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# FAREY GRAPH AND RATIONAL FIXED POINTS OF THE EXTENDED MODULAR GROUP

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Abstract. Fixed points of matrices have many applications in various areas of science and mathematics. The extended modular group $\overline{\Gamma}$ is the group of $2 \times 2$ matrices with integer entries and determinant $\pm 1$. There are strong connections between the extended modular group, continued fractions and Farey graph. The Farey graph is a graph with vertex set $\mathbb{Q}_\infty = \mathbb{Q} \cup \{\infty\}$. In this study we consider the elements in $\overline{\Gamma}$ that fix rationals. For a given rational number, we use its Farey neighbours to obtain the matrix representation of the element in $\overline{\Gamma}$ that fixes the given rational. Then we express such elements as words in terms of generators using the relations between the Farey graph and continued fractions. Finally we give the new block reduced form of these words which all blocks have Fibonacci numbers entries.

## 1. Introduction

The modular group $\Gamma = PSL(2,\mathbb{Z})$ is the projective special linear group of $2 \times 2$ matrices over the ring of integers with determinant one. This group is the quotient group $SL(2,\mathbb{Z})/\pm I$, hence each matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represents the same element with its negative $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$. The modular group acts on the upper half plane $\mathbb{H}$ via linear fractional transformations $z \to \frac{az+b}{cz+d}$. These transformations are orientation preserving isometries of $\mathbb{H}$. Modular group is generated by two elements;

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

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The presentation of $\Gamma$ is;

$$\Gamma =< T, S : T^2 = S^3 = I >\approx \mathbb{Z}_2 * \mathbb{Z}_3,$$

the free product of $\mathbb{Z}_2$ and $\mathbb{Z}_3$ where $S = TU = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. Let us denote the set $G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = -1 \right\}$. The corresponding transformations of elements in $G$ are anti-automorphisms. Thus the extended modular group can be defined as $\overline{\Gamma} = PSL(2, \mathbb{Z}) \cup G$. Hence, the extended modular group is the projective linear group $PGL(2, \mathbb{Z})$ and isomorphic to the free product of two dihedral groups of order four and six amalgamated with the cyclic group of order 2 i.e.

$$\overline{\Gamma} =< T, S, R : T^2 = S^3 = R^2 = (TR)^2 = (SR)^2 = I >\approx D_2 *_{\mathbb{Z}_2} D_3$$

where $R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ as a reflection map. So the modular group is normal in the extended modular group with index 2.

For each $V \in \overline{\Gamma}$; the number $z \in \mathbb{C} \cup \{\infty\}$ is called a fixed point of $V$ if $V(z) = z$ where $V(z)$ is the corresponding transformation. There is a relation between the number fixed points and trace of $V$. Elements of $\overline{\Gamma}$ are classified according to the number of fixed points. There are five types of elements in $\overline{\Gamma}$. Now we list the certain types of elements.

If $V \in \Gamma$ then $V$ has at most two fixed points. Also if;

- $|trV| > 2$, then there are two fixed points in $\mathbb{R} \cup \{\infty\}$ and $V$ is called a hyperbolic element.
- $|trV| = 2$, then there is one fixed point in $\mathbb{R} \cup \{\infty\}$ and $V$ is called a parabolic element.
- $|trV| < 2$, then there are two conjugate fixed points in $\mathbb{C} \cup \{\infty\}$ and $V$ is called an elliptic element.

If $V \in G$ then it has either two fixed points in the real line or the fixed point set is a circle perpendicular to real line. Also if;

- $|trV| \neq 0$, then there is one fixed point in $\mathbb{R} \cup \{\infty\}$ and $V$ is called a glide reflection.
- $|trV| = 0$, then the set of fixed points is a circle perpendecular to the real line and $V$ is called a reflection.

For more information see [1, 2, 11].

There are impressive relations between the modular group and continued fractions. In [25], Rosen defined $\lambda$ continued fractions for $\lambda \in \mathbb{R}$;

$$[r_0\lambda; r_1\lambda, ..., r_n\lambda] = r_0\lambda - \cfrac{1}{r_1\lambda - \cfrac{1}{r_2\lambda - \cfrac{1}{\ddots r_{n-1}\lambda - \frac{1}{r_n\lambda}}}}$$

In this expansion, for $i \leq n$, $C_i = \frac{p_i}{q_i} = [r_0\lambda; r_1\lambda, ..., r_i\lambda]$ is called $ith$ convergent of the expansion. And it can be seen by calculation $p_i.q_{i-1} - q_i.p_{i-1} = \pm 1$. Owing to this viewpoint, Rosen revealed a criteria for membership problem for Hecke groups $H(\lambda)$, a general class of modular group. He proved that an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H(\lambda)$ if and only if $\frac{a}{c}$ has a finite $\lambda$ continued fraction expansion. For $\lambda = 1$ this expression is called integer continued fraction and related to the modular group, on the contrary the membership problem for the modular group is obvious because $\Gamma = PSL(2, \mathbb{Z})$. On the other hand, for $\lambda = 1$ it is possible to make connections between Rosen's fractions and the Farey sequence.

