http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 72, Number 4, Pages 959–975 (2023)
DOI:10.31801/cfsuasmas.1089389
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: March 17, 2022; Accepted: July 30, 2023

QUASI HEMI-SLANT PSEUDO-RIEMANNIAN SUBMERSIONS IN PARA-COMPLEX GEOMETRY

Esra BAŞARIR NOYAN[1] and Yılmaz GÜNDÜZALP[2]

[1,2]Department of Mathematics, Dicle University, 21280, Sur, Diyarbakır, TÜRKİYE

Abstract. We introduce a new class of pseudo-Riemannian submersions which are called quasi hemi-slant pseudo-Riemannian submersions from para-Kaehler manifolds to pseudo-Riemannian manifolds as a natural generalization of slant submersions, semi-invariant submersions, semi-slant submersions and hemi-slant Riemannian submersions in our study. Also, we give non-trivial examples of such submersions. Further, some geometric properties with two types of quasi hemi-slant pseudo-Riemannian submersions are investigated.

1. Introduction

A $C^\infty-$submersion ψ can be defined according to the following conditions. A pseudo-Riemannian submersion ([12], [16], [13], [17], [26]), an almost Hermitian submersion ([27], [29]), bi-slant submanifold ([3], [5]), a slant submersion ([7], [11], [1], [19], [23]), bi-slant submersion ([21]), an anti-invariant submersion ([8], [9], [10], [24]), a hemi-slant submersion ([28], [22]), a quasi-bi-slant Submersion ([20]), a semi-invariant submersion ([18], [25]), etc. As we know, Riemannian submersions were severally introduced by B. O'Neill ([17]) and A. Gray ([12]) in 1960s. In particular, by using the concept of almost Hermitian submersions, B. Watson ([30]) gave some differential geometric properties among fibers, base manifolds, and total manifolds. Some interesting results concerning para-Kaehler-like statistical submersions were obtained by G.E. Vîlcu ([29]).
Motivated by the above studies, we presented quasi hemi-slant pseudo-Riemannian submersions in para-complex geometry from para-Kaehler manifolds onto pseudo-Riemannian manifolds. We organized our work in three sections. In section 2, we gather basic concepts and definitions needed in the following parts. In section 3,

2020 *Mathematics Subject Classification.* 53C43, 53B20, 53C40.
Keywords. Quasi hemi-slant submersion, hemi-slant submersion, para-Kaehler manifold, pseudo-Riemannian submersions.

[1] bsrrnoyan@gmail.com; 0000-0001-6535-7498
[2] ygunduzalp@dicle.edu.tr-Corresponding author; 0000-0002-0932-949X.

We examined quasi hemi-slant pseudo-Riemannian submersions in para-complex geometry that satisfies certain conditions. We give some non-trivial examples of these submersions which satisfy the conditions of two types, while in we study the decomposition theorem of two types of the distributions.

2. Preliminaries

By a para-Hermitian manifold we mean a triple $(\mathcal{B}, \mathcal{P}, g_{\mathcal{B}})$, where $\mathcal{B}$ is connected differentiable manifold of $2n$- dimensional , $\mathcal{P}$ is a tensor field of type (1,1) and a pseudo-Riemannian metric $g_{\mathcal{B}}$ on $\mathcal{B}$, satisfying

$$\mathcal{P}^2 E_1 = E_1, \quad g_{\mathcal{B}}(\mathcal{P}E_1, \mathcal{P}E_2) = -g_{\mathcal{B}}(E_1, E_2) \tag{1}$$

where E_1, E_2 are vector fields on $\mathcal{B}$. Then we can say that $\mathcal{B}$ is a para-Kaehler manifold such that

$$\nabla \mathcal{P} = 0; \tag{2}$$

where ∇ denotes the Levi-Civita connection on $\mathcal{B}$ ([15]).

Let $(\mathcal{B}, g_{\mathcal{B}})$ and $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be two pseudo-Riemannian manifolds. Being a pseudo-Riemannian submersion $\psi : \mathcal{B} \to \tilde{\mathcal{B}}$ provides the following three properties;
(i) $\psi_{*|p}$ is onto for all $p \in \mathcal{B}$,
(ii) the fibres $\psi^{-1}(q)$, $q \in \tilde{\mathcal{B}}$, are $r-$ dimensional pseudo-Riemannian submanifolds of $\mathcal{B}$, where $r = dim(\mathcal{B}) - dim(\tilde{\mathcal{B}})$,
(iii) ψ_* preserves scalar products of vectors normal to fibres.

The vectors tangent to the fibres are called vertical and those normal to the fibres are called horizontal. A vector field U on $\mathcal{B}$ is called basic if U is horizontal and ψ- related to a vector field U_* on $\tilde{\mathcal{B}}$, i.e., $\psi_* U_p = U_{*\psi_p}$ for all $p \in \mathcal{B}$. We indicate by $\mathcal{V}$ the vertical distribution, by $\mathcal{H}$ the horizontal distribution and by v and h the vertical and horizontal projection. We know that $(\mathcal{B}, g_{\mathcal{B}})$ is called total manifold and $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ is called base manifold of the submersion $\psi : (\mathcal{B}, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$.

Now, let's denote O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$:

$$\mathcal{T}_U \mathcal{W} = h\nabla_{vU} v\mathcal{W} + v\nabla_{vU} h\mathcal{W} \tag{3}$$

and

$$\mathcal{A}_U \mathcal{W} = v\nabla_{hU} h\mathcal{W} + h\nabla_{hU} v\mathcal{W} \tag{4}$$

for every $U, \mathcal{W} \in \chi(\mathcal{B})$, on $\mathcal{B}$ where ∇ is the Levi-Civita connection of $g_{\mathcal{B}}$.
Further, a pseudo-Riemannian submersion $\psi : \mathcal{B} \to \tilde{\mathcal{B}}$ has totally geodesic fibers if and only if $\mathcal{T} \equiv 0$. Also, if $\mathcal{A}$ vanishes then the horizontal distribution is integrable(see [4], [6]). Using (3) and (4), we get

$$\nabla_U W = \mathcal{T}_U W + \hat{\nabla}_U W; \tag{5}$$

$$\nabla_U \zeta = \mathcal{T}_U \zeta + h\nabla_U \zeta; \tag{6}$$

$$\nabla_\zeta U = \mathcal{A}_\zeta U + v\nabla_\zeta U; \tag{7}$$

$$\nabla_\zeta \eta = \mathcal{A}_\zeta \eta + h\nabla_\zeta \eta, \tag{8}$$

for any $\zeta, \eta \in \Gamma((ker\psi_*)^\perp)$, $U, W \in \Gamma(ker\psi_*)$. Also, if ζ is basic then $h\nabla_U \zeta = h\nabla_\zeta U = \mathcal{A}_\zeta U$.

We can easily see that $\mathcal{T}$ is symmetric on the vertical distribution and $\mathcal{A}$ is alternating on the horizontal distribution such that

$$\mathcal{T}_{\mathcal{W}} U = \mathcal{T}_U \mathcal{W}, \quad \mathcal{W}, U \in \Gamma(ker\psi_*); \tag{9}$$

$$\mathcal{A}_Y V = -\mathcal{A}_V Y = \frac{1}{2} v[Y, V], \quad Y, V \in \Gamma((ker\psi_*)^\perp). \tag{10}$$

Also, it is easily seen that for any $\wp \in \Gamma(T\mathcal{B})$, $\mathcal{T}_\wp$ and $\mathcal{A}_\wp$ are skew-symmetric operators on $\Gamma(T\mathcal{B})$, such that

$$g_{\mathcal{B}}(\mathcal{T}_{\mathcal{W}} U, \mathcal{X}) = -g_{\mathcal{B}}(\mathcal{T}_{\mathcal{W}} \mathcal{X}, U) \tag{11}$$

$$g_{\mathcal{B}}(\mathcal{A}_{\mathcal{W}} U, \mathcal{X}) = -g_{\mathcal{B}}(\mathcal{A}_{\mathcal{W}} \mathcal{X}, U) \tag{12}$$

Definition 1. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian submersion. Let us assume that the total manifold as an almost para-Hermitian manifold and base manifold as a pseudo-Riemannian manifold. Then, there exists a pseudo-Riemannian submersion ψ is an invariant pseudo-Riemannian submersion if the vertical distribution is invariant with respect to $\mathcal{P}$, i.e., $\mathcal{P}(ker\psi_*) = (ker\psi_*)$([10].*

Definition 2. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian submersion. Let us assume that the total manifold as an almost para-Hermitian manifold and base manifold as a pseudo-Riemannian manifold. Then, there exists a pseudo-Riemannian submersion ψ such that $ker\psi_*$ is anti-invariant with respect to $\mathcal{P}$, i.e., $\mathcal{P}(ker\psi_*) \subseteq (ker\psi_*)^\perp$. So, we can say ψ is an anti-invariant pseudo-Riemannian submersion([8]).*

Definition 3. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian submersion. Let us assume that the total manifold as an almost para-Hermitian manifold and base manifold as a pseudo-Riemannian manifold. Then, there exists a pseudo-Riemannian submersion ψ is a semi-invariant pseudo-Riemannian submersion if there is a distribution $\mathtt{D}_1 \subseteq ker\psi_*$, such that*

$$ker\psi_* = \mathtt{D}_1 \oplus \mathtt{D}_2,$$

and

$$\mathcal{P}\mathtt{D}_1 = \mathtt{D}_1, \mathcal{P}\mathtt{D}_2 \subseteq (ker\psi_*)^\perp$$

where $\mathtt{D}_2$ is orthogonal complementary to $\mathtt{D}_1$ in $ker\psi_*$([2]).

