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SOME REFINEMENTS OF BEREZIN NUMBER INEQUALITIES VIA CONVEX FUNCTIONS

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Abstract. The Berezin transform $\widetilde{A}$ and the Berezin number of an operator A on the reproducing kernel Hilbert space over some set Ω with normalized reproducing kernel $\widehat{k}_\lambda$ are defined, respectively, by $\widetilde{A}(\lambda) = \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle$, $\lambda \in \Omega$ and $\operatorname{ber}(A) := \sup_{\lambda\in\Omega} \left|\widetilde{A}(\lambda)\right|$. A straightforward comparison between these characteristics yields the inequalities $\operatorname{ber}(A) \leq \frac{1}{2}\left(\|A\|_{\text{ber}} + \left\|A^2\right\|_{\text{ber}}^{1/2}\right)$. In this paper, we study further inequalities relating them. Namely, we obtained some refinements of Berezin number inequalities involving convex functions. In particular, for $A \in \mathcal{B}(\mathcal{H})$ and $r \geq 1$ we show that

$$\operatorname{ber}^{2r}(A) \leq \frac{1}{4}\left(\|A^*A + AA^*\|_{\text{ber}}^r + \|A^*A - AA^*\|_{\text{ber}}^r\right) + \frac{1}{2}\operatorname{ber}^r\left(A^2\right).$$

1. Introduction and Preliminaries

Recall that the reproducing kernel Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ (shortly, RKHS) is the Hilbert space of complex-valued functions on some set Ω such that the evaluation functional $f \to f(\lambda)$ is bounded on $\mathcal{H}$ for every $\lambda \in \Omega$. Then, by Riesz representation theorem for each $\lambda \in \Omega$ there exists a unique vector k_λ in $\mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$. The function k_λ is called the reproducing kernel of the space $\mathcal{H}$. It is well known that (see Aronzajn [2])

$$k_\lambda(z) = \sum_{n=0}^{\infty} \overline{e_n(\lambda)} e_n(z)$$

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for any orthonormal basis $\{e_n(z)\}_{n\geq 0}$ of the space $\mathcal{H}(\Omega)$. The normalized reproducing kernel is defined by $\widehat{k}_\lambda := \frac{k_\lambda}{\|k_\lambda\|_{\mathcal{H}}}$. For a bounded linear operator A acting in the RKHS $\mathcal{H}$, its Berezin symbol $\widetilde{A}$ (see Berezin [7]) is defined by the formula

$$\widetilde{A}(\lambda) := \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \ (\lambda \in \Omega).$$

The Berezin symbol is a function that is bounded by norm of the operator. Karaev [19] defined the Berezin set and the Berezin number of operator A, respectively by

$$\operatorname{Ber}(A) := \operatorname{Range}\left(\widetilde{A}\right) = \left\{\widetilde{A}(\lambda) : \lambda \in \Omega\right\}$$

and

$$\operatorname{ber}(A) := \sup_{\lambda\in\Omega} \left|\widetilde{A}(\lambda)\right|.$$

It is clear from definitions that $\widetilde{A}$ is a bounded function, $\operatorname{Ber}(A)$ lies in the numerical range $W(A)$, and so $\operatorname{ber}(A)$ does not exceed the numerical radius $w(A)$ of operator A. Recall that the numerical range and the numerical radius of operator A are defined, respectively, by

$$W(A) := \{\langle Ax, x\rangle : x \in \mathcal{H} \text{ and } \|x\| = 1\}$$

and

$$w(A) := \sup_{\|x\|=1} |\langle Ax, x\rangle|$$

(for more information, see [1, 9, 10, 15, 21, 22, 25–28, 31]). Berezin set and Berezin number of operators are new numerical characteristics of operators on the RKHS which are introduced by Karaev in [19].

Suppose that $B(\mathcal{H})$ denotes the C^*-algebra of all bounded linear operators on $\mathcal{H}$. It is well-known that

$$\operatorname{ber}(A) \leq w(A) \leq \|A\| \tag{1}$$

and

$$\frac{\|A\|}{2} \leq w(A)$$

for any $A \in B(\mathcal{H})$. But, Karaev [20] showed that

$$\frac{\|A\|}{2} \leq \operatorname{ber}(A)$$

is not hold for every $A \in B(\mathcal{H})$. Also, Berezin number inequalities were given by using the other inequalities in [11, 13, 17, 20, 32].