The Farey sequence of order $n$ is a complete and ordered set of reduced rational numbers in the interval $[0, 1]$ which have the denominators not exceeding $n$.

$$F_1 = \left\{ \frac{0}{1}, \frac{1}{1} \right\}$$

$$F_2 = \left\{ \frac{0}{1}, \frac{1}{2}, \frac{1}{1} \right\}$$

$$F_3 = \left\{ \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \right\}$$

$$F_4 = \left\{ \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1} \right\}$$

It can be seen that if $\frac{a}{c}$ and $\frac{b}{d}$ appears one after another in some $F_n$ then $ad - bc = \pm 1$. We called such rationals Farey neighbours. All Farey neighbours of a rational $x$ is denoted by $\mathcal{N}(x)$. The Farey sum of $\frac{a}{c}$ and $\frac{b}{d}$ defined as;

$$\frac{a}{c} \oplus \frac{b}{d} = \frac{a+b}{c+d}$$

All Farey neigbours of a rational number can be obtained by Farey sum. More precisely if a rational $\frac{p}{q}$ first appears in $F_n$ by the Farey sum of $\frac{a}{c}$ and $\frac{b}{d}$ in $F_{n-1}$ i.e. $\frac{a}{c} \oplus \frac{b}{d} = \frac{a+b}{c+d} = \frac{p}{q}$ then $\frac{a}{c}$ and $\frac{b}{d}$ are Farey neighbours of $\frac{p}{q}$. Here $\frac{a}{c}$ and $\frac{b}{d}$ are called the Farey parents of $\frac{p}{q}$, and conversely $\frac{p}{q}$ is called the Farey child of $\frac{a}{c}$ and $\frac{b}{d}$. If $\frac{a_i}{c_i}$ is a Farey neighbour of $\frac{p}{q}$ then $\frac{a_i}{c_i} \oplus \frac{p}{q}$ is also a Farey neighbour of $\frac{p}{q}$.

Observe that every $F_n$ includes $F_{n-1}$ and new members are obtained by Farey sum of its neighbours. For instance $\frac{1}{2} \in F_2$ is the Farey sum of $\frac{0}{1}$ and $\frac{1}{1}$ in $F_1$. This rule is known as the mediant rule. It should be noted that if the denominator of a Farey sum of two neighbours in $F_{n-1}$ exceeds $n$ then this will not be appear in $F_n$ since the definition of Farey sequence. Definition of Farey sequence can be extended to $\mathbb{Q}_\infty$ by assuming $\infty = \frac{1}{0}$. Hence for a given rational $\frac{a}{c}$; it is known that $\frac{a}{c}$ has finite integer continued fraction expansion. In addition $\frac{b}{d}$ is the penultimate convergent of the integer continued fraction expansion of $\frac{a}{c}$. This

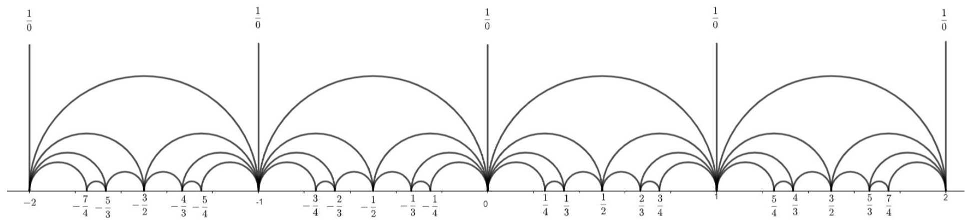

FIGURE 1. Farey graph

yields $ad - bc = \pm 1$; in other words $\frac{a}{c}$ and $\frac{b}{d}$ are Farey neighbours. As a result $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{\Gamma}$.

The Farey graph is a graph with vertex set $\mathbb{Q}_\infty$. And two reduced fractions $\frac{p}{q}$ and $\frac{r}{s}$ are adjacent if and only if $ps - rq = \pm 1$, i.e they are Farey neighbours. An edge between two vertices is drawn by a hyperbolic line in $\mathbb{H}$. The edges between $\frac{1}{0} = \infty$ and every integer $a$ are vertical lines. To construct the graph, first join the vertices $\frac{1}{0}, \frac{0}{1}$ and $\frac{1}{1}$ and obtain a big triangle. Then by induction if the endpoints of a long edge are $\frac{a}{c}$ and $\frac{b}{d}$, the label of the third vertex of the triangle is $\frac{a}{c} \oplus \frac{b}{d} = \frac{a+b}{c+d}$, see Figure 1.

## 2. MOTIVATION

There are numerous studies about modular and extended modular group in the literature, related to many branches of mathematics such as group theory, number theory automorphic functions, etc. Algebric structures of subgroups of modular and extended modular group and related topics are studied in [3, 4, 8, 17–21, 26, 31, 33, 34]. In recent years, many studies have contributed the theory of continued fractions related to the action of some subgroups of Möbius transformations. Series studied the relations between geodesics on the quotient of the hyperbolic plane by the modular group and continued fractions [28]. In [2], integer continued fraction expansions and geodesic expansions were studied from the perspective of graph theory. Short and Walker represented Rosen continued fractions by path in a class of graphs in hyperbolic geometry [30]. Same authors defined even integer continued fractions which all digits are even integers. And they studied connections between even integer continued fractions and the Farey graph [29].