We know that μ is the complementary orthogonal subbundle to $\mathcal{P}(ker\psi_*)$ in $(ker\psi_*)^\perp$.
Also we have;

$$(ker\psi_*)^\perp = \mathcal{P}\mathtt{D}_2 \oplus \mu.$$

From here we can say that μ is an invariant subbundle of $(ker\psi_*)^\perp$ with respect to the para-complex structure $\mathcal{P}$.

For any non-null vector field $U_2 \in (ker\psi_*)$, we get

$$\mathcal{P}U_2 = qU_2 + rU_2,$$

where qU_2 is vertical part and rU_2 is horizontal part.

If for non-null vector field $U_2 \in ker\psi_*$, the quotient $\frac{g_\mathcal{B}(qU_2,qU_2)}{g_\mathcal{B}(\mathcal{P}U_2,\mathcal{P}U_2)}$ is constant, i.e., it is independent of the choice of the point $\bar{q} \in \mathcal{B}$ and choice of the non-null vector field $U_2 \in \Gamma(ker\psi_*)$, we can say that ψ is a slant submersion. So, the angle is called the slant angle of the slant submersion ([10]).

Let $\psi : (\mathcal{B}, g_\mathcal{B}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper slant submersion. Let us assume that the total manifold as an almost para-Hermitian manifold and base manifold as a pseudo-Riemannian manifold. Then, we have;
type $\sim$1 if for every space-like (time-like) vector field $U_2 \in \Gamma(ker\psi_*)$, qU_2 is time-like (space-like), and $\frac{\|qU_2\|}{\|\mathcal{P}U_2\|} > 1$,
type $\sim$ 2 if for every space-like (time-like) vector field $U_2 \in \Gamma(ker\psi_*)$, qU_2 is time-like (space-like), and $\frac{\|qU_2\|}{\|\mathcal{P}U_2\|} < 1$([10]).

Theorem 1. *([10]) Let $\psi : (\mathcal{B}, g_\mathcal{B}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper slant submersion. Let us assume that the total manifold as an almost para-Hermitian manifold and base manifold as a pseudo-Riemannian manifold. Then,*
(a) *ψ is slant submersion of type-1 if and only if for any space-like (time-like) vector field $U_1 \in ker\psi_*$, qU_1 is time-like (space-like) and there exists a constant $\mu \in (1, +\infty)$ such that*

$$q^2 = \mu Id.$$

where Id is the identity operator. If ψ is a proper slant submersion of type-1, then $\mu = \cosh^2 \varphi$, with $\varphi > 0$.
(b) *ψ is slant submersion of type-1 if and only if for any space-like (time-like) vector field $U_1 \in ker\psi_*$, qU_1 is time-like (space-like) and there exists a constant $\mu \in (0, 1)$ such that*

$$q^2 = \mu Id.$$

where Id is identity operator. If ψ is a proper slant submersion of type-1, then $\mu = \cos^2 \varphi$, with $0 < \varphi < \frac{\pi}{2}$.

Definition 4. *Let $(\mathcal{B}, g_\mathcal{B}, \mathcal{P})$ be an almost para-Hermitian manifold and $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian manifold. A pseudo-Riemannian submersion $\psi : (\mathcal{B}, g_\mathcal{B}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ is known a semi-slant submersion if there is a distribution $\mathtt{D}_1 \in ker\psi_*$ such* *that*

$$ker\psi_* = \mathtt{D}_1 \oplus \mathtt{D}_2, \quad \mathcal{P}(\mathtt{D}_1) = \mathtt{D}_1$$

and the angle φ is known the semi-slant angle of the submersion where $\mathtt{D}_2$ is the orthogonal complement of $\mathtt{D}_1$ in $ker\psi_*$.

Definition 5. *Let $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ be an almost para-Hermitian manifold and $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian manifold. A pseudo-Riemannian submersion $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ is known a hemi-slant submersion if the vertical distribution $ker\psi_*$ of ψ accepts two orthogonal complementary distribution $\mathtt{D}^\varphi$ and $\mathtt{D}^\perp$, such that $\mathtt{D}^\varphi$ is slant and $\mathtt{D}^\perp$ is anti-invariant, i.e., we can show*

$$ker\psi_* = \mathtt{D}^\varphi \oplus \mathtt{D}^\perp$$

Therefore, the angle φ is known the hemi-slant angle of the submersion.

$\psi : \mathcal{B} \to \tilde{\mathcal{B}}$ is a differentiable map and $(\mathcal{B}, g_{\mathcal{B}})$ and $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be pseudo-Riemannian manifolds. Then, the second fundamental form of ψ is described by

$$(\nabla\psi_*)(\zeta, V) = \nabla^\psi_\zeta \psi_* V - \psi_*(\nabla_\zeta V) \tag{13}$$

for $\zeta, V \in \Gamma(\mathcal{B})$. When $trace(\nabla\psi_*) = 0$, we can say that ψ is *harmonic* and ψ is a *totally geodesic* map when $(\nabla\psi_*)(\zeta, V) = 0$ for $\zeta, V \in \Gamma(T\mathcal{B})$ ([14]). Recall that ∇^ψ is the pullback connection.

3. Quasi Hemi-Slant Submersions

Definition 6. *Let $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ be an almost para-Hermitian manifold and $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian manifold. A pseudo-Riemannian submersion $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ is known a quasi hemi-slant submersion if there are three orthogonal distributions $\mathtt{D}$, $\mathtt{D}^\varphi$ and $\mathtt{D}^\perp$, such that*

- $ker\psi_* = \mathtt{D} \oplus \mathtt{D}^\varphi \oplus \mathtt{D}^\perp$,
- $\mathcal{P}(\mathtt{D}) = \mathtt{D}$ *i.e., $\mathtt{D}$ is invariant,*
- *the angle φ between $\mathcal{P}U$ and $\mathtt{D}^\varphi$ is constant. Also, the angle φ is known slant angle.*
- $\mathtt{D}^\perp$ *is anti-invariant, $\mathcal{P}\mathtt{D}^\perp \subseteq (ker\psi_*)^\perp$.*

We can say that φ is quasi hemi-slant angle of $\mathcal{B}$.

Now, if we show the dimension of $\mathtt{D}$, $\mathtt{D}^\varphi$ and $\mathtt{D}^\perp$, by n_1, n_2 and n_3, respectively, we can easily notice the following situations:

(1) If $n_1 = 0$, then $\mathcal{B}$ is a hemi-slant submersion
(2) If $n_2 = 0$, then $\mathcal{B}$ is a semi-invariant submersion
(3) If $n_3 = 0$, then $\mathcal{B}$ is a semi-slant submersion

If we observe the three items above , we can say that also they are all examples of quasi hemi-slant submersion.

Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a quasi hemi-slant submersion with type-1 or 2. Then, we obtain;

$$TB = ker\psi_* \oplus (ker\psi_*)^{\perp} \tag{14}$$

For any non-null vector field $U \in (ker\psi_*)$, we get

$$U = KU + LU + RU, \tag{15}$$

where KU, LU and RU are projection morphisms of $ker\psi_*$ onto $\mathtt{D}$, $\mathtt{D}^{\varphi}$ and $\mathtt{D}^{\perp}$, respectively.
We denote endomorphisms ϕ, the projection morphisms f on $\mathcal{B}$. For non-null vector field $U \in (ker\psi_*)$, we have

$$\mathcal{P}U = \phi U + fU, \tag{16}$$

where $\phi U \in ker\psi_*$ and $fU \in (ker\psi_*)^{\perp}$.

