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UNIQUENESS OF THE SOLUTION TO THE INVERSE PROBLEM OF SCATTERING THEORY FOR SPECTRAL PARAMETER DEPENDENT KLEIN-GORDON S-WAVE EQUATION

Esra KIR ARPAT[1] and Turhan KOPRUBASI[2]

[1]Department of Mathematics, Gazi University, 06500 Ankara, TÜRKİYE
[2]Department of Mathematics, Kastamonu University, 37100 Kastamonu, TÜRKİYE

Abstract. In the present work, the inverse problem of the scattering theory for Klein-Gordon s-wave equation with a spectral parameter in the boundary condition is investigated. We define the scattering data set, and obtain the main equation of operator. Furthermore, the uniqueness of the solution of the inverse problem is proved.

1. Introduction

Scattering problems, which play a role in the structure of matter in Newtonian mechanics, are an important research topic of mathematical physics. Obtaining the scattering data by giving the potential function and investigating the properties of these scattering data is called the direct problem in scattering theory, while obtaining the potential function according to the scattering data is called the inverse problem. Therefore, the importance of inverse scattering problems in terms of natural sciences is an undeniable reality.

The inverse problem of scattering theory for the boundary value problem

$$-y'' + q(x)y = \lambda^2 y, \tag{1}$$
$$y(0) = 0 \tag{2}$$

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[1] esrakir@gazi.edu.tr; 0000-0002-6322-5130
[2] tkoprubasi@kastamonu.edu.tr-Corresponding author; 0000-0003-1551-1527.

was studied in [13] and the author obtained that the Jost function of (1)-(2) defined by

$$e(\lambda) = 1 + \int_0^\infty K(0,t)e^{i\lambda t}dt \ , \quad \lambda \in \overline{\mathbb{C}}_+ := \{\lambda : \lambda \in \mathbb{C}, \ \operatorname{Im}\lambda \geq 0\} .$$

has a finite number of simple zeros in $\mathbb{C}_+$. The scattering data of (1)-(2) is

$$\{S(\lambda), \lambda_k, m_k : k = 1, 2, ..., n\} ,$$

where λ_k are the zeros of Jost function, m_k^{-1} are the norm of the zeros of Jost function for $\lambda = \lambda_k$ in $L_2(0,\infty)$ and $S(\lambda)$ is scattering function of (1)-(2) given by

$$S(\lambda) := \frac{\overline{e(\lambda)}}{e(\lambda)} \ , \quad \lambda \in (-\infty, \infty).$$

As the potential function q is given, the problem of getting scattering data and investigating the properties of scattering data is called the direct problem for scattering theory. Oppositely, finding the potential function q according to the scattering data is known inverse problem of scattering theory. The direct and inverse scattering problems for a selfadjoint infinite system second-order difference equations with operator valued coefficients were considered in [11]. The uniqueness of the solution to the inverse problem of scattering theory for equation (1) with a spectral parameter in the boundary condition

$$y'(0) + (\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2)y(0) = 0$$

was studied bh Kh. R. Mamedov ([12]). Also, the solution to the inverse problem of scattering theory for spectral parameter dependent Sturm-Liouville operator system was founded uniquely by G. Bascanbaz Tunca and E. Kir Arpat in [15], and the scattering analysis of a transmission boundary value problem which consists of a discrete Schrodinger equation and transmission conditions was investigated in [5]. Furthermore, the scattering theory of impulsive Sturm-Liouville equations, impulsive discrete Dirac systems, impulsive Sturm-Liouville equation in Quantum-Calculus and Dirac operator with impulsive condition on whole axis were investigated in [1,4,8,9]. The scattering function of impulsive matrix difference operators and scattering properties of eigenparameter dependent discrete impulsive Sturm-Liouville equations were studied in [2,3,6]. But scattering theory of Klein-Gordon s-wave equation with boundary condition depends on spectral parameter has not been investigated yet.

Let L_μ denotes the Klein-Gordon s-wave operator of second order with boundary condition generated by

$$y'' + [\lambda - q(x)]^2 \, y = 0 \ , \quad 0 \leq x < \infty \tag{3}$$

and

$$y'(0,\lambda) + (\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2)y(0,\lambda) = 0,$$

where $\lambda = \mu^2$ is a complex spectral parameter, α_i are real numbers for $i = 0, 1, 2$, $\alpha_1 \leq 0$, $\alpha_2 > 0$, $(\alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2) \neq 0$ and q is a non-negative real valued function satisfying the following condition

$$\int_0^\infty x \left[|q(x)| + |q'(x)| \right] dx < \infty. \tag{4}$$

In this paper, we examine the inverse problem of scattering theory of L_μ under the condition (4).