The fixed points of automorphisms and anti-automorphisms of the extended complex plane have especially been of great interest in many fields of mathematics such as number theory, functional analysis, theory of complex functions, geometry and group theory [22, 24, 27]. Also fixed points of elements in $GL(2, \mathbb{R})$ in tropical algebra are discussed in [7]. In this study we focus on the fixed points of the elements in extended modular group $\overline{\Gamma}$.

Fixed points of an element $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{\Gamma}$ can be calculated by solving the equation $\frac{az+b}{cz+d} = z$ i.e.

$$z = \frac{a - d \pm \sqrt{(a+d)^2 - 4}}{2c} \tag{1}$$

where $ad - bc = 1$ in other words the corresponding transformation $V(z)$ is automorphism. And similarly fixed points of an anti-automorphism are

$$z = \frac{a - d \pm \sqrt{(a+d)^2 + 4}}{2c} \tag{2}$$

where $ad - bc = -1$. The action of extended modular group on extended rational numbers $\mathbb{Q}_\infty$ is intriguing. This action is defined as;

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} ap + bq \\ cp + dq \end{pmatrix}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{\Gamma}$ and the column vector $\begin{pmatrix} p \\ q \end{pmatrix}$ represents the rational number $\frac{p}{q}$. Fixed points of an element in $\overline{\Gamma}$ are rationals if and only if $a + d = 2$ or $-2$ for the equation (1) and $a + d = 0$ for the equation (2). This means that rational numbers are fixed only by parabolic or reflection elements.

In this study we establish relations between the Farey graph and elements of $\overline{\Gamma}$ that fixes a given rational $\frac{p}{q}$. Firstly we obtain matrix representation of the element fixing the rational $\frac{p}{q}$ via the Farey neighbours of $\frac{p}{q}$. Then, we consider the relations between paths in the Farey graph and integer continued fractions and obtain the element as a word of the generators $U$ and $T$. Afterwards, we express this word in block reduced forms and new block reduced forms, related to Fibonacci numbers. Finally, we give some relevant examples of our results.

## 3. Matrix Representations of the Parabolic and Reflection Elements

In this section we obtain the parabolic and reflection elements in $\overline{\Gamma}$ as matrices that fix a given rational. To do this, we use Farey neighbours.

**Theorem 1.** *Let $z = \frac{p}{q} \in \mathbb{Q}_\infty$ and $\frac{r}{s}, \frac{m}{k} \in \mathcal{N}(z)$ then the element*

$$V = \begin{pmatrix} ps - mq & pm - pr \\ qs - qk & pk - qr \end{pmatrix} \tag{3}$$

*fixes the rational $z$.*

*Proof.* Since $\frac{r}{s}, \frac{m}{k}$ are Farey neighbours of $z$, the elements $V_1 = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$ and $V_2 = \begin{pmatrix} p & m \\ q & k \end{pmatrix}$ belong to $\overline{\Gamma}$. Furhermore $V_1$ and $V_2$ both send $\infty$ to $\frac{p}{q}$. As a result $V = V_2.V_1^{-1}$ is the desired element. □

Let $\frac{p}{q}$ and $\frac{r}{s}$ are adjacent such that $\frac{r}{s} < \frac{p}{q}$ then $ps - qr = 1$ otherwise $-1$. The trace of the element mentioned in (3) is $ps - mq + pk - qr$. By the fact that $\frac{r}{s}, \frac{m}{k}$ are Farey neighbours of $\frac{p}{q}$ we have $ps - qr = \pm 1$ and $pk - mq = \pm 1$. Hence $tr\,(V) = 0, \pm 2$ and we have proved the following corollary.

**Corollary 1.** *Let $z = \frac{p}{q} \in \mathbb{Q}_\infty$ and $\frac{r}{s}, \frac{m}{k} \in \mathcal{N}\,(z)$. If $\frac{r}{s}$ and $\frac{m}{k}$ are at the same side of $\frac{p}{q}$ then the element in (3) is parabolic otherwise a reflection.*

We know that the fixed point set of a reflection map is a circle perpendicular to real axis. If the element $V$ mentioned in (3) is a reflection then we know from [5] that $V$ fixes the circle centered at $\left(\frac{ps-mq}{qs-qk}, 0\right)$ with radius $\frac{1}{|qs-qk|}$.

**Example 1.** *For the rational $\frac{8}{3}$ one can choose Farey neighbours as $\frac{5}{2}$ and $\frac{13}{5}$. Then, we have the parabolic element*

$$V = \begin{pmatrix} -23 & 64 \\ -9 & 25 \end{pmatrix}$$

*fixes $\frac{8}{3}$. And if one chooses the neighbours as $\frac{5}{2}$ and $\frac{11}{4}$ then the reflection element*

$$V' = \begin{pmatrix} -17 & 48 \\ -6 & 17 \end{pmatrix}$$

*fixes not only $\frac{8}{3}$ but also the circle centered at $\left(\frac{17}{6}, 0\right)$ with radius $r = \frac{1}{6}$.*

Suppose a Farey neighbour of $\frac{p}{q}$ is $\frac{r}{s}$. Then some other neighbours can be obtained by the mediant rule. The following two theorems based on this idea.