From (15) and (16) we get:

$$\begin{aligned} \mathcal{P}U &= \mathcal{P}(KU) + \mathcal{P}(LU) + \mathcal{P}(RU), \\ &= \phi(KU) + f(KU) + \phi(LU) + f(LU) + \phi(RU) + f(RU). \end{aligned}$$

Since $\mathcal{P}(\mathtt{D}) = (\mathtt{D})$ and $\mathcal{P}\mathtt{D}^{\perp} \subseteq (ker\psi_*)^{\perp}$ we obtain $f(KU) = 0$ and $\phi(RU) = 0$. Now, let us arrange the above equation

$$\mathcal{P}U = \phi(KU) + \phi(LU) + f(LU) + f(RU). \tag{17}$$

So, we have the following decomposition:

$$\mathcal{P}(ker\psi_*) = \mathtt{D} \oplus \phi\mathtt{D}^{\varphi} \oplus f\mathtt{D}^{\varphi} \oplus \mathcal{P}\mathtt{D}^{\perp}. \tag{18}$$

Since, $f\mathtt{D}^{\varphi} \subseteq (ker\psi_*)^{\perp}$ and $\mathcal{P}\mathtt{D}^{\perp} \subseteq (ker\psi_*)^{\perp}$, we have;

$$(ker\psi_*)^{\perp} = f\mathtt{D}^{\varphi} \oplus \mathcal{P}\mathtt{D}^{\perp} \oplus \mu$$

where μ is the orthogonal complementary distribution of $f\mathtt{D}^{\varphi} \oplus \mathcal{P}\mathtt{D}^{\perp}$ in $(ker\psi_*)^{\perp}$.
In adittion, for any non-null vector field $W \in (ker\psi_*)^{\perp}$ is decomposed as

$$\mathcal{P}W = BW + CW \tag{19}$$

where $BW \in \Gamma(\mathtt{D}^{\varphi} \oplus \mathtt{D}^{\perp})$ and $CW \in \Gamma(\mu)$.

Lemma 1. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ is a quasi hemi-slant submersion with type ∼1 or 2. Let us suppose the total manifold as an almost para-Hermitian manifold and base manifold as a pseudo-Riemannian manifold. Then, we obtain the following equations:*

(a) $\phi\mathtt{D}^{\varphi} = \mathtt{D}^{\varphi}$ **(b)** $\phi\mathtt{D}^{\perp} = \{0\}$
(c) $\mathtt{BfD}^{\varphi} = \mathtt{D}^{\varphi}$ **(d)** $\mathtt{BfD}^{\perp} = \mathtt{D}^{\perp}$.

Proof. For any non-null vector field $W \in \Gamma(\mathtt{D}^{\varphi})$, by (16), we have $\mathcal{P}W = \phi W + fW$. On the other hand, with the help of (18), $\mathcal{P}W \in \Gamma(\mathtt{D}^{\varphi})$, i.e., $fW = 0$. Thus, we obtain $\phi\mathtt{D}^{\varphi} = \mathtt{D}^{\varphi}$. For any non-null vector field $U \in \Gamma(\mathtt{D}^{\perp})$, by (16), we have $\mathcal{P}U = \phi U + fU$. Beside this, by using (18), $\mathcal{P}W \in (ker\psi_*)^{\perp}$, i.e., $\phi U = 0$. Thus, we obtain $\phi\mathtt{D}^{\perp} = \{0\}$. To prove (c) and (d), the same method above can be used. □

Lemma 2. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ is a quasi hemi-slant submersion with type ∼1 or 2. Let us suppose the total manifold as an almost para-Hermitian manifold and base manifold as a pseudo-Riemannian manifold. Then, we obtain the following equations:*
(**a**) $\phi^2\mathcal{Z} + Bf\mathcal{Z} = \mathcal{Z}$ (**b**) $C^2U + fBU = U$
(**c**) $\phi BU + BCU = \{0\}$ (**d**) $f\phi\mathcal{Z} + Cf\mathcal{Z} = \{0\}$ *for all non-null vectors $\mathcal{Z} \in \Gamma(ker\psi_*)$ and $U \in \Gamma(ker\psi_*)^{\perp}$.*

Proof. For any non-null vector field $\mathcal{Z} \in \Gamma(ker\psi_*)$, by (1), we have $\mathcal{P}^2\mathcal{Z} = \mathcal{Z}$. Using (16) and (19), we have $\mathcal{Z} = \phi^2\mathcal{Z} + f\phi\mathcal{Z} + Bf\mathcal{Z} + Cf\mathcal{Z}$. If this equation is considered as decomposed into the vertical and horizontal parts, we obtain (a) and (d). (b) and (c) can be proved with the same method above. □

Theorem 2. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a quasi hemi-slant submersion with type∼1. Let us suppose the total manifold as an almost para-Hermitian manifold and base manifold as a pseudo-Riemannian manifold. In this case, ψ is quasi-hemi-slant submersion such that:*

(a) $\phi^2\mathcal{Z} = \cosh^2\varphi\mathcal{Z}$
(b) $g_{\mathcal{B}}(\phi\mathcal{Z}, \phi Y) = -\cosh^2\varphi g_{\mathcal{B}}(\mathcal{Z}, Y)$
(c) $g_{\mathcal{B}}(f\mathcal{Z}, fY) = \sinh^2\varphi g_{\mathcal{B}}(\mathcal{Z}, Y)$

for any space-like(time-like) vector field $\mathcal{Z}, Y \in \Gamma(\mathtt{D}^{\varphi})$.

Proof. (a) If ψ is a quasi hemi-slant submersion of type 1, for any space-like vector field $\mathcal{Z} \in \Gamma(\mathtt{D}^{\varphi})$, $\phi\mathcal{Z}$ is timelike and by virtue of (1), $\mathcal{P}\mathcal{Z}$ is time-like. Then, there exists $\varphi > 0$ such that

$$\cosh\varphi = \frac{\|\phi\mathcal{Z}\|}{\|\mathcal{P}\mathcal{Z}\|} = \frac{\sqrt{-g_{\mathcal{B}}(\phi\mathcal{Z}, \phi\mathcal{Z})}}{\sqrt{-g_{\mathcal{B}}(\mathcal{P}\mathcal{Z}, \mathcal{P}\mathcal{Z})}}.$$

Using the above equation, (1) and (16), we get:

$$g_{\mathcal{B}}(\phi^2\mathcal{Z}, \mathcal{Z}) = -g_{\mathcal{B}}(\phi\mathcal{Z}, \phi\mathcal{Z}) = -\cosh^2\varphi g_{\mathcal{B}}(\mathcal{P}\mathcal{Z}, \mathcal{P}\mathcal{Z}) = \cosh^2\varphi g_{\mathcal{B}}(\mathcal{P}^2\mathcal{Z}, \mathcal{Z}).$$

From the above equation and (1), we obtain $\phi^2\mathcal{Z} = \cosh^2\varphi\mathcal{Z}$.
Everything works in a similar way for any time-like vector field $\mathcal{Z} \in \Gamma(\mathtt{D}^{\varphi})$.

(b) For any space-like(time-like) vector field $\mathcal{Z}, Y \in \Gamma(\mathtt{D}^{\varphi})$, by virtue of (1), we get $g_{\mathcal{B}}(\mathcal{P}\mathcal{Z}, Y) = -g_{\mathcal{B}}(\mathcal{Z}, \mathcal{P}Y)$. On the other hand, with the help of (16), we get $g_{\mathcal{B}}(\phi\mathcal{Z} + f\mathcal{Z}, Y) = -g_{\mathcal{B}}(\mathcal{Z}, \phi Y + fY)$. If we arrange the last equation, we

obtain $g_{\mathcal{B}}(\phi\mathcal{Z}, Y) = -g_{\mathcal{B}}(\mathcal{Z}, \phi Y)$. Beside this, if $Y = \phi Y$ is accepted, we obtain $g_{\mathcal{B}}(\phi\mathcal{Z}, \phi Y) = -g_{\mathcal{B}}(\mathcal{Z}, \phi^2 Y)$. Using Theorem 2(a), we get $g_{\mathcal{B}}(\phi\mathcal{Z}, \phi Y) = -\cosh^2\varphi g_{\mathcal{B}}(\mathcal{Z}, Y)$
To prove (c), the same method above can be used. □

Theorem 3. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a quasi hemi-slant submersion with type∼2. Let us suppose the total manifold as an almost para-Hermitian manifold and base manifold as a pseudo-Riemannian manifold. In this case, ψ is quasi hemi-slant submersion such that:*

(a) $\phi^2\mathcal{Z} = \cos^2\varphi\mathcal{Z}$
(b) $g_{\mathcal{B}}(\phi\mathcal{Z}, \phi Y) = -\cos^2\varphi g_{\mathcal{B}}(\mathcal{Z}, Y)$
(c) $g_{\mathcal{B}}(f\mathcal{Z}, fY) = -\sin^2\varphi g_{\mathcal{B}}(\mathcal{Z}, Y)$

for any space-like(time-like) vector field $\mathcal{Z}, Y \in \Gamma(\mathtt{D}^{\varphi})$.