2. Preliminaries

To be able to well defined mapping between λ and μ, we will study on the region $\operatorname{Re} \mu \geq 0$. If the condition (4) is satisfied, equation (3) has the following solutions

$$f^{(1)}(x, \mu) = f(x, \mu^2) = e^{i\left[\alpha(x) + \mu^2 x\right]} + \int_x^\infty K(x, t) e^{i\mu^2 t} dt, \tag{5}$$

$$\overline{f^{(1)}(x, \mu)} = \overline{f(x, \mu^2)} = e^{-i\left[\alpha(x) + \mu^2 x\right]} + \int_x^\infty K(x, t) e^{-i\mu^2 t} dt$$

for $\mu \in \mathbb{R}_1 := \{\mu : \operatorname{Re} \mu \geq 0,\ \operatorname{Im} \mu = 0\}$ and they have analytic continuation to $\overline{\mathbb{C}_1^+} := \{\mu \in \mathbb{C} : \operatorname{Re} \mu \geq 0,\ \operatorname{Im} \mu \geq 0\}$ and $\overline{\mathbb{C}_1^-} := \{\mu \in \mathbb{C} : \operatorname{Re} \mu \geq 0,\ \operatorname{Im} \mu \leq 0\}$, respectively where $\alpha(x) = \int_x^\infty q(t) dt$ and $K(x, t)$ is solution of integral equations of Volterra type which has continuous derivatives with respect to their arguments ([7]). Moreover, $K(x, t)$, $K_x(x, t)$, $K_t(x, t)$ satisfy the following inequalities

$$|K(x, t)| \leq c\omega\left(\tfrac{x+t}{2}\right) \exp\left(\gamma(x)\right),$$

$$|K_x(x, t)|, |K_t(x, t)| \leq c\left[\omega^2\left(\tfrac{x+t}{2}\right) + \theta\left(\tfrac{x+t}{2}\right)\right],$$

where

$$\omega(x) = \int_x^\infty \left[|q(t)|^2 + \left| q'(t) \right| \right] dt\ ,$$

$$\gamma(x) = \int_x^\infty \left[t\,|q(t)|^2 + 2\,|q(t)| \right] dt\ ,$$

$$\theta(x) = \tfrac{1}{4}\left[2\,|q(x)|^2 + \left| q'(x) \right| \right]$$

and $c>0$ is a constant. In addition, the function $K(x,t)$ and potential are related to

$$K(x,x)=2\int_x^\infty q(t)dt$$

([14]). Furthermore, $f^{(1)}(x,\mu)$ and $\overline{f^{(1)}(x,\mu)}$ are respectively analytic in $\mathbb{C}_1^+ := \{\mu\in\mathbb{C} : \operatorname{Re}\mu>0,\ \operatorname{Im}\mu>0\}$ and $\mathbb{C}_1^- := \{\mu\in\mathbb{C} : \operatorname{Re}\mu>0,\ \operatorname{Im}\mu<0\}$ and they are continuous on real and imaginary axes with respect to μ. The solutions $f^{(1)}(x,\mu)$ and $\overline{f^{(1)}(x,\mu)}$ are called Jost solutions of L_μ ([10]). From (5), $f^{(1)}(x,\mu)$ satisfies the asymptotic equalities

$$\begin{array}{l} f^{(1)}(x,\mu)=e^{i\mu^2x}\left[1+o(1)\right]\ ,\ x\to\infty\ , \\ f_x^{(1)}(x,\mu)=e^{i\mu^2x}\left[i\mu^2+o(1)\right]\ ,\ x\to\infty \end{array} \tag{6}$$

and

$$f^{(1)}(x,\mu)=e^{i\left[\alpha(x)+\mu^2x\right]}+o(1)\ ,\ \ |\mu|\to\infty \tag{7}$$

([14]). From (6), the Wronskian of the solutions of $f^{(1)}(x,\mu)$ and $\overline{f^{(1)}(x,\mu)}$ is

$$W\left[f^{(1)}(x,\mu),\overline{f^{(1)}(x,\mu)}\right]=\lim_{x\to\infty}W\left[f^{(1)}(x,\mu),\overline{f^{(1)}(x,\mu)}\right]=-2i\mu^2 \tag{8}$$

for $\mu\in\mathbb{R}_1$. Hence $f^{(1)}(x,\mu)$ and $\overline{f^{(1)}(x,\mu)}$ are the fundamental solutions of (3) for $\mu\in\mathbb{R}_1^*=\mathbb{R}_1\backslash\{0\}$.