**Theorem 2.** *Let $\frac{p}{q} \in \mathbb{Q}_\infty$ then the parabolic element that fixes $\frac{p}{q}$ is*

$$V = \begin{pmatrix} \pm 1 - pq & p^2 \\ -q^2 & \pm 1 + pq \end{pmatrix}$$

*Proof.* Let $\frac{p}{q} \in \mathbb{Q}_\infty$ and $\frac{r}{s}$ is a Farey neighbour of $\frac{p}{q}$. By the mediant rule we have another Farey neighbour $\frac{p+r}{q+s}$ on the same side with $\frac{r}{s}$. Using the same technique in the proof of Theorem 1 we have the element $V$ as stated. Additionally the trace of the element $V$ is $\pm 2$ with determinant 1 which proves $V$ is parabolic in $\Gamma$. □

Unlike the Theorem 1, Theorem 2 gives an algorithm to obtain a parabolic element that fixes a given rational, without using anything but the rational. Here we do similar things to obtain a reflection whose fixed circle includes a given rational.

**Theorem 3.** *Let $\frac{p}{q} \in \mathbb{Q}_\infty$ and $\frac{r}{s}$ is a Farey parent of $\frac{p}{q}$. Then the reflection element in $\overline{\Gamma}$ that fixes $\frac{p}{q}$ is*

$$V = \begin{pmatrix} ps - pq + qr & p^2 - 2pr \\ 2qs - q^2 & -qr + qp - ps \end{pmatrix}$$

*Proof.* Suppose $\frac{p}{q} \in \mathbb{Q}_\infty$ and $\frac{r}{s}$ is a Farey parent of $\frac{p}{q}$. Another Farey parent of $\frac{p}{q}$ which is at the opposite side of $\frac{r}{s}$ can be obtained by the madiant rule. So we have this parent as $\frac{p-r}{q-s}$. The elements $V_1 = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$ and $V_2 = \begin{pmatrix} p & p-r \\ q & q-s \end{pmatrix}$ belong to $\overline{\Gamma}$. Although one of them is automorphism, the other is anti-automorphism since the Farey parents are at the opposite side of $\frac{p}{q}$. Hence the element $V = V_2.V_1^{-1} \in \overline{\Gamma}$ fixes $\frac{p}{q}$. Since $trV = 0$, $V$ is a reflection that the fixed point set is a circle that centered at $\left(\frac{ps-pq+qr}{2qs-q^2}, 0\right)$ with radius $r = \frac{1}{|2qs-q^2|}$ which proves the result. □

So far to this point, we have focused on Farey neighbours. Now observe all the Farey neighbours of a given reduced rational $\frac{p}{q}$. Suppose $\frac{r}{s}$ and $\frac{m}{k}$ are Farey parents of $\frac{p}{q}$ such that $\frac{r}{s} < \frac{p}{q} < \frac{m}{k}$. Then $\frac{p}{q}$ appears in $F_q$ via $\frac{r}{s} \oplus \frac{m}{k} = \frac{p}{q}$. In other words, the hyperbolic line segment joining $\frac{r}{s}$ and $\frac{m}{k}$ covers all the neighbours. Consequently all neighbours of $\frac{p}{q}$ can be obtained by the mediant rule;

$$\frac{r}{s} < \frac{r}{s} \oplus \frac{p}{q} = \frac{p+r}{q+s} < \frac{p+r}{q+s} \oplus \frac{p}{q} = \frac{2p+r}{2q+s} < ... < \frac{p}{q} < ...\frac{p}{q} \oplus \frac{m}{k} = \frac{p+m}{q+k} < \frac{m}{k}$$

## 4. Farey Paths, Integer Continued Fractions and Blocks in Extended Modular Group $\overline{\Gamma}$

In this section, the relation between integer continued fractions and Farey paths is used to obtain the word form of the element in $\overline{\Gamma}$, which fixes a given rational number, in terms of generators. A path in a graph consists of consequtive adjacent vertices. So a Farey path $< v_1, v_2, ..., v_n >$ is a path such that $v_i = \frac{p_i}{q_i}$ for $i = 1, 2, ..., n$ are reduced rationals and since the consequtive $v_i$'s are adjacent $p_i.q_{i-1} - q_i.p_{i-1} = \pm 1$. The Farey graph is connected hence there is a natural distance between two rationals $v$ and $w$ that is $d(v, w)$, the minimum number of edges in any path from $v$ to $w$ in $F_n$. The distance of an integer to $\infty$ is $d(\infty, x) = 1$.

