], Theorem 3.1) but that the reverse does not have to be.

**Definition 2.16** ([6], Definition 5)**.** *For a non-empty subset $F$ of a quasi-ordered residuated system $\mathfrak{A}$ we say that a* comparative filter *of $\mathfrak{A}$ if (F2) and the following condition*

*(FC)* $(\forall u, v, z \in A)((u \rightarrow ((v \rightarrow z) \rightarrow v) \in F \wedge u \in F) \implies v \in F)$

*are valid.*

**Example 2.17.** *Let $\mathfrak{A}$ be a quasi-ordered residuated system as in Example 2.4. Then the set $F := \{1, a, b\}$ is a comparative filter of $\mathfrak{A}$.*

Since any comparative filter $F$ of $\mathfrak{A}$ satisfies the condition (F2), $F$ also satisfies the condition (F0): $1 \in F$.

**Proposition 2.18** ([6], Theorem 3.2)**.** *Let $F$ be a filter of a quasi-ordered residuated system $\mathfrak{A}$. Then $F$ is a comparative filter of $\mathfrak{A}$ if and only if the condition*

(16) $(\forall v, z \in A)((v \rightarrow z) \rightarrow v \in F \implies v \in F)$

*is valid.*

## 3. The Concept of Shift Filters

In this section, which is the main part of this article, we introduce and analyze the concept of shift filters of quasi-ordered residuated system.

**Definition 3.1.** *Let $\mathfrak{A}$ be a quasi-ordered residuated system. A non empty subset $F$ of $A$ is a* shift filter *of $\mathfrak{A}$ if it satisfies the conditions (F2) and the following condition*
*(SF)* $(\forall u,v,z \in A)((u \to (v \to z) \in F \wedge u \in F) \Longrightarrow ((z \to v) \to v) \to z \in F)$.

**Remark 3.2.** *In some other algebraic systems, request (SF) is recognized as a fantastic filter.*

**Example 3.3.** *Let $A = \{1,a,b,c\}$ and operations '$\cdot$' and '$\to$' defined on A as follows:*

| $\cdot$ | 1 | a | b | c |
|---|---|---|---|---|
| 1 | 1 | a | b | c |
| a | a | a | a | c |
| b | b | a | b | c |
| c | c | a | c | c |

and

| $\to$ | 1 | a | b | c |
|---|---|---|---|---|
| 1 | 1 | a | b | c |
| a | 1 | 1 | 1 | c |
| b | 1 | a | 1 | c |
| c | 1 | a | b | 1 |

.

*Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows*

$$\preccurlyeq = \{(1,1),(a,a),(b,b),(c,c),(a,1),(b,1),(c,1),(a,b)\}.$$

*Then the subsets $\{1,b\}$ is a shift filter of $\mathfrak{A}$.*

**Example 3.4.** *Let $A = \{1,a,b,c\}$ and operations '$\cdot$' and '$\to$' be defined on A as follows:*

| $\cdot$ | 1 | a | b | c |
|---|---|---|---|---|
| 1 | 1 | a | b | c |
| a | a | a | c | c |
| b | b | c | b | c |
| c | c | c | c | c |

and

| $\to$ | 1 | a | b | c |
|---|---|---|---|---|
| 1 | 1 | a | b | c |
| a | 1 | 1 | 1 | 1 |
| b | 1 | a | 1 | c |
| c | 1 | a | b | 1 |

.

*Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ is a quasi-ordered residuated systems, where the relation '$\preccurlyeq$' is defined as follows*

$$\preccurlyeq = \{(1,1),(a,a),(b,b),(c,c),(a,1),(b,1),(c,1),(a,b),(a,c)\}.$$

*Then the subsets $\{1,b\}$ is a filter of $\mathfrak{A}$ but it is not a shift filter of $\mathfrak{A}$. For example, for $u = b$, $v = a$ and $z = c$, we have $b \to (a \to c) = b \to 1 = 1 \in \{1,b\}$ and $b \in \{1,b\}$, but $((c \to a) \to a) \to c = (a \to a) \to c = 1 \to c = c \notin \{1,b\}$.*

It can be verified that a shift filter of a quasi-ordered residuated system $\mathfrak{A}$ has the following property:

**Proposition 3.5.** *Let $F$ be a shift filter of a quasi-ordered residuated system $\mathfrak{A}$. Then*
(17) $(\forall u,v \in A)(u \to v \in F \Longrightarrow ((v \to u) \to u) \to v \in F)$.