Let $\varphi^{(1)}(x,\mu)=\varphi(x,\mu^2)$ denotes the solution of (3) satisfying the initial conditions

$$\begin{array}{rcl} \varphi^{(1)}(0,\mu) & = & \varphi(0,\mu^2)=1, \\ \varphi_x^{(1)}(0,\mu) & = & \varphi_x(0,\mu^2)-(\alpha_0+\alpha_1\mu^2+\alpha_2\mu^4). \end{array}$$

Definition 1.

$$\begin{array}{rcl} W\left[\varphi^{(1)}(x,\mu),f^{(1)}(x,\mu)\right] & = & \varphi^{(1)}(0,\mu)f_x^{(1)}(0,\mu)-\varphi_x^{(1)}(0,\mu)f^{(1)}(0,\mu) \\ & = & f_x^{(1)}(0,\mu)+(\alpha_0+\alpha_1\mu^2+\alpha_2\mu^4)f^{(1)}(0,\mu) \\ & = & F(\mu^2)=F_1(\mu) \end{array} \tag{9}$$

is called Jost function of L_μ ([10]).

Theorem 1. *Under the condition (4), Jost function has following asymptotic equality*

$$F_1(\mu)\approx\left\{\begin{array}{ccc} i\mu^2(1-i\alpha_1)e^{i\alpha(0)} & , & \alpha_1\neq0,\ |\mu|\to\infty \\ \alpha_2\mu^4 & , & \alpha_1=0,\ |\mu|\to\infty \end{array}\right. , \tag{10}$$

where $\alpha_1\le0$ and $\alpha_2>0$.

Proof. This aymptotic equality can be seen smoothly from (7) and Definition 1. □

3. Main Equation of L_μ

Definition 2. *We can define scattering function using Jost function as follows for $\mu \in \mathbb{R}_1$:*

$$S_1(\mu) = S(\mu^2) = \frac{\overline{F(\mu^2)}}{F(\mu^2)} = \frac{\overline{F_1(\mu)}}{F_1(\mu)}. \tag{11}$$

Theorem 2. *Under the condition (4), the scattering function satisfies following asymptotic equality*

$$S_1(\mu) = 1 + O\left(\frac{1}{\mu^2}\right), \quad |\mu| \to \infty. \tag{12}$$

Proof. The proof can be easily attained using definition of scattering function and (7). □

Lemma 1. *Under the condition (4),*

$$F_1(\mu) = f_x^{(1)}(0,\mu) + (\alpha_0 + \alpha_1\mu^2 + \alpha_2\mu^4) f^{(1)}(0,\mu) \neq 0$$

for all $\mu \in \mathbb{R}_1^*$.

Proof. Let $F_1(\mu_0) = 0$ for any $\mu_0 \in \mathbb{R}_1^*$. Then, we obtain

$$f_x^{(1)}(0,\mu_0) = -(\alpha_0 + \alpha_1\mu_0 + \alpha_2\mu_0^4) f^{(1)}(0,\mu_0).$$

Also,

$$W\left[\overline{f^{(1)}(x,\mu)}, f^{(1)}(x,\mu)\right] = 2i\mu^2.$$

for all $\mu \in \mathbb{R}_1$. So,

$$f_x^{(1)}(0,\mu_0)\overline{f^{(1)}(0,\mu_0)} - f^{(1)}(0,\mu_0)\overline{f_x^{(1)}(0,\mu_0)} = 2i\mu_0^2$$

and, we get

$$-(\alpha_0+\alpha_1\mu_0^2+\alpha_2\mu_0^4) f^{(1)}(0,\mu_0)\overline{f^{(1)}(0,\mu_0)} + (\alpha_0+\alpha_1\mu_0^2+\alpha_2\mu_0^4)\overline{f^{(1)}(0,\mu_0)} f^{(1)}(0,\mu_0) = 2i\mu_0^2.$$

From last equation, we can write

$$2i\mu_0^2 = 0.$$

But this is a contradiction because of $\mu_0 \in \mathbb{R}_1^*$. □

Lemma 2. *The following equation*

$$\frac{2i\mu^2\varphi^{(1)}(x,\mu)}{f_x^{(1)}(0,\mu) + (\alpha_0 + \alpha_1\mu^2 + \alpha_2\mu^4) f^{(1)}(0,\mu)} = \overline{f^{(1)}(x,\mu)} - S_1(\mu) f^{(1)}(x,\mu) \tag{13}$$

holds. Furthermore, $\overline{S_1(\mu)} = [S_1(\mu)]^{-1}$.