**Lemma 1.** *[25] Let $\frac{p}{q} = [r_0; r_1, r_2, ..., r_n]$ be a reduced rational number then;*

$$U^{r_0}TU^{r_1}TU^{r_2}T...U^{r_n}T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

**Theorem 4.** *Let $\frac{p}{q}$ be a reduced rational and have an integer continued fraction expansion as $[r_0; r_1, r_2, ..., r_n]$, then the parabolic element fixing $\frac{p}{q}$ is*

$$U^{r_0}TU^{r_1}TU^{r_2}T...U^{r_n}T.U.TU^{-r_n}TU^{-r_{n-1}}T...U^{-r_1}TU^{-r_0} \quad (4)$$

*Proof.* Let $\frac{p}{q} = [r_0; r_1, r_2, ..., r_n]$. By Lemma 1, we have

$$\begin{aligned} & U^{r_0}TU^{r_1}TU^{r_2}T...U^{r_n}T.U.TU^{-r_n}TU^{-r_{n-1}}T...U^{-r_1}TU^{-r_0}\begin{pmatrix} p \\ q \end{pmatrix} \\ = & \quad U^{r_0}TU^{r_1}TU^{r_2}T...U^{r_n}T.U\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ = & \quad U^{r_0}TU^{r_1}TU^{r_2}T...U^{r_n}T\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ = & \quad \begin{pmatrix} p \\ q \end{pmatrix} \end{aligned}$$

Since conjugacy preserves the trace we have

$$\begin{aligned} & tr\left(U^{r_0}TU^{r_1}TU^{r_2}T...U^{r_n}T.U.TU^{-r_n}TU^{-r_{n-1}}T...U^{-r_1}TU^{-r_0}\right) \\ & = tr\left(U\right) = 2 \end{aligned}$$

which proves the element given in (4) is parabolic. □

We know from [9] that stabilizer of a point in $\Gamma$ is an infinite cyclic group. So we can give the following corollary.

**Corollary 2.** *Let $\frac{p}{q} = [r_0; r_1, r_2, ..., r_n] \in \mathbb{Q}$; then for all $0 \neq k \in \mathbb{Z}$;*

$$U^{r_0}TU^{r_1}TU^{r_2}T...U^{r_n}T.U^{k}.TU^{-r_n}TU^{-r_{n-1}}T...U^{-r_1}TU^{-r_0}$$

*is a parabolic element in $\Gamma$ whose fixed point is $\frac{p}{q}$.*

Now we obtain a reflection element as a word in generators of $\overline{\Gamma}$ that fixes a given rational $\frac{p}{q}$.

**Theorem 5.** *Let $\frac{p}{q}$ be a reduced rational and have an integer continued fraction expansion as $[r_0; r_1, r_2, ..., r_n]$, then the reflection element in $\overline{\Gamma}$ fixing $\frac{p}{q}$ is*

$$U^{r_0}TU^{r_1}TU^{r_2}T...U^{r_n}T.RTU.TU^{-r_n}TU^{-r_{n-1}}T...U^{-r_1}TU^{-r_0}$$

*Proof.* We have $RTU = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ as a reflection map. Furthermore $RTU\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The rest of the proof follows similar to the proof of Theorem 4. □

**Example 2.** *Choose the rational $\frac{8}{5}$. The integer continued fraction expansion of $\frac{8}{5}$ is*

$$\frac{8}{5} = 2 - \frac{1}{3 - \frac{1}{2}} = [2; 3, 2].$$

*Then the parabolic element fixing $\frac{8}{5}$ is*

$$U^2TU^3TU^2TUTU^{-2}TU^{-3}TU^{-2} = \begin{pmatrix} -39 & 64 \\ -25 & 41 \end{pmatrix}.$$

*And the reflection element is*

$$U^2TU^3TU^2TRTUTU^{-2}TU^{-3}TU^{-2} = \begin{pmatrix} -89 & 104 \\ -55 & 89 \end{pmatrix}$$

Here we mention about relations between paths in the Farey graph and integer continued fractions. The convergents of a certain continued fraction expansion of a reduced rational $\frac{p}{q} = [r_0; r_1, ..., r_n]$, are defined as $C_i = \frac{p_i}{q_i} = [r_0; r_1, ..., r_i]$ for $0 \leq i \leq n$, where $C_0 = \frac{p_0}{q_0} = \frac{r_0}{1}$ and $C_n = \frac{p_n}{q_n} = \frac{p}{q}$. Furthermore we know that $p_i.q_{i-1} - q_i.p_{i-1} = \pm 1$. Hence every consequtive pair $C_i$ and $C_{i-1}$ are Farey neighbours. Also, since $C_0 = r_0 \in \mathbb{Z}$ and every integer is adjacent to infinity with a vertical line, $< \infty, C_0, C_1, ..., C_{n-1}, C_n >$ is a path from $\infty$ to $\frac{p}{q}$. To sum up every integer fraction expansion of a rational $\frac{p}{q}$ is related to a path from $\infty$ to $\frac{p}{q}$. Moreover the shortest integer continued fraction of $\frac{p}{q}$ is related to a geodesic path from $\infty$ to $\frac{p}{q}$. In Theorem 4 and Theorem 5, the integer continued fraction expansion of a given rational is related to an element in $\overline{\Gamma}$ that fixes the rational. It is possible to make connections with Farey paths.