Huban et al. [18, Theorem 2.14] improved the inequality (1) by proving that

$$\operatorname{ber}(A) \leq \frac{1}{2}\left(\|A\|_{\text{ber}} + \left\|A^2\right\|_{\text{ber}}^{1/2}\right) \tag{2}$$

for any $A \in \mathcal{B}(\mathcal{H})$.

It has been shown in [17] that if $A \in \mathcal{B}(\mathcal{H})$, then

$$\frac{1}{4}\|A^*A + AA^*\| \leq \operatorname{ber}^2(A) \leq \frac{1}{2}\|A^*A + AA^*\|. \tag{3}$$

The following estimate of the Berezin numbers has been given in [16],

$$\operatorname{ber}(A) \leq \frac{1}{2}\sqrt{\|AA^* + A^*A\|_{\operatorname{ber}} + 2\operatorname{ber}(A^2)} \leq \|A\|_{\operatorname{ber}}. \tag{4}$$

The inequality (4) also refines the inequality (2). This can be seen by using the fact that

$$\|AA^* + A^*A\|_{\operatorname{ber}} \leq \|A\|_{\operatorname{ber}}^2 + \left\|A^2\right\|_{\operatorname{ber}}. \tag{5}$$

In this work, inspired by the numerical radius inequalities in [29], an extension of the inequality (3) is proved. In particular, for $A \in \mathcal{B}(\mathcal{H})$ and $r \geq 1$ we prove that

$$\operatorname{ber}^{2r}(A) \leq \frac{1}{4}\left(\|A^*A + AA^*\|_{\operatorname{ber}}^r + \|A^*A - AA^*\|_{\operatorname{ber}}^r\right) + \frac{1}{2}\operatorname{ber}^r\left(A^2\right).$$

Other general related results are also established.

2. Main Results

In order to achieve our goal, we need the following series of corollaries.

Lemma 1. *([23]) Let A be an operator in $\mathcal{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$ be any vectors.*

(i) If $0 \leq \alpha \leq 1$, then $|\langle Ax, y\rangle|^2 \leq \left\langle |A|^{2\alpha} x, x\right\rangle \left\langle |A^*|^{2(1-\alpha)} y, y\right\rangle$.

(ii) If f and g are non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$, $(t \geq 0)$, then $|\langle Ax, y\rangle| \leq \|f(|A|)x\| \|g(|A^*|)y\|$.

Lemma 2. *([24]) Let A be a self-adjoint operator in $\mathcal{B}(\mathcal{H})$ with the spectrum contained in the interval J, and let h be convex function on J. Then for any unit vector $x \in \mathcal{H}$,*

$$h(\langle Ax, x\rangle) \leq \langle h(A)x, x\rangle.$$

In [31, Lemma 2.4], the authors present an improvement of the Young inequality as follows:

Lemma 3. *Let $a, b > 0$ and $\min\{a, b\} \leq m \leq M \leq \max\{a, b\}$. Then*

$$\sqrt{ab} \leq \frac{2\sqrt{Mm}}{M+m}\frac{a+b}{2}. \tag{6}$$

In 1941, R.P. Boas [8] and in 1944, independently, R. Bellman [6] proved the following generalization of Bessel's inequality.

Lemma 4. *If $a, b_1, ..., b_n$ are elements of an inner product space $(\mathcal{H}, \langle ., . \rangle)$, then the following inequality holds:*

$$\sum_{i=1}^{n} |\langle a, b_i \rangle|^2 \leq \|a\|^2 \left(\max_{1 \leq i \leq n} \|b_i\|^2 + \left(\sum_{1 \leq i \neq j \leq n}^{n} |\langle b_i, b_j \rangle|^2 \right)^{\frac{1}{2}} \right).$$

In particulary, the case $n = 2$ in the above reduces to

$$|\langle a, b_1 \rangle|^2 + |\langle a, b_2 \rangle|^2 \leq \|a\|^2 \left(\max \left(\|b_1\|^2, \|b_2\|^2 \right) + |\langle b_1, b_2 \rangle| \right). \tag{7}$$