Proof. Since $f^{(1)}(x,\mu)$ and $\overline{f^{(1)}(x,\mu)}$ are basic solutions of L_μ,

$$\varphi^{(1)}(x,\mu) = c_1 f^{(1)}(x,\mu) + c_2\overline{f^{(1)}(x,\mu)}. \tag{14}$$

From (14),

$$c_1(\mu) f^{(1)}(0,\mu) + c_2(\mu)\overline{f^{(1)}(0,\mu)} = 1$$

and

$$c_1 f_x^{(1)}(x,\mu) + c_2\overline{f_x^{(1)}(x,\mu)} = -(\alpha_0+\alpha_1\mu^2+\alpha_2\mu^4).$$

By finding $c_1(\mu)$ and $c_2(\mu)$ from last two equations and using (8), we can obtain (13). In addition, we hold easily

$$\overline{S_1(\mu)} = \frac{F_1(\mu)}{\overline{F_1(\mu)}} = [S_1(\mu)]^{-1}$$

from (11). □

Lemma 3. *The all zeros of Jost function $F_1(\mu)$ are finite and on the imaginary axis. Also, they are simply on the upper imaginary axis.*

Proof. Using asymptotic equality (10), Lemma 1, uniqueness theorems for analytic functions and Bolzano-Weierstrass Theorem we can easily reach finiteness of the zeros of Jost function. Now, we will show that the zeros of $F_1(\mu)$ are on the imaginary axis. Let μ_0 be an arbitrary zero of $F_1(\mu)$. We can write

$$0 = F_1(\mu_0) = f_x^{(1)}(0,\mu_0) + (\alpha_0+\alpha_1\mu_0^2+\alpha_2\mu_0^4)f^{(1)}(0,\mu_0)$$

and

$$\begin{cases} f_{xx}^{(1)}(x,\mu_0) + \left[\mu_0^4 - 2\mu_0^2 q(x) + q^2(x)\right] f_x^{(1)}(x,\mu_0) = 0, \\ \overline{f_{xx}^{(1)}(x,\mu_0)} + \left[\overline{\mu_0^4} - 2\overline{\mu_0^2} q(x) + q^2(x)\right]\overline{f_x^{(1)}(x,\mu_0)} = 0 \end{cases}$$

from (3) and (9). By using the last equalities together the definition of Wronskian and the partial integration method, we find that

$$\begin{aligned} 0 \;=\; & \left(\mu_0^2 - \overline{\mu_0^2}\right)\Bigg\{\alpha_1\left|f^{(1)}(0,\mu_0)\right|^2 + \left(\mu_0^2+\overline{\mu_0^2}\right)\left[\alpha_2 + \int_0^\infty \left|f^{(1)}(x,\mu_0)\right|^2 dx\right] \\ & -2\int_0^\infty q(x)\left|f^{(1)}(x,\mu_0)\right|^2 dx\Bigg\} \end{aligned}$$

and then

$$\begin{aligned} 0 \;=\; & \left(\mu_0^2 - \overline{\mu_0^2}\right)\Bigg\{\alpha_1\left|f^{(1)}(0,\mu_0)\right|^2 + \left[(\operatorname{Re}\mu_0)^2 - (\operatorname{Im}\mu_0)^2\right]\left[\alpha_2 + \int_0^\infty \left|f^{(1)}(x,\mu_0)\right|^2 dx\right] \\ & -2\int_0^\infty q(x)\left|f^{(1)}(x,\mu_0)\right|^2 dx\Bigg\}. \end{aligned}$$

The last equality is satisfied if $\mu_0^2 - \overline{\mu_0^2} = 0$ and $(\operatorname{Re}\mu_0)^2 = 0$, i.e. $\operatorname{Re}\mu_0 = 0$. So, all zeros of $F_1(\mu)$ are on the imaginary axis. Finally, to get the simplicity of any zero $\mu_0 = i\omega_0$, $\omega_0 > 0$, we need to prove that