Proof. This proof can be done using the techniques of the proof of Theorem 2.

Let's consider para-complex structure on R^{2n}_n :

$$P(\frac{\partial}{\partial y_{2i}}) = \frac{\partial}{\partial y_{2i-1}}, \ \ P(\frac{\partial}{\partial y_{2i-1}}) = \frac{\partial}{\partial y_{2i}}, \ \ g = (dy^1)^2 - (dy^2)^2 + (dy^3)^2 - ... - (dy^{2n})^2$$

here $i \in \{1, ..., n\}$. Also, $(y_1, y_2, ..., y_{2n})$ denotes the cartesian coordinates over R^{2n}_n. □

We can easily present non-trivial examples of proper quasi hemi-slant pseudo-Riemannian submersions of type∼1 and 2.

Example 1. *Let's determine map $\psi : R^{10}_5 \to R^5_2$*

$$\psi(y_1, ..., y_{10}) = (y_2 \sinh\beta + y_3 \cosh\beta, y_4, y_6, y_9, y_{10}),$$

So, ψ is a proper quasi hemi-slant pseudo-Riemannian submersion with type ~ 1. By direct calculations, we have

$\mathtt{D} = <\frac{\partial}{\partial y_7}, \frac{\partial}{\partial y_8}>$

$\mathtt{D}^{\varphi} = <\cosh\beta\frac{\partial}{\partial y_2} - \sinh\beta\frac{\partial}{\partial y_3}, \frac{\partial}{\partial y_1}>$

$\mathtt{D}^{\perp} = <\frac{\partial}{\partial y_5}>$

with hemi-slant angle φ with $\phi^2 = \cosh^2\beta I$.

Example 2. *Let's determine map $\psi : R^{10}_5 \to R^5_2$*

$$\psi(y_1, ..., y_{10}) = (y_1 \sin\alpha + y_3 \cos\alpha, y_2 \sin\beta + y_4 \cos\beta, y_6, y_9, y_{10})$$

So, ψ is a proper quasi hemi-slant pseudo-Riemannian submersion with type ~ 2. By direct calculations, we get

$$\mathtt{D} =< \frac{\partial}{\partial y_7}, \frac{\partial}{\partial y_8} >$$

$$\mathtt{D}^{\varphi} =< -\cos\alpha\frac{\partial}{\partial y_1} + \sin\alpha\frac{\partial}{\partial y_3}, -\cos\beta\frac{\partial}{\partial y_2} + \sin\beta\frac{\partial}{\partial y_4} >$$

$\mathtt{D}^{\perp} =< \frac{\partial}{\partial y_5} >$ *with hemi-slant angle φ with $\phi^2 = \cos^2(\alpha - \beta)I$.*

Lemma 3. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a quasi hemi-slant pseudo-Riemannian submersion with type ∼1 or 2. Let us suppose the total manifold as a para-Kaehler manifold and base manifold as a pseudo-Riemannian manifold. So, we obtain the following equations.*

$$\hat{\nabla}_U \phi W + \mathcal{T}_U fW = \phi\hat{\nabla}_U W + \mathcal{B}\mathcal{T}_U W \tag{20}$$

$$\mathcal{T}_U \phi W + \mathcal{H}\nabla_U fW = f\hat{\nabla}_U W + \mathcal{C}\mathcal{T}_U W \tag{21}$$

$$\mathcal{V}\nabla_{\mathcal{X}}\mathcal{B}\mathcal{Y} + \mathcal{A}_{\mathcal{X}}\mathcal{C}\mathcal{Y} = \phi\mathcal{A}_{\mathcal{X}}\mathcal{Y} + \mathcal{B}\mathcal{H}\nabla_{\mathcal{X}}\mathcal{Y} \tag{22}$$

$$\mathcal{A}_{\mathcal{X}}\mathcal{B}\mathcal{Y} + \mathcal{H}\nabla_{\mathcal{X}}\mathcal{C}\mathcal{Y} = f\mathcal{A}_{\mathcal{X}}\mathcal{Y} + \mathcal{C}\mathcal{H}\nabla_{\mathcal{X}}\mathcal{Y} \tag{23}$$

$$\hat{\nabla}_U \mathcal{B}\mathcal{X} + \mathcal{T}_U \mathcal{C}\mathcal{X} = \phi\mathcal{T}_U \mathcal{X} + \mathcal{B}\mathcal{H}\nabla_U \mathcal{X} \tag{24}$$

$$\mathcal{T}_U \mathcal{B}\mathcal{X} + \mathcal{H}\nabla_U \mathcal{C}\mathcal{X} = f\mathcal{T}_U \mathcal{X} + \mathcal{C}\mathcal{H}\nabla_U \mathcal{X}, \tag{25}$$

for any non-null vector fields $U, W \in \Gamma(ker\psi_*)$ and $\mathcal{X}, \mathcal{Y} \in \Gamma(ker\psi_*)^{\perp}$.

Proof. For any non-null vector fields $U, W \in \Gamma(ker\psi_*)$, using (2), we get

$$\mathcal{P}\nabla_U W = \nabla_U \mathcal{P}W$$

Hence, using (5)∼(6)∼(16) and (19), we get

$$\mathcal{B}\mathcal{T}_U W + \mathcal{C}\mathcal{T}_U W + \phi\hat{\nabla}_U W + f\hat{\nabla}_U W = \mathcal{T}_U \phi W + \hat{\nabla}_U \phi W + \mathcal{T}_U fW + \mathcal{H}\nabla_U fW$$

Taking the vertical and horizontal parts of this equation, we get (20) and (21). The other assertions can be obtained by using (7)∼(8)∼(16) and (19).

Now we can show

$$(\nabla_U \phi)W = \hat{\nabla}_U \phi W - \phi\hat{\nabla}_U W$$

$$(\nabla_U f)W = \mathcal{H}\nabla_U fW - f\hat{\nabla}_U W,$$

$$(\nabla_X B)\zeta = \hat{\nabla}_X B\zeta - B\mathcal{H}\nabla_X \zeta$$

$$(\nabla_X C)\zeta = \mathcal{H}\nabla_X C\zeta - C\mathcal{H}\nabla_X \zeta$$

for any non-null vector fields $U, W \in ker\psi_*$ and $X, \zeta \in (ker\psi_*)^{\perp}$.
The above assertions can be obtained by using (20)∼(21)∼(22) and (23), respectively. □

Lemma 4. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a quasi-hemi-slant pseudo-Riemannian submersion with type ∼1 and type ∼ 2. Let us suppose the total manifold as a para-Kaehler manifold and base manifold as a pseudo-Riemannian manifold. So, we obtain the following equations.*

$$(\nabla_U \phi)W = \mathcal{B}\mathcal{T}_U W - \mathcal{T}_U fW \tag{26}$$

$$(\nabla_U f)W = \mathcal{C}\mathcal{T}_U W - \mathcal{T}_U \phi W \tag{27}$$

$$(\nabla_X B)\zeta = \phi\mathcal{A}_X \zeta - \mathcal{A}_X \mathcal{B}\zeta \tag{28}$$

$$(\nabla_X C)\zeta = f\mathcal{A}_X \zeta - \mathcal{A}_X \mathcal{C}\zeta \tag{29}$$

for any non-null vector fields $U, W \in ker\psi_*$ and $X, \zeta \in (ker\psi_*)^\perp$.

Proof. The proof is simple.
If ϕ and f are parallel with respect to ∇ on $\mathcal{B}$, from (26) and (27), we have

$$\mathcal{B}\mathcal{T}_U W = \mathcal{T}_U fW \text{ and } \mathcal{C}\mathcal{T}_U W = \mathcal{T}_U \phi W \text{ for any } U, W \in \Gamma(TB).$$ □

Theorem 4. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper quasi hemi-slant pseudo-Riemannian submersion with type∼1 or 2 from a para-Kaehler manifold to a pseudo-Riemannian manifold. The invariant distribution $\mathtt{D}$ is integrable if and only if*

$$g_{\mathcal{B}}(\mathcal{T}_W \phi U - \mathcal{T}_U \phi W, fL\zeta + fR\zeta) = g_{\mathcal{B}}(\mathcal{V}\nabla_U \phi W - \mathcal{V}\nabla_W \phi U, \phi L\zeta) \tag{30}$$

for any non-null vector fields $U, W \in \Gamma(\mathtt{D})$ and $\zeta \in \Gamma(\mathtt{D}^\varphi \oplus \mathtt{D}^\perp)$.