*Proof.* Let $F$ be a shift filter of $\mathfrak{A}$. If we put $u = 1$, $v = u$ and $z = v$ in (SF), we get

$$(1 \to (u \to v) \in F \wedge 1 \in F) \Longrightarrow ((v \to u) \to u) \to v \in F$$

whence it follows

$$u \to v \in F \Longrightarrow ((v \to u) \to u) \to v \in F$$

by (F0) and Lemma 2.12. □

Let us show now that the condition (17) is sufficient for a filter $F$ of a quasi-ordered system $\mathfrak{A}$ satisfying the condition (17) to be a shift filter of $\mathfrak{A}$.

**Theorem 3.6.** *Let $F$ be a filter of a quasi-ordered residuated system $\mathfrak{A}$ and suppose that $F$ satisfies the condition (17). Then $F$ is a shift filter of $\mathfrak{A}$.*

*Proof.* Suppose that $F$ is a filter of $\mathfrak{A}$ that satisfies the condition (17). Let $u,v,z \in A$ be such that $u \to (v \to z) \in F$ and $u \in F$. Then $v \to z$ by (F3). Thus $((z \to v) \to v) \to z \in F$ by (17). So, $F$ is a shift filter of $\mathfrak{A}$. □

Our second theorem on this class of filters of quasi-ordered residuated systems is the following:

**Theorem 3.7.** *Every comparative filter of a quasi-ordered residuared system $\mathfrak{A}$ is a shift filter of $\mathfrak{A}$.*

*Proof.* Suppose that $F$ is a comparative filter of $\mathfrak{A}$. To prove that $F$ is a shift filter of $\mathfrak{A}$, we will show that $F$ satisfies the condition (17). For this purpose, let us take elements $u,v \in A$ such that $u \to v \in F$.

From $u \to v \in F$ and from the valid formula (14) in the form $u \to v \preccurlyeq ((v \to u) \to u) \to ((v \to u) \to v)$, it follows $((v \to u) \to u) \to ((v \to u) \to v) \in F$ according (F2), which it is equivalent to

$$(v \to u) \to (((v \to u) \to u) \to v \in F$$

according to (13).

On the other hand, from the valid formula (11), in the form $(v \preccurlyeq (v \to u) \to u) \to v$, with respect to (14), we obtain

$$(((v \to u) \to u) \to v) \to u \preccurlyeq (v \to u).$$

From here, by acting with $((v \to u) \to u) \to v$ on the last inequality by the right, taking into account the valid formula (14), we obtain

$$(v \to u) \to (((v \to u) \to u) \to v) \preccurlyeq$$
$$((((v \to u) \to u) \to v) \to u) \to (((v \to u) \to u) \to v).$$

From here it follows

$$((((v \to u) \to u) \to v) \to u) \to (((v \to u) \to u) \to v) \in F.$$

Since $F$ is a comparative filter in $\mathfrak{A}$, we get $((v \to u) \to u) \to v \in F$ in accordance with (16). Therefore, $F$ is a shift filter. □

The following example shows that any shift filter of a quasi-ordered residuated system $\mathfrak{A}$ does not have to be a comparative filter of $\mathfrak{A}$.