## 5. Block Reduced Forms in the Extended Modular Group $\overline{\Gamma}$

Every element in $\overline{\Gamma}$ can be expressed as a word of $T, S$ and $R$ denoted by $W(T, S, R)$. Consider the blocks

$$TS = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad TS^2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Using these blocks every reduced word $W(T, S, R)$ in $\overline{\Gamma}$ where the sum of exponents of $R$ is an even number can be expressed as;

$$S^i (TS)^{m_0} \left(TS^2\right)^{n_0} ... (TS)^{m_k} \left(TS^2\right)^{n_k} T^j,$$

and every reduced word $W(T, S, R)$ in $\overline{\Gamma}$ where the sum of exponents of $R$ is an odd number can be expressed as;

$$S^i (TS)^{m_0} \left(TS^2\right)^{n_0} ... (TS)^{m_k} \left(TS^2\right)^{n_k} T^j R$$

Here $i = 0, 1, 2$, $j = 0, 1$, $m_0$ and $n_k$ may be zero and other exponents are positive integers. This representetion is known as the *block reduced form* [13]. For example, the block reduced form of the word $W(T, S, R) = TSTSTSSTSST$ is $(TS)^2 . \left(TS^2\right)^2 T$. And the block reduced form of the word $W(T, S, R) = RTS^2RTS^2R$ is $(TS) . \left(TS^2\right)^2 R$. Trace classes of the modular group and extended modular group are studied in [6, 13] by using the block reduced form. In this section we give the block reduced form of the element in $\overline{\Gamma}$ fixing a given rational $\frac{p}{q}$.

**Theorem 6.** *Let $\frac{p}{q}$ be a reduced rational number and have an integer continued fraction expansion $[r_0; r_1, ..., r_n]$ then the block form of the parabolic element fixing $\frac{p}{q}$ is*

$$\begin{aligned} W(T,S,R) &= (TS)^{r_0-1}\left(TS^2\right)(TS)^{r_1-2}\left(TS^2\right)...(TS)^{r_{n-1}-2}\left(TS^2\right). \\ &\quad (TS)^{r_n-1}\left(TS^2\right)^{-1}(TS)^{-r_n-1}\left(TS^2\right)(TS)^{-r_{n-1}-2}\left(TS^2\right). \\ &\quad ...(TS)^{-r_1-2}\left(TS^2\right)(TS)^{-r_0-1} \end{aligned}$$

*Proof.* By Theorem 4, we know that

$$U^{r_0}TU^{r_1}TU^{r_2}T...U^{r_n}T.U.TU^{-r_n}TU^{-r_{n-1}}T...U^{-r_1}TU^{-r_0}$$

fixes $\frac{p}{q}$. Considering $U = TS$ we have

$$\begin{aligned} W(T,S,R) &= (TS)^{r_0}.T.(TS)^{r_1}.T...(TS)^{r_{n-1}}.T.(TS)^{r_n}.T \\ &\quad (TS).T.(TS)^{-r_n}.T.(TS)^{-r_{n-1}}.T.....(TS)^{-r_1}.T.(TS)^{-r_0} \\ &= (TS)^{r_0-1}TS.T.TS.(TS)^{r_1-2}TS.T...TS(TS)^{r_{n-1}-2}TS. \\ &\quad T.TS(TS)^{r_n-1}.T(TS).T.(TS)^{-r_n-1}TS.T.TS(TS)^{-r_n-2}. \\ &\quad TS.T.....TS(TS)^{-r_1-2}TS.T.TS(TS)^{-r_0-1} \end{aligned}$$

Using the relations $T^2 = I$ and $\left(TS^2\right)^{-1} = ST$,

$$\begin{aligned} W(T,S,R) &= (TS)^{r_0-1}.\left(TS^2\right).(TS)^{r_1-2}.\left(TS^2\right)...\left(TS^2\right).(TS)^{r_{n-1}-2}. \\ &\quad \left(TS^2\right).(TS)^{r_n-1}.\left(TS^2\right)^{-1}(TS)^{-r_n-1}.\left(TS^2\right).(TS)^{-r_{n-2}}. \\ &\quad \left(TS^2\right).....\left(TS^2\right).(TS)^{-r_1-2}.\left(TS^2\right).(TS)^{-r_0-1} \end{aligned}$$

□

**Theorem 7.** *Let $\frac{p}{q}$ be a reduced rational number and have an integer continued fraction expansion $[r_0; r_1, ..., r_n]$ then the block form of the reflection element fixing $\frac{p}{q}$ is*

$$\begin{aligned} W(T,S,R) &= (TS)^{r_0-1}.\left(TS^2\right).(TS)^{r_1-2}.\left(TS^2\right).....(TS)^{r_{n-1}-2}.\left(TS^2\right). \\ &\quad (TS)^{r_n}.\left(TS^2\right)^{-r_n-2}.(TS).\left(TS^2\right)^{-r_{n-1}-2}.(TS).... \\ &\quad (TS).\left(TS^2\right)^{-r_1-2}.(TS)\left(TS^2\right)^{-r_0-1}.R \end{aligned}$$

*Proof.* From Theorem 5, the reflection element fixing $\frac{p}{q}$ is

$$U^{r_0}TU^{r_1}TU^{r_2}T...U^{r_n}T.RTU.TU^{-r_n}TU^{-r_{n-1}}T...U^{-r_1}TU^{-r_0}.$$