Proof. For any non-null vector fields $U, W \in \Gamma(\mathtt{D})$ and $\zeta \in \Gamma(\mathtt{D}^\varphi \oplus \mathtt{D}^\perp)$. Then using (1),(2),(5) and (16) obtained:

$$\begin{aligned} g_{\mathcal{B}}([U, W], \zeta) &= -g_{\mathcal{B}}(\nabla_U \mathcal{P}W, \mathcal{P}\zeta) + g_{\mathcal{B}}(\nabla_W \mathcal{P}U, \mathcal{P}\zeta) \\ &= -g_{\mathcal{B}}(\nabla_U \phi W, \mathcal{P}\zeta) + g_{\mathcal{B}}(\nabla_W \phi U, \mathcal{P}\zeta) \\ &= g_{\mathcal{B}}(\mathcal{T}_W \phi U - \mathcal{T}_U \phi W, fL\zeta + fR\zeta) \\ &+ g_{\mathcal{B}}(\mathcal{V}\nabla_W \phi U - \mathcal{V}\nabla_U \phi W, \phi L\zeta). \end{aligned} \tag{31}$$

So, the proof is complete. □

Theorem 5. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper quasi hemi-slant pseudo-Riemannian submersion with type∼1 or 2 from a para-Kaehler manifold to a pseudo-Riemannian manifold. The slant distribution $\mathtt{D}^\varphi$ is integrable if and only if*

$$\begin{aligned} g_{\mathcal{B}}(\mathcal{T}_U f\phi W - \mathcal{T}_W f\phi U, \mathcal{X}) &= g_{\mathcal{B}}(\mathcal{T}_U fW - \mathcal{T}_W fU, \phi K\mathcal{X}) \\ &+ g_{\mathcal{B}}(\mathcal{H}\nabla_U fW - \mathcal{H}\nabla_W fU, fR\mathcal{X}) \end{aligned} \tag{32}$$

for any non-null vector fields $U, W \in \Gamma(\mathtt{D}^\varphi)$ and $\mathcal{X} \in \Gamma(\mathtt{D} \oplus \mathtt{D}^\perp)$.

Proof. We only give its proof ψ is type$\sim$1. For any non-null vector fields $U, W \in \Gamma(\mathtt{D}^{\varphi})$ and $\mathcal{X} \in \Gamma(\mathtt{D} \oplus \mathtt{D}^{\perp})$. Then using (1),(2),(6), (16) and Theorem 2(a), we get:

$$\begin{aligned} g_{\mathcal{B}}([U,W],\mathcal{X}) &= -g_{\mathcal{B}}(\nabla_U \mathcal{P}W, \mathcal{P}\mathcal{X}) + g_{\mathcal{B}}(\nabla_W \mathcal{P}U, \mathcal{P}\mathcal{X}) \\ &= -g_{\mathcal{B}}(\nabla_U \phi W, \mathcal{P}\mathcal{X}) - g_{\mathcal{B}}(\nabla_U fW, \mathcal{P}\mathcal{X}) \\ &+ g_{\mathcal{B}}(\nabla_W \phi U, \mathcal{P}\mathcal{X}) + g_{\mathcal{B}}(\nabla_W fU, \mathcal{P}\mathcal{X}) \\ &= -\cosh^2\varphi g_{\mathcal{B}}([U,W],\mathcal{X}) \\ &- g_{\mathcal{B}}(\mathcal{T}_U f\phi W - \mathcal{T}_W f\phi U, \mathcal{X}) \\ &+ g_{\mathcal{B}}(\mathcal{T}_U fW + \mathcal{H}\nabla_U fW, \phi K\mathcal{X} + fR\mathcal{X}) \\ &- g_{\mathcal{B}}(\mathcal{T}_W fU + \mathcal{H}\nabla_W fU, \phi K\mathcal{X} + fR\mathcal{X}). \end{aligned}$$

Then, we have;

$$\begin{aligned} (1+\cosh^2\varphi) g_{\mathcal{B}}([U,W],\mathcal{X}) &= g_{\mathcal{B}}(\mathcal{T}_U fW - \mathcal{T}_W fU, \phi K\mathcal{X}) \\ &+ g_{\mathcal{B}}(\mathcal{H}\nabla_U fW - \mathcal{H}\nabla_W fU, fR\mathcal{X}) \\ &- g_{\mathcal{B}}(\mathcal{T}_U f\phi W - \mathcal{T}_W f\phi U, \mathcal{X}) \end{aligned}$$

which completes proof. □

Corollary 1. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper quasi hemi-slant pseudo-Riemannian submersion with type$\sim$1 or 2 from a para-Kaehler manifold to a pseudo-Riemannian manifold. If for any non-null vector fields $U, W \in \Gamma(\mathtt{D}^{\varphi})$ and $\mathcal{X} \in \Gamma(\mathtt{D} \oplus \mathtt{D}^{\perp})$*

$$\mathcal{H}\nabla_U fW - \mathcal{H}\nabla_W fU \in \Gamma(f\mathtt{D}^{\varphi} \oplus \mu)$$
$$\mathcal{T}_U f\phi W - \mathcal{T}_W f\phi U \in \Gamma(\mathtt{D}^{\varphi})$$
$$\mathcal{T}_U fW - \mathcal{T}_W fU \in \Gamma(\mathtt{D}^{\perp} \oplus \mathtt{D}^{\varphi})$$

Theorem 6. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper quasi hemi-slant pseudo-Riemannian submersion with type$\sim$1 or 2 from a para-Kaehler manifold to a pseudo-Riemannian manifold. The slant distribution $\mathtt{D}^{\perp}$ is integrable.*

Proof. The proof of Theorem 6 is similar to those given in ([28]). Therefore we skip its proof. □

Corollary 2. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper quasi hemi-slant pseudo-Riemannian submersion with type$\sim$1 or 2 from a para-Kaehler manifold to a pseudo-Riemannian manifold. In this case, for any non-null vector fields $U, W \in \Gamma(D^{\perp})$ we get*

$$\mathcal{T}_U PW = \mathcal{T}_W PU. \tag{33}$$

Proof. Using Lemma 1(b), from (20), we obtain

$$\mathcal{T}_U fW = \phi(\hat{\nabla}_U W) + \mathcal{B}\mathcal{T}_W U \tag{34}$$

If we take $U = W$ in (34) and subtracting it from (34), we get

$$\mathcal{T}_U fW - \mathcal{T}_W fU = \phi\,[U,W] \tag{35}$$

By Theorem 6 and Lemma 1(b), we get $\phi[U,W]=0$ from (35). This gives (33), since $fU=PU$ for every non-null vector field $U\in \mathtt{D}^{\perp}$. □

Theorem 7. *Let $\psi:(\mathcal{B},g_{\mathcal{B}},\mathcal{P})\to(\tilde{\mathcal{B}},g_{\tilde{\mathcal{B}}})$ be a proper quasi hemi-slant pseudo-Riemannian submersion with type∼1 from a para-Kaehler manifold to a pseudo-Riemannian manifold. In this case, the horizontal distribution $(ker\psi_*)^{\perp}$ describes a totally geodesic foliation on $\mathcal{B}$ if and only if*

$$\begin{aligned} g_{\mathcal{B}}(\mathcal{A}_{\mathcal{W}}\mathcal{Z},K\zeta+\cosh^2\varphi L\zeta) &= -g_{\mathcal{B}}(\mathcal{H}\nabla_{\mathcal{W}}\mathcal{Z},f\phi K\zeta+f\phi L\zeta)\\ &\quad +g_{\mathcal{B}}(\mathcal{A}_{\mathcal{W}}B\mathcal{Z}+\mathcal{H}\nabla_{\mathcal{W}}C\mathcal{Z},f\zeta) \end{aligned} \tag{36}$$

for any non-null vector fields $\mathcal{W},\mathcal{Z}\in(ker\psi_*)^{\perp}$ and $\zeta\in(ker\psi_*)$.