**Example 3.8.** *Let $A = \{1,a,b,c\}$ and operations '$\cdot$' and '$\to$' be defined on $A$ as follows:*

| $\cdot$ | 1 | a | b | c |
|---|---|---|---|---|
| 1 | a | a | b | c |
| a | a | a | b | c |
| b | b | b | b | c |
| c | c | c | c | c |

*and*

| $\to$ | 1 | a | b | c |
|---|---|---|---|---|
| 1 | 1 | a | b | c |
| a | 1 | 1 | a | b |
| b | 1 | a | 1 | b |
| c | 1 | 1 | 1 | 1 |

*Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ is a quasi-ordered residuated systems, where the relation '$\preccurlyeq$' is defined as follows:*

$$\preccurlyeq = \{(1,1),(a,a),(b,b),(c,c),(a,1),(b,1),(c,1),(b,a),(c,a),(c,b)\}.$$

*Then the subsets $\{1\}$ is a shift filter of $\mathfrak{A}$ but it is not a comparative filter of $\mathfrak{A}$. For example, for $u = 1$, $v = a$ and $z = b$, we have $1 \to ((a \to b) \to a) = 1 \to (a \to a) = 1 \to 1 = 1 \in \{1\}$ and $1 \in \{1\}$, but $a \notin \{1\}$.*

**Theorem 3.9.** *Let $F$ be an implicative filter of a quasi-ordered residuated system $\mathfrak{A}$ satisfying*

(18) $(\forall u,v \in A)((u \to v) \to v \in F \implies (v \to u) \to u \in F)$.

*Then $F$ is a shift of $\mathfrak{A}$.*

*Proof.* The proof of this theorem is obtained by combining Theorem 4 in [6] and Theorem 3.7. □

The following example shows that any shift filter of a quasi-ordered residuated system $\mathfrak{A}$ does not have to be an implicative filter of $\mathfrak{A}$.

**Example 3.10.** *Let $A = \{1,a,b,c\}$ and operations '$\cdot$' and '$\to$' defined on $A$ as follows:*

| $\cdot$ | 1 | a | b | c |
|---|---|---|---|---|
| 1 | a | a | b | c |
| a | a | a | b | c |
| b | b | b | b | c |
| c | c | c | c | c |

and

| $\to$ | 1 | a | b | c |
|---|---|---|---|---|
| 1 | 1 | a | b | c |
| a | 1 | 1 | a | b |
| b | 1 | 1 | 1 | b |
| c | 1 | 1 | 1 | 1 |

*Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ is a quasi-ordered residuated systems, where the relation '$\preccurlyeq$' is defined as follows:*

$$\preccurlyeq = \{(1,1),(a,a),(b,b),(c,c),(a,1),(b,1),(c,1),(b,a),(c,a),(c,b)\}.$$

*Then the subsets $\{1\}$ is a shift filter of $\mathfrak{A}$ but it is not an implicative filter of $\mathfrak{A}$. For example, for $u = b$, $v = b$ and $z = c$, we have $b \to (b \to c) = 1 \in \{1\}$ and $b \to b = 1 \in \{1\}$, but $b \to c = b \notin \{1\}$.*

We end this section with the following theorem.

**Theorem 3.11.** *The family $\mathfrak{F}_s(A)$ of all shift filters of a quasi-ordered residuated system $\mathfrak{A}$ forms a complete lattice.*

*Proof.* Let $\{F_k\}_{k\in\Lambda}$ be a family of shift filters of $\mathfrak{A}$ where $\Lambda$ is index set. It is clear that $1 \in \bigcap_{k\in\Lambda} F_k$. Let $u, v \in \bigcap_{k\in\Lambda} F_k$ and $u \preccurlyeq v$. Then $u \in F_k$ and $u \preccurlyeq v$ for any $k \in \Lambda$. Thus $v \in F_k$ by (F2) since $F_k$ is a shift filter in $\mathfrak{A}$. Hence $v \in \bigcap_{k\in\Lambda} F_k$.

Let $u, v, z \in A$ be such that $u \to (v \to z) \in \bigcap_{k\in\Lambda} F_k$ and $u \in \bigcap_{k\in\Lambda} F_k$. Then $u \to (v \to z) \in F_k$ and $u \in F_k$ for any $k \in \Lambda$. Thus $((z \to v) \to v) \to z \in F_k$ for all $k \in \Lambda$. Hence $((z \to v) \to v) \to z \in \bigcap_{k\in\Lambda} F_k$. So, the intersection $\bigcap_{k\in\Lambda} F_k$ satisfies the condition (SF). Therefore $\bigcap_{k\in\Lambda} F_k$ is a shift filter of $\mathfrak{A}$.