We recall the following refinement of the Cauchy-Schwarz inequality obtained by Dragomir in [9]. If a, b, e are vectors in $\mathcal{H}$ and $\|e\| = 1$, then we have

$$|\langle a, b \rangle| \leq |\langle a, e \rangle \langle e, b \rangle| + |\langle a, b \rangle - \langle a, e \rangle \langle e, b \rangle| \leq \|a\| \|b\|. \tag{8}$$

From the inequality (8) we deduce that

$$|\langle a, e \rangle \langle e, b \rangle| \leq \frac{1}{2} \left(\|a\| \|b\| + |\langle a, b \rangle| \right). \tag{9}$$

Let $\widehat{k}_\lambda$ be a normalized reproducing kernel. Then, by taking $e = \widehat{k}_\lambda$, $a = A\widehat{k}_\lambda$ and $b = A^* \widehat{k}_\lambda$ in the inequality (9), we get

$$\left| \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^2 \leq \frac{1}{2} \left(\left\| A\widehat{k}_\lambda \right\| \left\| A^* \widehat{k}_\lambda \right\| + \left| \left\langle A^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| \right) \tag{10}$$

and

$$\sup_{\lambda \in \Omega} \left| \widetilde{A}(\lambda) \right|^2 \leq \sup_{\lambda \in \Omega} \frac{1}{2} \left(\left\| A\widehat{k}_\lambda \right\|^2 + \left| \left\langle A^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| \right)$$

which is equivalent to

$$\mathrm{ber}^2(A) \leq \frac{1}{2} \left(\|A\|_{\mathrm{Ber}}^2 + \mathrm{ber}\left(A^2\right) \right). \tag{11}$$

In addition to this, we have the following related inequality:

Theorem 1. *Let $A \in \mathcal{B}(\mathcal{H})$, f, g be non-negative continuous functions on $[0, \infty)$ satisfying $f(t)\, g(t) = t$, $(t \geq 0)$, and h be a non-negative increasing convex function on $[0, \infty)$. If*

$$0 < f^2\left(\left|A^2\right|\right) \leq m < M \leq g^2\left(\left|\left(A^2\right)^*\right|\right),$$

or

$$0 < g^2\left(\left|\left(A^2\right)^*\right|\right) \leq m < M \leq f^2\left(\left|A^2\right|\right),$$

then

$$h\left(\mathrm{ber}\left(A^2\right)\right) \leq \frac{2\sqrt{Mm}}{M+m} \left\| \frac{h\left(f^2\left(\left|A^2\right|\right)\right) + h\left(g^2\left(\left|\left(A^2\right)^*\right|\right)\right)}{2} \right\|_{\mathrm{ber}}. \tag{12}$$

Proof. Let $\widehat{k}_\lambda$ be a normalized reproducing kernel. Then, we have

$$
\begin{aligned}
& h\left(\left|\left\langle A^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\right) \\
& \leq h\left(\sqrt{\left\langle f^2\left(\left|A^2\right|\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\left\langle g^2\left(\left|\left(A^2\right)^*\right|\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}\right) \\
& \text{(by Lemma 1 (ii))} \\
& \leq h\left(\frac{2\sqrt{Mm}}{M+m}\left(\frac{\left\langle f^2\left(\left|A^2\right|\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle+\left\langle g^2\left(\left|\left(A^2\right)^*\right|\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}{2}\right)\right) \\
& \text{(by the inequality (6))} \\
& \leq \frac{2\sqrt{Mm}}{M+m}h\left(\frac{\left\langle f^2\left(\left|A^2\right|\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle+\left\langle g^2\left(\left|\left(A^2\right)^*\right|\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}{2}\right) \\
& \leq \frac{2\sqrt{Mm}}{M+m}\left(\frac{h\left(\left\langle f^2\left(\left|A^2\right|\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right)+h\left(\left\langle g^2\left(\left|\left(A^2\right)^*\right|\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right)}{2}\right) \\
& \leq \frac{2\sqrt{Mm}}{M+m}\left(\frac{\left\langle h\left(f^2\left(\left|A^2\right|\right)\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle+\left\langle h\left(g^2\left(\left|\left(A^2\right)^*\right|\right)\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}{2}\right) \\
& \text{(by Lemma 2)} \\
& = \frac{2\sqrt{Mm}}{M+m}\left\langle\frac{h\left(f^2\left(\left|A^2\right|\right)\right)+h\left(g^2\left(\left|\left(A^2\right)^*\right|\right)\right)}{2}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle.
\end{aligned}
$$