After substituting $U = TS$ in the word above, we have

$$\begin{aligned} W(T,S,R) &= (TS)^{r_0}T(TS)^{r_1}T...(TS)^{r_{n-1}}T(TS)^{r_n}T \\ &\quad RT(TS)T(TS)^{-r_n}T(TS)^{-r_{n-1}}T...(TS)^{-r_1}T(TS)^{-r_0} \\ &= (TS)^{r_0-1}TSTTS(TS)^{r_1-2}TSTTS...TS(TS)^{r_{n-1}-2}TS \\ &\quad TTS(TS)^{r_n-1}TRST(TS)(TS)^{-r_n-2}TSTTS(TS)^{-r_{n-1}-2} \end{aligned}$$

$$TST...TS\,(TS)^{-r_1-2}\,TSTTS\,(TS)^{-r_0-1}$$

Since $(TR)^2=(SR)^2=I$ we obtain $TR=RT$ and $SR=RS^2$. Hence,

$$\begin{aligned} W(T,S,R) &= (TS)^{r_0-1}\left(TS^2\right)(TS)^{r_1-2}\left(TS^2\right)...(TS)^{r_{n-1}-2}\left(TS^2\right) \\ & \quad (TS)^{r_n}\left(TS^2\right)^{-r_n-2}(TS)\left(TS^2\right)^{-r_{n-1}-2}(TS)... \\ & \quad (TS)\left(TS^2\right)^{-r_1-2}(TS)\left(TS^2\right)^{-r_0-1}R \end{aligned}$$

□

We can obtain elements which fix a given rational $\frac{p}{q}$ in terms of $TS$ and $TS^2$ by finding a path from $\infty$ to $\frac{p}{q}$ in the Farey graph. We explain this with an example:

**Example 3.** *Suppose the given rational is $\frac{-10}{3}$. Then one may choose the path $<\infty,-3,\frac{-13}{4},\frac{-10}{3}>$, see Figure 2. We know the consequtive vertices in this path are consequtive convergents of the integer continued fraction expansion of the rational $\frac{-10}{3}$ i.e., $C_0=-3$, $C_1=\frac{-13}{4}$ and $C_2=\frac{-10}{3}$. Hence, we obtain the integer continued fraction expansion as*

$$-3-\cfrac{1}{4-\frac{1}{1}}=[-3,4,1]$$

*Using the values $r_0=-3$, $r_1=4$ and $r_2=1$ in Theorem 6 we have the parabolic element fixing $\frac{-10}{3}$ in blocks $TS$ and $TS^2$ as follows:*

$$\begin{aligned} W(T,S,R) &= (TS)^{-4}\left(TS^2\right)(TS)^2\left(TS^2\right)(TS)^0\left(TS^2\right)^{-1}(TS)^{-2} \\ & \quad \left(TS^2\right)(TS)^{-6}.\left(TS^2\right)(TS)^2 \end{aligned}$$

*We can reduce this word by the presentation of $\Gamma$ as;*

$$W(T,S,R)=S^2.\left(TS^2\right)^2.(TS)^2.\left(TS^2\right)^4.(TS)^3$$

*For the reflection element fixing $\frac{-10}{3}$ we use Theorem 7;*

$$\begin{aligned} W(T,S,R) &= (TS)^{-4}\left(TS^2\right)(TS)^2\left(TS^2\right)(TS)^1\left(TS^2\right)^{-3}(TS) \\ & \quad \left(TS^2\right)^{-6}(TS)\left(TS^2\right)^2R \end{aligned}$$

*The block reduced form of this word can be obtained by the relators of $\overline{\Gamma}$;*

$$W(T,S,R)=S^2.\left(TS^2\right)^2.(TS)^3.\left(TS^2\right)^3.(TS)^3.\left(TS^2\right)^3.R$$

## 6. Fibonacci Sequence and New Block Reduced Forms

Jones and Thornton obtained relations between elements of extended modular group and Fibonacci numbers in [10]. Özgür defined two new sequences which are generalizations of Fibonacci and Lucas sequences for the Hecke group $H(\sqrt{q})$ where $q\geq 5$ prime [32]. Also there are some results for Modular group and Pell Fibonacci and Lucas numbers in [14–16, 23]. Koruoğlu and Şahin used a generalized version of Fibonacci sequence to get relations with extended Hecke groups $\overline{H}(\lambda)$ in [12]. In

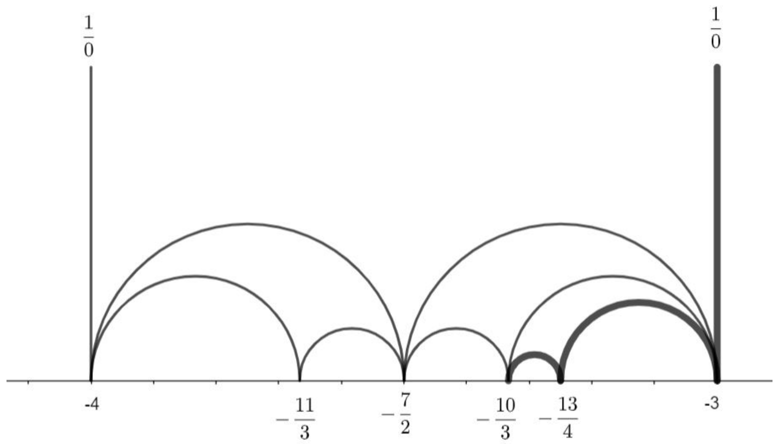


FIGURE 2. The path $< \infty, -3, \frac{-13}{4}, \frac{-10}{3} >$

same study they give an application to extended modular group $\overline{\Gamma}$. They considered the following elements:

$$f = RTS = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad h = TSR = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