$$\frac{\partial F_1(\mu_0)}{\partial \mu} \neq 0.$$

From equation (3), we have

$$\begin{aligned}
\overline{f_{xx}^{(1)}(x,\mu)} + q^2(x)\overline{f^{(1)}(x,\mu)} &= 2\mu^2 q(x)\overline{f^{(1)}(x,\mu)} - \mu^4\overline{f^{(1)}(x,\mu)}\ , \\
\left(\dot{f}_{xx}^{(1)}\right)(x,\mu) + q^2(x)\left(\dot{f}^{(1)}\right)(x,\mu) &= 4\mu q(x) f^{(1)}(x,\mu) + 2\mu^2 q(x)\left(\dot{f}^{(1)}\right)(x,\mu) \\
&\quad -4\mu^3 f^{(1)}(x,\mu) - \mu^4\left(\dot{f}^{(1)}\right)(x,\mu)
\end{aligned}$$

and then

$$4\mu\int_0^\infty \left[q(x)-\mu^2\right]\left|f^{(1)}(x,\mu)\right|^2 dx = \left(\dot{f}^{(1)}\right)(0,\mu)\overline{f_x^{(1)}(0,\mu)} - \left(\dot{f}_x^{(1)}\right)(0,\mu)\overline{f^{(1)}(0,\mu)}\ ,$$

where $\left.\frac{\partial f^{(1)}(x,\mu)}{\partial\mu}\right|_{x=0} := \left(\dot{f}^{(1)}\right)(0,\mu)$ and $\mu = i\omega$, $\omega \geq 0$. Also, we find the following equation

$$\begin{aligned}
4i\omega\int_0^\infty \left[q(x)+\omega^2\right]\left|f^{(1)}(x,i\omega)\right|^2 dx &= \left(\dot{f}^{(1)}\right)(0,i\omega)\overline{f_x^{(1)}(0,i\omega)} \\
&\quad -\left(\dot{f}_x^{(1)}\right)(0,i\omega)\overline{f^{(1)}(0,i\omega)}. \qquad (15)
\end{aligned}$$

By the definition of $F_1(\mu)$, we hold

$$\begin{aligned}
f_x^{(1)}(0,\mu) &= F_1(\mu) - (\alpha_0 + \alpha_1\mu^2 + \alpha_2\mu^4) f^{(1)}(0,\mu)\ , \\
\left(\dot{f}_x^{(1)}\right)(0,\mu) &= (\dot{F}_1)(\mu) - \left(2\alpha_1\mu + 4\alpha_2\mu^3\right) f^{(1)}(0,\mu) - (\alpha_0 + \alpha_1\mu^2 + \alpha_2\mu^4)\left(\dot{f}^{(1)}\right)(0,\mu).
\end{aligned}$$

These derivatives are taken into account in the equation (15) with $\mu_0 = i\omega_0$, $\omega_0 > 0$,

$$\begin{aligned}
4i\omega_0\int_0^\infty \left[q(x)+\omega_0^2\right]\left|f^{(1)}(x,i\omega_0)\right|^2 dx &= -(\dot{F}_1)(i\omega_0)\overline{f^{(1)}(0,i\omega_0)} \\
&\quad +i\left(2\alpha_1\omega_0 - 4\alpha_2\omega_0^3\right)\left|f^{(1)}(0,i\omega_0)\right|^2 (16)
\end{aligned}$$

and from (3.6)

$$-(\dot{F}_1)(i\omega_0)\overline{f^{(1)}(0,i\omega_0)} = i\left[\left(-2\alpha_1\omega_0 + 4\alpha_2\omega_0^3\right)\left|f^{(1)}(0,i\omega_0)\right|^2\right.$$

$$+4\omega_0\int_0^\infty \left[q(x)+\omega_0^2\right]\left|f^{(1)}(x,i\omega_0)\right|^2 dx\Bigg]. \quad (17)$$

If $f^{(1)}(0,i\omega_0)=0$ in (17), then it is occured that $f^{(1)}(x,i\omega_0)\equiv 0$ but this can not be. So, it is clear that the left side of (17) is nonzero. Therefore, it is attained that $(\dot{F}_1)(\mu_0)\neq 0$ with $F_1(\mu_0)=0$. So, the zeros of Jost function are simply on the upper imaginary axis. □

Lemma 4. *If the function*

$$F_{S_1}(x)=\frac{1}{\pi}\int_0^\infty \mu\left[1-S_1(\mu)\right]e^{i\mu^2 x}d\mu \quad (18)$$

is Fourier transformation of $\mu\left[1-S_1(\mu)\right]$ *for all* $x\geq 0$, *it belongs to the* $L_2(0,\infty)$ *space.*