Proof. For any non-null vectors $\mathcal{W},\mathcal{Z}\in(ker\psi_*)^{\perp}$ and $\zeta\in(ker\psi_*)$, we get:

$$g_{\mathcal{B}}(\nabla_{\mathcal{W}}\mathcal{Z},\zeta)=g_{\mathcal{B}}(\nabla_{\mathcal{W}}\mathcal{Z},K\zeta+L\zeta+R\zeta)$$

Then using (1), (2), (7), (8), (16), (17) and Theorem 2(a), we get

$$\begin{aligned} g_{\mathcal{B}}(\nabla_{\mathcal{W}}\mathcal{Z},\zeta) &= -g_{\mathcal{B}}(\nabla_{\mathcal{W}}\mathcal{P}\mathcal{Z},\mathcal{P}K\zeta)-g_{\mathcal{B}}(\nabla_{\mathcal{W}}\mathcal{P}\mathcal{Z},\mathcal{P}L\zeta)\\ &\quad - g_{\mathcal{B}}(\nabla_{\mathcal{W}}\mathcal{P}\mathcal{Z},\mathcal{P}R\zeta)\\ &= g_{\mathcal{B}}(\mathcal{A}_{\mathcal{W}}\mathcal{Z},K\zeta+BfK\zeta+\cosh^2\varphi L\zeta)\\ &\quad + g_{\mathcal{B}}(\mathcal{H}\nabla_{\mathcal{W}}\mathcal{Z},f\phi K\zeta+f\phi L\zeta)\\ &\quad - g_{\mathcal{B}}(\mathcal{A}_{\mathcal{W}}B\mathcal{Z}+\mathcal{H}\nabla_{\mathcal{W}}C\mathcal{Z},fK\zeta+fL\zeta+fR\zeta). \end{aligned}$$

Since $fK\zeta=0$ and $fK\zeta+fL\zeta+fR\zeta=f\zeta$, we obtain;

$$\begin{aligned} g_{\mathcal{B}}(\nabla_{\mathcal{W}}\mathcal{Z},\zeta) &= g_{\mathcal{B}}(\mathcal{A}_{\mathcal{W}}\mathcal{Z},K\zeta+\cosh^2\varphi L\zeta)\\ &\quad + g_{\mathcal{B}}(\mathcal{H}\nabla_{\mathcal{W}}\mathcal{Z},f\phi K\zeta+f\phi L\zeta)\\ &\quad - g_{\mathcal{B}}(\mathcal{A}_{\mathcal{W}}B\mathcal{Z}+\mathcal{H}\nabla_{\mathcal{W}}C\mathcal{Z},f\zeta) \end{aligned}$$

which gives proof. □

Similarly, the following conclusion is obtained.

Theorem 8. *Let $\psi:(\mathcal{B},g_{\mathcal{B}},\mathcal{P})\to(\tilde{\mathcal{B}},g_{\tilde{\mathcal{B}}})$ be a proper quasi hemi-slant pseudo-Riemannian submersion with type∼1 from a para-Kaehler manifold to a pseudo-Riemannian manifold. In this case, the vertical distribution $(ker\psi_*)$ describes a totally geodesic foliation on $\mathcal{B}$ if and only if*

$$\begin{aligned} g_{\mathcal{B}}(\mathcal{T}_U\zeta+\cosh^2\varphi\mathcal{T}_UL\zeta,\mathcal{W}) &= g_{\mathcal{B}}(\mathcal{H}\nabla_Uf\phi K\zeta+\mathcal{H}\nabla_Uf\phi L\zeta,\mathcal{W})\\ &\quad +g_{\mathcal{B}}(\mathcal{T}_Uf\zeta,B\mathcal{W})+g_{\mathcal{B}}(\mathcal{H}\nabla_Uf\zeta,C\mathcal{W}). \end{aligned} \tag{37}$$

for any non-null vector fields $U,\zeta\in\Gamma(ker\psi_*)$ and $\mathcal{W}\in\Gamma(ker\psi_*)^{\perp}$.

Using Theorem 7 and Theorem 8, we get the Theorem 9.

Theorem 9. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \rightarrow (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper quasi hemi-slant pseudo-Riemannian submersion with type∼1 from a para-Kaehler manifold to a pseudo-Riemannian manifold. In this case, the total space is a locally product $\mathcal{B}_{ker\psi_*} \times \mathcal{B}_{ker\psi_*^{\perp}}$ where $\mathcal{B}_{ker\psi_*}$ and $\mathcal{B}_{ker\psi_*^{\perp}}$ are leaves of $(ker\psi_*)$ and $(ker\psi_*)^{\perp}$, respectively, if and only if (36) and (37) are satisfied.*

Theorem 10. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \rightarrow (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper quasi hemi-slant pseudo-Riemannian submersion with type∼1 or 2 from a para-Kaehler manifold to a pseudo-Riemannian manifold. In this case, the invariant distribution $\mathtt{D}$ describes a totally geodesic foliation on $\mathcal{B}$ if and only if*

$$g_{\mathcal{B}}(\mathcal{T}_{\mathcal{W}}\phi\mathcal{Z}, fLY + fRY) = -g_{\mathcal{B}}(\mathcal{V}\nabla_{\mathcal{W}}\phi\mathcal{Z}, \phi LY) \tag{38}$$

and

$$g_{\mathcal{B}}(\mathcal{T}_{\mathcal{W}}\phi\mathcal{Z}, C\xi) = -g_{\mathcal{B}}(\mathcal{V}\nabla_{\mathcal{W}}\phi\mathcal{Z}, B\xi) \tag{39}$$

Proof. For all non-null vectors $\mathcal{W}, \mathcal{Z} \in \Gamma(\mathtt{D})$ and $Y \in \Gamma(\mathtt{D}^{\varphi_1} \oplus \mathtt{D}^{\varphi_2})$ and $\xi \in \Gamma(ker\psi_*)^{\perp}$. Then using (1),(2),(5),(16) and $f\mathcal{Z} = 0$, we get:

$$\begin{aligned} g_{\mathcal{B}}(\nabla_{\mathcal{W}}\mathcal{Z}, Y) &= -g_{\mathcal{B}}(\nabla_{\mathcal{W}}\mathcal{P}\mathcal{Z}, \mathcal{P}Y) \\ &= -g_{\mathcal{B}}(\nabla_{\mathcal{W}}\mathcal{P}\mathcal{Z}, \mathcal{P}LY + \mathcal{P}RY) \\ &= -g_{\mathcal{B}}(\mathcal{T}_{\mathcal{W}}\phi\mathcal{Z}, fLY + fRY) - g_{\mathcal{B}}(\mathcal{V}\nabla_{\mathcal{W}}\phi\mathcal{Z}, \phi LY) \end{aligned}$$

Then, again using (1),(2),(5),(16),(19) and $f\mathcal{Z} = 0$, we get:

$$\begin{aligned} g_{\mathcal{B}}(\nabla_{\mathcal{W}}\mathcal{Z}, \xi) &= -g_{\mathcal{B}}(\nabla_{\mathcal{W}}\mathcal{P}\mathcal{Z}, \mathcal{P}\xi) \\ &= -g_{\mathcal{B}}(\nabla_{\mathcal{W}}\phi\mathcal{Z}, B\xi + C\xi) \\ &= -g_{\mathcal{B}}(\mathcal{T}_{\mathcal{W}}\phi\mathcal{Z}, C\xi) - g_{\mathcal{B}}(\mathcal{V}\nabla_{\mathcal{W}}\phi\mathcal{Z}, B\xi). \end{aligned}$$

So, the proof is complete. □

Theorem 11. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \rightarrow (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper quasi hemi-slant pseudo-Riemannian submersion with type∼1 or 2 from a para-Kaehler manifold to a pseudo-Riemannian manifold. In this case, the slant distribution $\mathtt{D}^{\varphi}$ describes a totally geodesic foliation on $\mathcal{B}$ if and only if*

$$g_{\mathcal{B}}(\mathcal{T}_U f\phi V, Y) = g_{\mathcal{B}}(\mathcal{T}_U fV, \phi KY) + g_{\mathcal{B}}(\mathcal{H}\nabla_U fV, fRY) \tag{40}$$

and

$$g_{\mathcal{B}}(\mathcal{H}\nabla_U f\phi V, \xi) = g_{\mathcal{B}}(\mathcal{H}\nabla_U fV, C\xi) + g_{\mathcal{B}}(\mathcal{T}_U fV, B\xi) \tag{41}$$

for any non-null vector fields $U, V \in \Gamma(\mathtt{D}^{\varphi})$ and $Y \in \Gamma(\mathtt{D} \oplus \mathtt{D}^{\perp})$ and $\xi \in \Gamma(ker\psi_*)^{\perp}$.