Therefore,

$$
h\left(\left|\left\langle A^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\right) \leq \frac{2\sqrt{Mm}}{M+m}\left\langle\frac{h\left(f^2\left(\left|A^2\right|\right)\right)+h\left(g^2\left(\left|\left(A^2\right)^*\right|\right)\right)}{2}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle.
$$

By taking the supremum over $\lambda \in \Omega$ above inequality, we deduce the desired result

$$
h\left(\operatorname{ber}\left(A^2\right)\right) \leq \frac{2\sqrt{Mm}}{M+m}\left\|\frac{h\left(f^2\left(\left|A^2\right|\right)\right)+h\left(g^2\left(\left|\left(A^2\right)^*\right|\right)\right)}{2}\right\|_{\text{ber}}.
$$

This finalizes the proof. $\square$

The following result may be stated as well.

Corollary 1. *Let $A \in \mathcal{B}(\mathcal{H})$, f, g be non-negative continuous functions on $[0,\infty)$ satisfying $f(t)g(t) = t$, $(t \geq 0)$, and $r \geq 1$. If*

$$
0 < f^2\left(\left|A^2\right|\right) \leq m < M \leq g^2\left(\left|\left(A^2\right)^*\right|\right),
$$

or

$$0 < g^2\left(\left|\left(A^2\right)^*\right|\right) \leq m < M \leq f^2\left(\left|A^2\right|\right),$$

then

$$\operatorname{ber}^r\left(A^2\right) \leq \frac{2\sqrt{Mm}}{M+m}\left\|\frac{f^{2r}\left(\left|A^2\right|\right)+g^{2r}\left(\left|\left(A^2\right)^*\right|\right)}{2}\right\|_{\text{ber}}.$$

Remark 1. *By taking $r=1$ in Corollary 1, then it follows from the inequality (11) that*

$$\operatorname{ber}^2(A) \leq \frac{1}{2}\left(\left\|A^2\right\|_{\text{Ber}}+\frac{2\sqrt{Mm}}{M+m}\left\|\frac{f^2\left(\left|A^2\right|\right)+g^2\left(\left|\left(A^2\right)^*\right|\right)}{2}\right\|_{\text{ber}}\right).$$

For various operators, the following conclusion is true.

Theorem 2. *Let $A, B, C \in \mathcal{B}(\mathcal{H})$, $A, B \geq 0$, $0 \leq \alpha \leq 1$, and h be a non-negative increasing sub-multiplicative convex function on $[0,\infty)$. If*

$$0 < B^{2(1-\alpha)} \leq m < M \leq A^{2\alpha}$$

or

$$0 < A^{2\alpha} \leq m < M \leq B^{2(1-\alpha)},$$

then

$$h\left(\operatorname{ber}\left(A^\alpha C B^{1-\alpha}\right)\right) \leq \frac{2\sqrt{Mm}}{M+m} h\left(\|C\|_{\text{ber}}\right)\left\|\frac{h\left(B^{2(1-\alpha)}\right)+h\left(A^{2\alpha}\right)}{2}\right\|_{\text{ber}}. \tag{13}$$

Proof. Let $\widehat{k}_\lambda$ be a normalized reproducing kernel. Then, by the Cauchy-Schwarz, we have