The $k^{th}$ power of $f$ and $h$ are;

$$f^k = \begin{pmatrix} f_{k-1} & f_k \\ f_k & f_{k+1} \end{pmatrix} \quad \text{and} \quad h^k = \begin{pmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{pmatrix}$$

where $f_k$ denotes the $kth$ Fibonacci number. Hence every element in extended modular group can be expressed as a word in $f$ and $h$. This reduced word called *New Block Reduced Form*. The relations between block reduced forms and new block reduced forms are;

$$TS = Rf = hR \tag{5}$$

$$TS^2 = Rh = fR \tag{6}$$

It is proved that every block reduced word has a New Block Reduced Form. From this viewpoint we can express the element given in Theorem 6 and Theorem 7 in new block reduced form. We explain this with an example.

In example 3 the parabolic element fixing $\frac{-10}{3}$ is;

$$S^2 \left(TS^2\right)^2 (TS)^2 \left(TS^2\right)^4 (TS)^3$$

Using the relations 5 and 6 and $S^2 = TfR$; we can write this word;

$$TfR. (Rh.fR) . (Rf.hR) . (Rh.fR.Rh.fR) . (Rf.hR.Rf)$$

Since $R^2 = I$ we have the block reduced form;

$$T.f.h.f^2.h^2.f.h.f^2.h.f = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & f_1 \\ f_1 & f_2 \end{pmatrix} \cdot \begin{pmatrix} f_2 & f_1 \\ f_1 & 0 \end{pmatrix} \begin{pmatrix} f_1 & f_2 \\ f_2 & f_3 \end{pmatrix} \begin{pmatrix} f_3 & f_2 \\ f_2 & f_1 \end{pmatrix} \begin{pmatrix} 0 & f_1 \\ f_1 & f_2 \end{pmatrix} \begin{pmatrix} f_2 & f_1 \\ f_1 & 0 \end{pmatrix} \begin{pmatrix} f_1 & f_2 \\ f_2 & f_3 \end{pmatrix} \begin{pmatrix} f_2 & f_1 \\ f_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & f_1 \\ f_1 & f_2 \end{pmatrix}$$

It is stated in the same example that the reflection element fixing $\frac{-10}{3}$ is;

$$S^2 \left(TS^2\right)^2 (TS)^3 \left(TS^2\right)^3 (TS)^3 \left(TS^2\right)^3 R$$

Following the same procedure above we have the new block reduced form of this word as;

$$T.f.h. \left(f^2.h\right)^4 .f = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & f_1 \\ f_1 & f_2 \end{pmatrix} \begin{pmatrix} f_2 & f_1 \\ f_1 & 0 \end{pmatrix} \left[ \begin{pmatrix} f_1 & f_2 \\ f_2 & f_3 \end{pmatrix} \begin{pmatrix} f_2 & f_1 \\ f_1 & f_0 \end{pmatrix} \right]^4 \begin{pmatrix} 0 & f_1 \\ f_1 & f_2 \end{pmatrix}$$

## 7. Conclusion

In this article, elements in the extended modular group $\overline{\Gamma}$ which fix rationals, are considered. Matrix representations of parabolic and reflection elements which fix a given rational are mentioned in Section 3 via Farey neighbours. In Section 4 relationship between Farey paths and elements of $\overline{\Gamma}$ which have rational fixed points, is established. And these elements obtained as words in generators $U$ and $T$. Then, block reduced form of these words are given in Section 5. We use new block reduced forms in Section 6 to establish relations with Fibonacci numbers. As a summary of this work we give a final example, see Table 1.

| | |
|---|---|
| Path | $< \infty, 0, \frac{1}{2}, \frac{3}{7} >$ |
| ICF | $[0; -2, 3]$ |
| W(U,T) for parabolic element | $T.U^{-2}.T.U^3.T.U.T.U^{-3}.T.U^2.T$ |
| BRF for parabolic element | $\left(TS^2\right)^2 . (TS)^2 . \left(TS^2\right)^4 . (TS)^2 .T$ |
| NBRF for parabolic element | $f.h^2.f^2.h.f.h^2.f.T$ |
| W(U,T) for reflection element | $T.U^{-2}.T.U^3.T.R.T.U.T.U^{-3}.T.U^2.T$ |
| BRF for reflection element | $\left(TS^2\right)^2 . (TS)^3 . \left(TS^2\right) . (TS)^3 . \left(TS^2\right)^2 .T.R$ |
| NBRF for reflection element | $f.h^2.f.h^3.f.h^2.f.T$ |

Table 1. Elements in $\overline{\Gamma}$ fixing $\frac{3}{7}$

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