Let $\mathfrak{X}$ be the family of all shift filters containing the union $\bigcup_{k\in\Lambda} F_k$. Then $\cap\mathfrak{X}$ is a shift filter of $\mathfrak{A}$ according to the first part of this proof.

If we put $\sqcap_{k\in\Lambda} F_k = \bigcap_{k\in\Lambda} F_k$ and $\sqcup_{k\in\Lambda} F_k = \cap\mathfrak{X}$, then $(\mathfrak{F}_s(A), \sqcap, \sqcup)$ is a complete lattice. □

Let $\mathfrak{A}$ be a quasi-ordered residuated system. Before embarking on further conclusions, let us recall the terms 'minimum filter' and 'maximum filter' in a quasi-ordered residuated system: We shall say that a filter $A$ is a minimal filter of $\mathfrak{A}$ if there is no a filter $B$ of $\mathfrak{A}$ such that $B \subset A$. Also, dually, we shall say that a filter $A$ is a maximal filter of $\mathfrak{A}$ if there is no a filter $B$ of $\mathfrak{A}$ such that $A \subset B$. It is easy to conclude that if $A$ and $B$ are two minimum interiors filters of a quasi-ordered residuated system $\mathfrak{A}$, then $A \cap B = \emptyset$, because, otherwise, according to the previous theorem, $A \cap B$ would be a filter of $\mathfrak{A}$ contained in $A$ and contained in $B$, which is impossible.

**Corollary 3.12.** *Let $\mathfrak{A}$ be a quasi-ordered residuated system. For any subset $T$ of $A$, there is the unique minimum shift filter of $\mathfrak{A}$ that contains $T$.*

*Proof.* The proof of this Corollary follows directly from the second part of the proof of the previous theorem. □

**Corollary 3.13.** *Let $\mathfrak{A}$ be a quasi-ordered residuated system. For any element $x$ of $A$, there is the unique minimum shift filter of $\mathfrak{A}$ that contains $x$.*

*Proof.* The proof of this Corollary follows from the previous Corollary if we take $T = \{x\}$. □

## 4. Conclusion

The concept of quasi-ordered residuated systems was introduced in 2018 by S. Bonzio and I. Chajda. as a structure $\mathfrak{A} = \langle A, \cdot, \to, 1, R\rangle$, where $(A, \cdot)$ is a commutative semigroup with the identity 1 as the top element in this ordered monoid under a quasi-order $R$. In such algebraic systems, the author introduced the concept of filters, and then several types of filters such as implicative [4], associated [5] and comparative filters [6]. It is shown that a comparative filter is an implicative filter and vice versa does not have to be.

The concept of shift filters of such algebraic systems was introduced and analyzed in this paper. Also, this class of filters was compared with previously introduced filters. It is shown (Theorem 3.2) that every comparative filter is a shift filter and vice versa does not have to be. In addition, it is shown (Theorem 3.3) that if the implicative filter $F$ satisfies the added condition

$$(\forall u, v \in A)((u \to v) \to v \in F \implies (v \to u) \to u \in F))$$

then $F$ is a shift filter. The reverse, of course, does not have to be.

In our paper [8], we analyze a quasi-ordered residuated system (which we call the 'strong quasi-ordered residuated system') in which implicative and comparative filters are coincide. It is a quasi-ordered residuated system in which the formula

$$(\forall u, v \in A)((u \to v) \to v \preccurlyeq (v \to u) \to u \wedge (v \to u) \to u \preccurlyeq (u \to v) \to v)$$

is a valid formula. We also analyze the possibility of the existence of some new types of filters in such systems as prime and irreducible filters and their interrelationships ([9, 10]).

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## Competing interests

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