$$\begin{aligned}
&h\left(\left|\left\langle A^\alpha C B^{1-\alpha}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\right)\\
&= h\left(\left|\left\langle C B^{1-\alpha}\widehat{k}_\lambda, A^\alpha\widehat{k}_\lambda\right\rangle\right|\right)\\
&\leq h\left(\|C\|_{\text{ber}}\left\|B^{1-\alpha}\widehat{k}_\lambda\right\|\left\|A^\alpha\widehat{k}_\lambda\right\|\right)\\
&\text{(by } h \text{ sub-multiplicativity)}\\
&= h\left(\|C\|_{\text{ber}}\sqrt{\left\langle B^{1-\alpha}\widehat{k}_\lambda, B^{1-\alpha}\widehat{k}_\lambda\right\rangle\left\langle A^\alpha\widehat{k}_\lambda, A^\alpha\widehat{k}_\lambda\right\rangle}\right)\\
&\text{(by the inequality (6))}\\
&= h\left(\|C\|_{\text{ber}}\sqrt{\left\langle B^{2(1-\alpha)}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\left\langle A^{2\alpha}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle}\right)\\
&\leq h\left(\|C\|_{\text{ber}}\right) h\left(\sqrt{\left\langle B^{2(1-\alpha)}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\left\langle A^{2\alpha}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle}\right)
\end{aligned}$$

$$\leq h\left(\|C\|_{\text{ber}}\right) h\left(\frac{2\sqrt{Mm}}{M+m}\left(\frac{\left\langle B^{2(1-\alpha)}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle+\left\langle A^{2\alpha}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}{2}\right)\right)$$

$$\leq \frac{2\sqrt{Mm}}{M+m} h\left(\|C\|_{\text{ber}}\right) h\left(\frac{\left\langle B^{2(1-\alpha)}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\left\langle A^{2\alpha}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}{2}\right)$$

(by Lemma 2)

$$\leq \frac{2\sqrt{Mm}}{M+m} h\left(\|C\|_{\text{ber}}\right)\frac{h\left(\left\langle B^{2(1-\alpha)}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right)+h\left(\left\langle A^{2\alpha}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right)}{2}$$

$$\leq \frac{2\sqrt{Mm}}{M+m} h\left(\|C\|_{\text{ber}}\right)\frac{\left\langle h\left(B^{2(1-\alpha)}\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle+\left\langle h\left(A^{2\alpha}\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}{2}$$

$$= \frac{2\sqrt{Mm}}{M+m} h\left(\|C\|_{\text{ber}}\right)\left\langle\left(\frac{h\left(B^{2(1-\alpha)}\right)+h\left(A^{2\alpha}\right)}{2}\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle,$$

So,

$$h\left(\left|\left\langle A^\alpha CB^{1-\alpha}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\right)\leq \frac{2\sqrt{Mm}}{M+m} h\left(\|C\|_{\text{ber}}\right)\left\langle\left(\frac{h\left(B^{2(1-\alpha)}\right)+h\left(A^{2\alpha}\right)}{2}\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle,$$

and

$$\sup_{\lambda\in\Omega} h\left(\left|\left(\widetilde{A^\alpha CB^{1-\alpha}}\right)(\lambda)\right|\right)\leq \frac{2\sqrt{Mm}}{M+m} h\left(\|C\|_{\text{ber}}\right)\sup_{\lambda\in\Omega}\left\langle\left(\frac{h\left(B^{2(1-\alpha)}\right)+h\left(A^{2\alpha}\right)}{2}\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle$$

which is equivalent to

$$h\left(\text{ber}\left(A^\alpha CB^{1-\alpha}\right)\right)\leq \frac{2\sqrt{Mm}}{M+m} h\left(\|C\|_{\text{ber}}\right)\left\|\frac{h\left(B^{2(1-\alpha)}\right)+h\left(A^{2\alpha}\right)}{2}\right\|_{\text{ber}},$$

which proves the desired inequalities. □

Corollary 2. *Let* $A,B,C\in\mathcal{B}(\mathcal{H})$, $A,\ B\geq 0$, *and* $0\leq\alpha\leq 1$, *and let* $r\geq 1$. *If*

$$0<B^{2(1-\alpha)}\leq m<M\leq A^{2\alpha},$$

or

$$0<A^{2\alpha}\leq m<M\leq B^{2(1-\alpha)},$$

then

$$\text{ber}^r\left(A^\alpha CB^{1-\alpha}\right)\leq \frac{2\sqrt{Mm}}{M+m}\|C\|_{\text{ber}}^r\left\|\frac{\left(A^{2r\alpha}\right)+\left(B^{2r(1-\alpha)}\right)}{2}\right\|_{\text{ber}}.$$

As a consequence of the above, we can present the following inequality.