Proof. From (12), we can easily verify that

$$\mu\left[1-S_1(\mu)\right]\approx O\left(\frac{1}{\mu}\right)\ , \quad |\mu|\to\infty.$$

It follows that $\mu\left[1-S_1(\mu)\right]\in L_2(0,\infty)$ and hence the function $F_{S_1}(x)$ also belongs to the space $L_2(0,\infty)$. □

Definition 3. *For* $k=1,2,...,n$,

$$m_k^{-1}=\frac{\left[f^{(1)}(0,\mu_k)\right]^2}{\mu_k^2}\left\{\frac{1}{\left|f^{(1)}(0,\mu_k)\right|^2}\int_0^\infty\left[q(x)-\mu_k^2\right]\left|f^{(1)}(x,\mu_k)\right|^2dx-\frac{\alpha_1+2\alpha_2\mu_k^2}{2}\right\},$$

where μ_k *are zeros of Jost function on the upper imaginary axis.*

Lemma 5. *The kernel function* $K(x,t)$ *satisfies the main equation of* L_μ

$$e^{i\alpha(x)}G(x+y)+K(x,y)+\int_x^\infty K(x,t)G(t+y)dt=0\ , \quad (x<y), \quad (19)$$

where

$$G(x)=\sum_{k=1}^n m_k e^{i\mu_k^2 x}+F_{S_1}(x). \quad (20)$$

Proof. Lets rewrite (13) as follows

$$\frac{2i\mu^2\varphi^{(1)}(x,\mu)}{F_1(\mu)}=\overline{f^{(1)}(x,\mu)}-S_1(\mu)f^{(1)}(x,\mu),$$

and substitute $f^{(1)}(x,\mu)$ in this by its expression (13), we get that

$$\begin{aligned}\frac{2i\mu^2\varphi^{(1)}(x,\mu)}{F_1(\mu)} &= e^{-i\left[\alpha(x)+\mu^2 x\right]}+\int\limits_x^\infty K(x,t)e^{-i\mu^2 t}dt \\ &- S_1(\mu)\left[e^{i\left[\alpha(x)+\mu^2 x\right]}+\int\limits_x^\infty K(x,t)e^{i\mu^2 t}dt\right].\end{aligned}$$

Also, by making the necessary arrangements and using (18), we reach

$$\frac{2i}{\pi}\int\limits_0^\infty \frac{\mu^3\varphi^{(1)}(x,\mu)e^{i\mu^2 y}}{F_1(\mu)}d\mu = e^{i\alpha(x)}F_{S_1}(x+y)+K(x,y)+\int\limits_x^\infty K(x,t)F_{S_1}(t+y)dt. \quad (21)$$

By using Jordan Lemma and Residue Theorem,

$$\begin{aligned}\frac{2i}{\pi}\int\limits_0^\infty \frac{\mu^3\varphi^{(1)}(x,\mu)e^{i\mu^2 y}}{F_1(\mu)}d\mu &= 2\pi i\frac{2i}{\pi}\sum_{k=1}^n \operatorname{Res}(F_1,\mu_k) \\ &= -\sum_{k=1}^n \frac{4\mu_k^3\varphi^{(1)}(x,\mu_k)e^{i\mu_k^2 y}}{(\dot{F_1})(\mu_k)}\end{aligned}$$

and then

$$\frac{2i}{\pi}\int\limits_0^\infty \frac{\mu^3\varphi^{(1)}(x,\mu)}{F_1(\mu)}e^{i\mu^2 y}d\mu = \sum_{k=1}^n m_k f^{(1)}(x,\mu_k)e^{i\mu_k^2 y}$$

because of the fact that $\varphi^{(1)}(x,\mu_k)$ and $f^{(1)}(x,\mu_k)$ are linearly dependent with $\varphi^{(1)}(x,\mu_k)=\frac{f^{(1)}(x,\mu_k)}{f^{(1)}(0,\mu_k)}$ since $F_1(\mu_k)=0$. If we consider the last equation and (21) together, we get

$$\sum_{k=1}^n m_k\left[f^{(1)}(x,\mu_k)e^{i\mu_k^2 y}\right] = e^{i\alpha(x)}F_{S_1}(x+y)+K(x,y)+\int\limits_x^\infty K(x,t)F_{S_1}(t+y)dt,$$

and from (20), we obtain the main equation (19). □

Clearly, to form the main equation, it suffices to know the function $G(x)$. On the other hand, to find the function $G(x)$, it suffices to know only the set of values