Proof. We will show it when ψ is type∼1. For all non-null vectors $U, V \in \Gamma(\mathtt{D}^{\varphi})$ and $Y \in \Gamma(\mathtt{D} \oplus \mathtt{D}^{\perp})$ and $\xi \in \Gamma(ker\psi_*)^{\perp}$. Then using (1),(2),(6),(16) and Theorem 2(a), we get:

$$\begin{aligned} g_{\mathcal{B}}(\nabla_U V, Y) &= -g_{\mathcal{B}}(\nabla_U \phi V, \mathcal{P}Y) - g_{\mathcal{B}}(\nabla_U fV, \mathcal{P}Y) \\ &= \cosh^2\varphi\, g_{\mathcal{B}}(\nabla_U V, Y) + g_{\mathcal{B}}(\mathcal{T}_U f\phi V, Y) \end{aligned}$$

$$- \quad g_{\mathcal{B}}(\mathcal{T}_U fV, \phi KY) - g_{\mathcal{B}}(\mathcal{H}\nabla_U fV, fRY).$$

Hence we obtain;

$$\begin{aligned} -\sinh^2\varphi_1 g_{\mathcal{B}}(\nabla_U V, Y) &= g_{\mathcal{B}}(\mathcal{T}_U f\phi V, Y) - g_{\mathcal{B}}(\mathcal{T}_U fV, \phi KY) \\ &- g_{\mathcal{B}}(\mathcal{H}\nabla_U fV, fRY). \end{aligned}$$

Similarly, using (1),(2),(6),(16),(19) and Theorem 3.4(a), we get:

$$\begin{aligned} g_{\mathcal{B}}(\nabla_U V, \xi) &= -g_{\mathcal{B}}(\nabla_U \phi V, \mathcal{P}\xi) - g_{\mathcal{B}}(\nabla_U fV, \mathcal{P}\xi) \\ &= \cosh^2\varphi_1 g_{\mathcal{B}}(\nabla_U V, \xi) + g_{\mathcal{B}}(\mathcal{H}\nabla_U f\phi V, \xi) \\ &- g_{\mathcal{B}}(\mathcal{H}\nabla_U fV, C\xi) - g_{\mathcal{B}}(\mathcal{T}_U fV, B\xi). \end{aligned}$$

Hence, arrive at

$$\begin{aligned} -\sinh^2\varphi_1 g_{\mathcal{B}}(\nabla_U V, \xi) &= g_{\mathcal{B}}(\mathcal{H}\nabla_U f\phi V, \xi) - g_{\mathcal{B}}(\mathcal{H}\nabla_U fV, C\xi) \\ &- g_{\mathcal{B}}(\mathcal{T}_U fV, B\xi) \end{aligned}$$

which gives proof. □

Theorem 12. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper quasi-hemi-slant pseudo-Riemannian submersion with type∼1 or 2 from a para-Kaehler manifold to a pseudo-Riemannian manifold. In this case, the anti-invariant distribution $\mathtt{D}^{\perp}$ describes a totally geodesic foliation on $\mathcal{B}$ if and only if*

$$g_{\mathcal{B}}(\mathcal{A}_U\zeta, f\phi KV + f\phi LV) = -g_{\mathcal{B}}(\mathcal{H}\nabla_U f\zeta, fV) \tag{42}$$

and

$$g_{\mathcal{B}}(\mathcal{A}_U\mathcal{P}\zeta, B\xi) = -g_{\mathcal{B}}(\mathcal{H}\nabla_U\mathcal{P}\zeta, C\xi) \tag{43}$$

for any non-null vector fields $U, \zeta \in \Gamma(\mathtt{D}^{\perp})$ and $V \in \Gamma(\mathtt{D} \oplus \mathtt{D}^{\varphi})$ and $\xi \in \Gamma(ker\psi_*)^{\perp}$.

Proof. We will show it when ψ is type∼1. For all non-null vectors $U, \zeta \in \Gamma(\mathtt{D}^{\perp})$ and $KV + LV \in \Gamma(\mathtt{D} \oplus \mathtt{D}^{\varphi})$ and $\xi \in \Gamma(ker\psi_*)^{\perp}$. Then using (1),(16),(19) and Theorem 2(a), we get:

$$\begin{aligned} g_{\mathcal{B}}(\nabla_U\zeta, V) &= -g_{\mathcal{B}}(\nabla_U\mathcal{P}\zeta, \mathcal{P}V) = -g_{\mathcal{B}}(\nabla_U\mathcal{P}\zeta, \phi V) - g_{\mathcal{B}}(\nabla_U\mathcal{P}\zeta, fV) \\ &= \cosh^2\varphi g_{\mathcal{B}}(\nabla_U\zeta, LV) - g_{\mathcal{B}}(\nabla_U\zeta, KV) + g_{\mathcal{B}}(\nabla_U\zeta, BfKV) \\ &- g_{\mathcal{B}}(\nabla_U\zeta, f\phi KV) - g_{\mathcal{B}}(\nabla_U\zeta, f\phi LV) \\ &- g_{\mathcal{B}}(\nabla_U\mathcal{P}\zeta, fV). \end{aligned} \tag{44}$$

We know that $g_{\mathcal{B}}(\nabla_U\zeta, V) = g_{\mathcal{B}}(\nabla_U\zeta, KV) + g_{\mathcal{B}}(\nabla_U\zeta, LV)$ and using (8) and (16) from equation (44), we arrive at;

$$\begin{aligned} g_{\mathcal{B}}(\nabla_U\zeta, -\sinh^2\varphi LV - BfKV) &= -g_{\mathcal{B}}(\mathcal{A}_U\zeta, f\phi KV + f\phi LV) \\ &- g_{\mathcal{B}}(\mathcal{H}\nabla_U f\zeta, fV) \end{aligned} \tag{45}$$

which gives (42). Similarly, using (8) and (19), we get:

$$g_{\mathcal{B}}(\nabla_U\zeta, \xi) = -g_{\mathcal{B}}(\nabla_U\mathcal{P}\zeta, \mathcal{P}\xi) = -g_{\mathcal{B}}(\mathcal{A}_U\mathcal{P}\zeta, B\xi) - g_{\mathcal{B}}(\mathcal{H}\nabla_U\mathcal{P}\zeta, C\xi) \tag{46}$$

which gives (43). □

Now, from Theorem 10, Theorem 11 and Theorem 12 we arrive at the Theorem 13. This is decomposition theorem for the fiber:

Theorem 13. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper quasi-hemi-slant pseudo-Riemannian submersion with type∼1 or 2 from a para-Kaehler manifold to a pseudo-Riemannian manifold. In this case, the fibers of ψ are locally product $\mathcal{B}_{\mathtt{D}} \times \mathcal{B}_{\mathtt{D}^{\varphi}} \times \mathcal{B}_{\mathtt{D}^{\perp}}$ are leaves of $\mathtt{D}$, $\mathtt{D}^{\varphi}$ and $\mathtt{D}^{\perp}$,respectively, if and only if the conditions (38),(39),(40), (41),(42) and (43) hold.*

Theorem 14. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper quasi hemi-slant pseudo-Riemannian submersion with type∼1 from a para-Kaehler manifold to a pseudo-Riemannian manifold. In this case, ψ is a totally geodesic map on $\mathcal{B}$ if and only if*

$$\begin{aligned} & g_{\mathcal{B}}(\cosh^2\varphi \nabla_U LW + \mathcal{H}\nabla_U f\phi LW, Y) \\ &= g_{\mathcal{B}}(\mathcal{V}\nabla_U \mathcal{P}KW + \mathcal{T}_U fLW + \mathcal{T}_U fRW, \mathcal{P}Y) \\ &+ g_{\mathcal{B}}(\mathcal{T}_U \mathcal{P}KW + \mathcal{H}\nabla_U fLW + \mathcal{H}\nabla_U fRW, CY) \end{aligned} \tag{47}$$

and

$$\begin{aligned} & g_{\mathcal{B}}(\cosh^2\varphi \nabla_Y LU + \mathcal{H}\nabla_Y f\phi LU, Z) \\ &= g_{\mathcal{B}}(\mathcal{V}\nabla_Y \mathcal{P}KU + \mathcal{A}_Y fLU + \mathcal{A}_Y \mathcal{P}RU, BZ) \\ & g_{\mathcal{B}}(\mathcal{A}_Y \mathcal{P}KU + \mathcal{H}\nabla_Y fLU + \mathcal{H}\nabla_Y fRU, CZ) \end{aligned} \tag{48}$$

For any non-null vector fields $U, W \in \Gamma(ker\psi_*)$ and $Y, Z \in \Gamma(ker\psi_*)^{\perp}$.