Corollary 3. *Suppose that the assumptions of Corollary 2 are satisfied. Then*

$$\mathrm{ber}^r\left(A^{1/2}CB^{1/2}\right) \le \frac{2\sqrt{Mm}}{M+m}\left\|C\right\|_{\mathrm{ber}}^r\left\|\frac{A^r+B^r}{2}\right\|_{\mathrm{ber}}. \tag{14}$$

We can give the following corollary whose proof can be reached by using similar techniques from Theorem 3.4 and Lemma 3.5 in [30].

Corollary 4. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be invertible self-adjoint operators and $C \in \mathcal{B}(\mathcal{H})$. Then*

$$\mathrm{ber}^r\left(A^{1/2}CB^{1/2}\right) \le \left\|C\right\|_{\mathrm{ber}}^r\left\|\frac{A^r+B^r}{2}\right\|_{\mathrm{ber}}. \tag{15}$$

Remark 2. *Therefore, inequality (14) essentially gives a refinement of the inequality of (15) since $\frac{2\sqrt{Mm}}{M+m} \le 1$.*

The following result is of interest in itself.

Theorem 3. *Let $A \in \mathcal{B}(\mathcal{H})$, and let h be a non-negative increasing convex function on $[0,\infty)$.*

$$h\left(\mathrm{ber}^2(A)\right) \le \frac{1}{4}\left(h\left(\left\|A^*A+AA^*\right\|_{\mathrm{ber}}\right)+h\left(\left\|A^*A-AA^*\right\|_{\mathrm{ber}}\right)\right)+\frac{1}{2}h\left(\mathrm{ber}\left(A^2\right)\right).$$

In particular, for any $r \ge 1$,

$$\mathrm{ber}^{2r}(A) \le \frac{1}{4}\left(\left\|A^*A+AA^*\right\|_{\mathrm{ber}}^r+\left\|A^*A-AA^*\right\|_{\mathrm{ber}}^r\right)+\frac{1}{2}\mathrm{ber}^r\left(A^2\right).$$

Proof. Let $\lambda \in \Omega$ be an arbitrary. Put $b_1 = A\widehat{k}_\lambda$, $b_2 = A^*\widehat{k}_\lambda$, and $a = \widehat{k}_\lambda$ in the inequality (7). Since $\max(a,b) = \frac{|a+b|+|a-b|}{2}$, we get

$$\begin{aligned}
&\left|\left\langle \widehat{k}_\lambda, A\widehat{k}_\lambda\right\rangle\right|^2+\left|\left\langle \widehat{k}_\lambda, A^*\widehat{k}_\lambda\right\rangle\right|^2 \\
&\le \max\left(\left\|A\widehat{k}_\lambda\right\|^2, \left\|A^*\widehat{k}_\lambda\right\|^2\right)+\left|\left\langle A\widehat{k}_\lambda, A^*\widehat{k}_\lambda\right\rangle\right| \\
&= \frac{1}{2}\left(\left|\left\langle A^*A+AA^*\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|+\left|\left\langle A^*A-AA^*\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\right)+\left|\left\langle A^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|.
\end{aligned} \tag{16}$$

Applying the AM-GM inequality for the left hand side of the above inequality, we get

$$\begin{aligned}
&\left|\left\langle A^*\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| \\
&\le \frac{1}{4}\left(\left|\left\langle A^*A+AA^*\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|+\left|\left\langle A^*A-AA^*\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\right)+\frac{1}{2}\left|\left\langle A^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|.
\end{aligned}$$

Whence,

$$\begin{aligned}
&h\left(\left|\left\langle A^*\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\right) \\
&\le h\left(\frac{1}{4}\left(\left|\left\langle A^*A+AA^*\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|+\left|\left\langle A^*A-AA^*\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\right)+\frac{1}{2}\left|\left\langle A^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\right)
\end{aligned}$$

$$= h\left(\frac{\frac{1}{2}\left|\left\langle A^*A + AA^*\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| + \left|\left\langle A^*A - AA^*\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| + \left|\left\langle A^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|}{2}\right)$$

$$\leq \frac{1}{2}\left(h\left(\frac{\left|\left\langle A^*A + AA^*\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| + \left|\left\langle A^*A - AA^*\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|}{2}\right) + h\left(\left|\left\langle A^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\right)\right)$$

$$\leq \frac{1}{4}\left(h\left(\left|\left\langle A^*A + AA^*\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\right) + h\left(\left|\left\langle A^*A - AA^*\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\right)\right) + \frac{1}{2}h\left(\left|\left\langle A^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\right).$$