$$\{S_1(\mu),(0<\mu<\infty);\mu_k;m_k,(k=1,2,...,n)\}.$$

which is called the scattering data for L_μ. Given the scattering data, we can use formula (20) to construct the function $G(x)$ and write out the main equation (19) for the unknown function $K(x,y)$. Solving this equation, we find the Kernel $K(x,y)$ of the transformation operator, and hence the potential

$$q(x)=-\frac{1}{2}\frac{d}{dx}K(x,x).$$

Theorem 3. *The equation (19) has a unique solution $K(x,y) \in L_1[x,\infty)$.*

Proof. We need to show that the homogeneous equation

$$\psi(y) + \int_x^\infty \psi(t) G(t+y)\,dt = 0 \tag{22}$$

has only the zero solution in $L_2(0,\infty)$.

We assume that (22) has a nonzero solution. Multiplying $\psi(y)$ both sides of (22) and integrating,

$$\int_x^\infty \psi^2(y)dy + \int_x^\infty \psi(y) \int_x^\infty \psi(t) G(t+y)\,dtdy = 0.$$

After that,

$$\begin{aligned} 0 &= \int_x^\infty \psi^2(y)dy + \int_x^\infty \psi(y) \int_x^\infty \psi(t) F_S(t+y)\,dtdy \\ &\quad + \int_x^\infty \psi(y) \int_x^\infty \psi(t) \sum_{k=1}^n m_k e^{i\mu_k^2(t+y)}\,dtdy \end{aligned}$$

from (20). Using (18) in last equation,

$$\begin{aligned} 0 &= \int_x^\infty \psi^2(y)dy + \int_x^\infty \psi(y) \int_x^\infty \psi(t) \sum_{k=1}^n m_k e^{i\mu_k^2(t+y)}\,dtdy \\ &\quad + \int_x^\infty \psi(y) \int_x^\infty \psi(t) \left[\frac{1}{\pi} \int_0^\infty \mu\left[1 - S_1(\mu)\right] e^{i\mu^2(t+y)}\,d\mu \right] dtdy. \end{aligned} \tag{23}$$

In (23) interchanging integrals and using the uniform convergence of

$$\sum_{k=1}^n m_k e^{i\mu_k^2(t+y)} \psi(t),$$

(23) can be integrated by terms. So we obtain following equation

$$\begin{aligned} 0 &= \int_x^\infty \psi^2(y)dy + \sum_{k=1}^n m_k \left[\int_x^\infty \psi(t) e^{i\mu_k^2 t}\,dt \right]^2 \\ &\quad + \frac{1}{\pi} \int_0^\infty \mu\left[1 - S_1(\mu)\right] \left[\int_x^\infty \psi(t) e^{i\mu^2 t}\,dt \right]^2 d\mu. \end{aligned} \tag{24}$$

On the other hand, by using Parseval equation of Fourier transformation in (24),

$$\begin{aligned} 0 &= \frac{1}{\pi}\int_0^\infty \mu \left|\Phi(\mu)\right|^2 d\mu + \sum_{k=1}^n m_k \left[\Phi(\mu_k)\right]^2 \\ &\quad + \frac{1}{\pi}\int_0^\infty \mu \left[1 - S_1(\mu)\right] \left[\Phi(\mu)\right]^2 d\mu, \end{aligned} \tag{25}$$

where Parseval equation of

$$\Phi(\mu) = \int_x^\infty \psi(t) e^{i\mu^2 t} dt$$

is

$$\int_x^\infty \psi^2(y) dy = \frac{1}{\pi}\int_0^\infty \mu \left|\Phi(\mu)\right|^2 d\mu.$$

Since

$$\arg\mu = 0,\ \arg(m_k) = \eta_1(\mu),\ \arg\left[\Phi(\mu)\right] = \eta_2(\mu) \text{ and } \arg\left[1 - S_1(\mu)\right] = \eta_3(\mu),$$

(25) rewrite as polar form

$$\begin{aligned} 0 &= \sum_{k=1}^n |m_k| \left|\Phi(\mu_k)\right|^2 e^{i[\eta_1(\mu_k) + 2\eta_2(\mu_k)]} \\ &\quad + \frac{1}{\pi}\int_{-\infty}^\infty |\mu| \left|\Phi(\mu)\right|^2 \left\{1 + |1 - S_1(\mu)|\, e^{i[2\eta_2(\mu) + \eta_3(\mu)]}\right\} d\mu. \end{aligned} \tag{26}$$