Proof. For any non-null vector fields $U, W \in \Gamma(ker\psi_*)$ and $Y, Z \in \Gamma(ker\psi_*)^{\perp}$. Then, using (1),(2),(5),(16),(19) and Theorem 2(a) we get:

$$\begin{aligned} g_{\mathcal{B}}(\nabla_U W, Y) &= -g_{\mathcal{B}}(\nabla_U \mathcal{P}W, \mathcal{P}Y) \\ &= -g_{\mathcal{B}}(\nabla_U \mathcal{P}KW, \mathcal{P}Y) - g_{\mathcal{B}}(\nabla_U \mathcal{P}LW, \mathcal{P}Y) \\ &- g_{\mathcal{B}}(\nabla_U \mathcal{P}RW, \mathcal{P}Y) \\ &= -g_{\mathcal{B}}(\mathcal{V}\nabla_U \mathcal{P}KW + \mathcal{T}_U fLW + \mathcal{T}_U fRW, \mathcal{P}Y) \\ &+ g_{\mathcal{B}}(\cosh^2\varphi \nabla_U LW + \mathcal{H}\nabla_U f\phi LW, Y) \\ &- g_{\mathcal{B}}(\mathcal{T}_U \mathcal{P}KW + \mathcal{H}\nabla_U fLW + \mathcal{H}\nabla_U fRW, CY) \end{aligned}$$

Then, again using (1),(7),(8),(16),(19) and Theorem 2(a), we get:

$$\begin{aligned} g_{\mathcal{B}}(\nabla_Y U, Z) &= -g_{\mathcal{B}}(\nabla_Y \mathcal{P}U, \mathcal{P}Z) \\ &= -g_{\mathcal{B}}(\nabla_Y \mathcal{P}KU, \mathcal{P}Z) - g_{\mathcal{B}}(\nabla_Y \mathcal{P}LU, \mathcal{P}Z) \\ &- g_{\mathcal{B}}(\nabla_Y \mathcal{P}RU, \mathcal{P}Z) \\ &= -g_{\mathcal{B}}(\mathcal{V}\nabla_Y \mathcal{P}KU + \mathcal{A}_Y fLU + \mathcal{A}_Y fRU, BZ) \\ &- g_{\mathcal{B}}(\cosh^2\varphi \nabla_Y LU + \mathcal{H}\nabla_Y f\phi LU, Z) \end{aligned}$$

$$- \quad g_{\mathcal{B}}(\mathcal{A}_Y \mathcal{P}KU + \mathcal{H}\nabla_Y fLU + \mathcal{H}\nabla_Y fRU, CZ).$$

Therefore, a pseudo-Riemannian submersion ψ is said to be totally umbilical if

$$\mathcal{T}_{U_1}U_2 = g(U_1, U_2)H, \tag{49}$$

here H is the mean curvature vector field of the fibre in $\mathcal{B}$ for all non-null vector fields $U_1, U_2 \in \Gamma(ker\psi_*)$. The fibre is said to be minimal if $H = 0$([4]). □

Theorem 15. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper quasi-hemi-slant pseudo-Riemannian submersion from a para-Kaehler manifold to a pseudo-Riemannian manifold with totally umbilical fibers. In that case, either the anti-invariant distribution $dim(\mathtt{D}^\perp) = 1$ or the mean curvature vector field H of any fiber $\psi^{-1}(\bar{q})$, $\bar{q} \in \mathcal{B}$ is perpendicular to $P\mathtt{D}^\perp$. Eventually, if ϕ is parallel, then $H \in \Gamma(\mu)$. Moreover, if f is parallel, then $\mathcal{T} \equiv 0$.*

Proof. The proof is obtained by simple calculations. □

Author Contribution Statements The authors jointly worked on the results and they read and approved the final manuscript.

Declaration of Competing Interests The authors declare no potential conflict of interests.

References

[1] Akyol, M. A., Şahin, B., Conformal slant submersions, *Hacettepe Journal of Mathematics and Statistics*, 48(1) (2019), 28-44. https://doi.org/10.15672/HJMS.2017.506
[2] Akyol, M. A., Gündüzalp, Y., Semi-invariant semi-Riemannian submersions, *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, 67(1) (2018), 80-92. https://doi.org/10.1501/Commua1_0000000832
[3] Alegre, P., Carriazo, A., Bi-slant submanifolds of para-Hermitian manifolds, *Mathematics*, 7(7) (2019), 618. https://doi.org/10.3390/math7070618
[4] Baditoiu, G., Ianus, S., Semi-Riemannian submersions from real and complex pseudo-hyperbolic spaces, *Diff. Geom. and Appl.*, 16(1) (2002), 79-94. https://doi.org/10.1016/S0926-2245(01)00070-5
[5] Carriazo, A., Bi-slant immersions, *Proc. ICRAMS* 2000, (2000), 88-97.
[6] Falcitelli, M., Ianus, S., Pastore, A. M., Riemannian Submersions and Related Topics, World Scientific, 2004.
[7] Gündüzalp, Y., Slant submersions in paracontact geometry, *Hacettepe Journal of Mathematics and Statistics*, 49(2) (2020), 822-834. https://doi.org/10.15672/hujms.458085
[8] Gündüzalp, Y., Anti-invariant semi-Riemannian submersions from almost para-Hermitian manifolds, *Journal of Function Spaces and Applications*, 2013 (2013). https://doi.org/10.1155/2013/720623
[9] Gündüzalp, Y., Anti-invariant Pseudo-Riemannian submersions and Clairaut submersions from Paracosymplectic manifolds, *Mediterr. J. Math.*, 16 (2019), 1-18. https://doi.org/10.1007/s00009-019-1359-1
[10] Gündüzalp, Y., Neutral slant submersions in paracomplex geometry, *Afrika Matematika*, 32 (2021), 1095-1110. https://doi.org/10.1007/s13370-021-00884-8
[11] Gündüzalp, Y., Slant submersions from almost product Riemannian manifolds, *Turkish Journal of Mathematics*, 37(5) (2013), 863-873. https://doi.org/10.3906/mat-1205-64

[12] Gray, A., Pseudo-Riemannian almost product manifolds and submersions, *J. Math. Mech.*, 16 (1967), 715-737.

[13] Ianus, S., Mazzocco, R., Vilcu, G. E., Riemannian submersions from quaternionic manifolds, *Acta Appl. Math.*, 104 (2008), 83-89. https://doi.org/10.1007/s10440-008-9241-3

[14] Ianus, S., Vilcu, G. E., Voicu, R. C., Harmonic maps and Riemannian submersions between manifolds endowed with special structures, *Banach Center Publications,* 93 (2011), 277-288.

[15] Ivanov, S., Zamkovoy, S., Para-Hermitian and para-quaternionic manifolds, *Diff. Geom. and Its Appl.*, 23 (2005), 205-234. https://doi.org/10.1016/j.difgeo.2005.06.002

[16] Lee, C. W., Lee, J. W., Şahin, B., Vilcu, G-E., Optimal inequalities for Riemannian maps and Riemannian submersions involving Casorati curvatures, *Annali di Matematica*, 200 (2021), 1277–1295. https://doi.org/10.1007/s10231-020-01037-7

[17] O'Neill, B., The fundamental equations of a submersion, *Michigan Math. J.*, 13 (1966), 459–469. https://doi.org/10.1307/mmj/1028999604

[18] Özdemir, F., Sayar, C., Taştan, H. M., Semi-invariant submersions whose total manifolds are locally product Riemannian, *Quaestiones Mathematicae*, 40(7) (2017), 909-926. https://doi.org/10.2989/16073606.2017.1335657

[19] Sepet, S. A., Ergüt, M., Pointwise slant submersions from cosymplectic manifolds, *Turkish Journal of Mathematics*, 40(3) (2016), 582-593. https://doi.org/10.3906/mat-1503-98

[20] Prasad, R., Shukla, S. S., Kumar, S., On Quasi-bi-slant submersions, *Mediterr. J. Math.*, 16 (2019), 1-18. https://doi.org/10.1007/s00009-019-1434-7

[21] Sayar, C., Akyol, M. A., Prasad, R., Bi-slant submersions in complex geometry, *International Journal of Geometric Methods in Modern Physics*, 17(04) (2020), 2050055. https://doi.org/10.1142/S0219887820500553

[22] Sari, R., Akyol, M. A., Hemi-slant ξ -Riemannian submersions in contact geometry, *Filomat*, 34(11) (2020), 3747-3758. https://doi.org/10.2298/FIL2011747S

[23] Şahin, B., Slant submersions from almost Hermitian manifolds, *Bull. Math. Soc.Sci. Math. Roumanie Tome.*, 54(102) (2011), 93-105.

[24] Şahin, B., Anti-invariant Riemannian submersions from almost Hermitian manifolds, *Central European J.Math.*, 8(3) (2010), 437-447. https://doi.org/10.2478/s11533-010-0023-6

[25] Şahin, B., Semi-invariant submersions from almost Hermitian manifold, *Canadian Mathematical Bulletin*, 56(1) (2013), 173-183. https://doi.org/10.4153/CMB-2011-144-8

[26] Şahin, B., Riemannian Submersions, Riemannian Maps in Hermitian Geometry, and Their Applications, Academic Press, 2017.

[27] Şahin, B., Riemannian submersions from almost Hermitian manifolds, *Taiwanese J. Math.*, 17(2) (2013), 629-659. https://doi.org/10.11650/tjm.17.2013.2191

[28] Taştan, H. M., Şahin, B., Yanan, Ş., Hemi-slant submersions, *Mediterr. J. Math.*, 13 (2016), 2171–2184. https://doi.org/10.1007/s00009-015-0602-7

[29] Vilcu, G. E., Almost product structures on statistical manifolds and para-Kähler-like statistical submersions, *Bulletin des Sciences Mathématiques*, 171 (2021), 103018. https://doi.org/10.1016/j.bulsci.2021.103018

[30] Watson, B., Almost Hermitian submersions, *J. Differential Geom.*, 11 (1976), 147-165. https://doi.org/10.4310/jdg/1214433303