Therefore,

$$h\left(\left|\left\langle A^*\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\right)$$

$$\leq \frac{1}{4}\left(h\left(\left|\left\langle A^*A + AA^*\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\right) + h\left(\left|\left\langle A^*A - AA^*\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\right)\right) + \frac{1}{2}h\left(\left|\left\langle A^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\right).$$

By taking the supremum over $\lambda \in \Omega$ above inequality, we have

$$h\left(\mathrm{ber}^2(A)\right) \leq \frac{1}{4}\left(h\left(\|A^*A + AA^*\|_{\mathrm{ber}}\right) + h\left(\|A^*A - AA^*\|_{\mathrm{ber}}\right)\right) + \frac{1}{2}h\left(\mathrm{ber}\left(A^2\right)\right).$$

This completes the proof. □

Corollary 5. *Let $A \in \mathcal{B}(\mathcal{H})$ be an invertible operator. Then*

$$\mathrm{ber}(A) \leq \sqrt{\frac{1}{2}\|A\|_{\mathrm{ber}}^2 + \frac{3}{4}\|A^2\|_v - \frac{1}{4}\|A^{-1}\|_{\mathrm{ber}}^{-2}}.$$

Proof. By using similar techniques from [22], we get

$$\|A^*A - AA^*\|_{\mathrm{ber}} \leq \|A\|_{\mathrm{ber}}^2 - \left\|A^{-1}\right\|_{\mathrm{ber}}^{-2}. \tag{17}$$

On the other hand, from Theorem 3, we have

$$\mathrm{ber}^2(A) \leq \frac{1}{4}\left(\|A^*A + AA^*\|_{\mathrm{ber}} + \|A^*A - AA^*\|_{\mathrm{ber}}\right) + \frac{1}{2}\mathrm{ber}(A^2).$$

Hence

$$\begin{aligned} \mathrm{ber}^2(A) &\leq \frac{1}{4}\left(\|A^*A + AA^*\|_{\mathrm{ber}} + \|A^*A - AA^*\|_{\mathrm{ber}}\right) + \frac{1}{2}\mathrm{ber}(A^2) \\ &\leq \frac{1}{4}\left(\|A^*A + AA^*\|_{\mathrm{ber}} + \|A\|^2 - \left\|A^{-1}\right\|_{\mathrm{ber}}^{-2}\right) + \frac{1}{2}\mathrm{ber}(A^2) \\ &\quad \text{(by the inequality (17))} \\ &\leq \frac{1}{4}\left(2\|A\|_{\mathrm{ber}}^2 + \left\|A^2\right\|_{\mathrm{ber}} - \left\|A^{-1}\right\|_{\mathrm{ber}}^{-2}\right) + \frac{1}{2}\mathrm{ber}(A^2) \\ &\quad \text{(by the inequality (5))} \\ &\leq \frac{1}{2}\|A\|_{\mathrm{ber}}^2 + \frac{3}{4}\left\|A^2\right\|_{\mathrm{ber}} - \frac{1}{4}\left\|A^{-1}\right\|_{\mathrm{ber}}^{-2} \\ &\quad \text{(by the inequality (1))} \end{aligned}$$

as required. □

The following upper bound for the nonnegative difference $\mathrm{ber}^2(A) - \mathrm{ber}(A^2)$ can be obtained:

Corollary 6. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\mathrm{ber}^2(A) - \mathrm{ber}(A^2) \leq \frac{1}{4}\left(\left\||A|^2 + |A^*|^2\right\|_{\mathrm{ber}} + \left\||A|^2 - |A^*|^2\right\|_{\mathrm{ber}}\right).$$

For more recent results concerning Berezin radius inequalities for operators and other related results, we suggest [3–5, 12, 14, 16, 33].

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