Real part of (26) is

$$\begin{aligned} 0 &= \sum_{k=1}^n |m_k| \left|\Phi(\lambda_k)\right|^2 \cos\left(\eta_1(\mu_k) + 2\eta_2(\mu_k)\right) \\ &\quad + \frac{1}{\pi}\int_{-\infty}^\infty |\mu| \left|\Phi(\mu)\right|^2 \left\{1 + |1 - S_1(\mu)| \cos\left[2\eta_2(\mu) + \eta_3(\mu)\right]\right\} d\mu. \end{aligned}$$

Therefore, the last equation is equal to zero only situation is

$$\Phi(\mu) = 0 \ \text{ and so } \psi(t) = 0.$$

But this is a contradiction. So, the equation (19) has a unique solution for finite x. □

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References

[1] Aygar, Y., Bairamov, E., Scattering theory of impulsive Sturm-Liouville equation in quantum calculus, *Bull. Malays. Math. Sci. Soc.,* 42(6) (2019), 3247-3259. https://doi.org/10.1007/s40840-018-0657-2

[2] Aygar, Y., Bairamov, E., Ozbey, G. G., On the spectral and scattering properties of eigenparameter dependent discrete impulsive Sturm-Liouville equations, *Turkish J. Math.,* 45(2) (2021), 988-1000. https://doi.org/10.3906/mat-2101-45

[3] Bairamov, E., Aygar, Y., Cebesoy, S., Investigation of spectrum and scattering function of impulsive matrix difference operators, *Filomat,* 33(5) (2019), 1301-1312. https://doi.org/10.2298/FIL1905301B

[4] Bairamov, E., Aygar, Y., Eren, B., Scattering theory of impulsive Sturm-Liouville equations, *Filomat,* 31(17) (2017), 5401-5409. https://doi.org/10.2298/FIL1717401B

[5] Bairamov, E., Aygar, Y., Karslıoglu, D., Scattering analysis and spectrum of discrete Schrödinger equations with transmission conditions, *Filomat,* 31(17) (2017), 5391–5399. https://doi.org/10.2298/FIL1717391B

[6] Bairamov, E., Aygar, Y., Oznur, G. B., Scattering properties of eigenparameter-dependent impulsive Sturm-Liouville equations, *Bull. Malays. Math. Sci. Soc.,* 43(3) (2020), 2769-2781. https://doi.org/10.1007/s40840-019-00834-5

[7] Bairamov, E., Celebi, A. O., Spectral properties of the Klein-Gordon s-wave equation with complex potential, *Indian J. Pure Appl. Math.,* 28(6) (1997), 813-824.

[8] Bairamov, E., Solmaz, S., Spectrum and scattering function of the impulsive discrete Dirac systems, *Turkish J. Math.,* 42(6) (2018), 3182-3194. https://doi.org/10.3906/mat-1806-5

[9] Bairamov, E., Solmaz, S., Scattering theory of Dirac operator with the impulsive condition on whole axis, *Math. Methods Appl. Sci.,* 44(9) (2021), 7732-7746. https://doi.org/10.1002/mma.6645

[10] Jaulent, M., Jean, C., The inverse s-wave scattering problem for a class of potentials depending on energy, *Comm. Math. Phys.,* 28 (1972), 177-220. https://doi.org/10.1007/BF01645775

[11] Maksudov, F. G., Bairamov, E., Orujeva, R. U., An inverse scattering problem for an infinite Jacobi matrix with operator elements (Russian), *Dokl. Akad. Nauk.,* 323 (3) (1992), 415-419.

[12] Mamedov, Kh. R., Uniqueness of the solution to the inverse problem of scattering theory for the Sturm-Liouville operator with a spectral parameter in the boundary condition, *Mathematical Notes,* 74(1) (2003), 136-140. https://doi.org/10.4213/mzm587

[13] Marchenko, V. A., Sturm-Liouville Operators and Applications, Birkhauser Verlag, Basel, 1986. https://doi.org/10.1007/978-3-0348-5485-6

[14] Tunca, G. B., Spectral properties of the Klein-Gordon s-wave equation with spectral parameter-dependent boundary condition, *Int. J. Math. Math. Sci.,* 27 (2004), 1437-1445. https://doi.org/10.1155/S0161171204203088

[15] Tunca, G. B., Arpat, E. K., Uniqueness of the solution to the inverse problem of scattering theory for the Sturm-Liouville operator system with a spectral parameter in the boundary condition, *Gazi Univ. J. Sci.,* 29(1) (2016), 135-142. https://doi.org/10.1155/S0161171204203088