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# Properties of Double Fuzzy $b$-Open Sets

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## Abstract

In this paper, we introduce and study the concept of $(r,s)$-fuzzy $b$-border, $(r,s)$-fuzzy $b$-exterior and $(r,s)$-fuzzy $b$-frontier. Some of its interesting properties and characterizations are examined.

***Keywords:*** *Double fuzzy topology, $(r,s)$-fuzzy b-open set, $(r,s)$-fuzzy b-closed set.*

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## 1. Introduction

The concept of fuzzy sets was introduce by Zadeh [10]. Later on, Chang [2] introduced the concept of fuzzy topology, then the generalizations of the concept of fuzzy topology have been done by many authors. In [1], Atanassove introduced the idea of intuitionistic fuzzy sets, then Coker [3, 4], introduced the concept of intuitionistic fuzzy topological spaces. On the other hand, as a generalization of fuzzy topological spaces Samanta and Mondal [9], introduced the concept of intuitionistic gradation of openness. In 2005, the term intuitionistic is ended by Garcia and Rodabaugh [7]. They proved that the term intuitionistic is unsuitable in mathematics and applications and they replaced it by double. In this paper, we introduce and study the concept of $(r,s)$-fuzzy $b$-border, $(r,s)$-fuzzy $b$-exterior and $(r,s)$-fuzzy $b$-frontier. Some of its interesting properties and characterizations are examined.

## 2. Preliminaris

Throughout this paper, Let $X$ be a non-empty set, $I$ the unit interval [0, 1], $I_0 = (0,1]$ and $I_1 = [0,1)$. The family of all fuzzy sets on $X$ is denoted by $I^X$. By $\bar{0}$ and $\bar{1}$, we denote the smallest and the greatest fuzzy sets on $X$. For a fuzzy set $\lambda \in I^X$, $\bar{1}-\lambda$ denotes its complement. Given a function $f : I^X \longrightarrow I^Y$ and its inverse $f^{-1} : I^Y \longrightarrow I^X$ are defined by $f(\lambda)(y) = \bigvee_{f(x)=y} \lambda(x)$ and $f^{-1}(\mu)(x) = \mu(f(x))$, for each $\lambda \in I^X, \mu \in I^Y$ and $x \in X$, respectively. All other notations are standard notations of fuzzy set theory.

**Definition 2.1.** *[4, 9] A double fuzzy topology on X is a pair of maps $\tau, \tau^\star : I^X \to I$, which satisfies the following properties:*

1. *$\tau(\lambda) \leq \underline{1} - \tau^\star(\lambda)$ for each $\lambda \in I^X$.*
2. *$\tau(\lambda_1 \wedge \lambda_2) \geq \tau(\lambda_1) \wedge \tau(\lambda_2)$ and $\tau^\star(\lambda_1 \wedge \lambda_2) \leq \tau^\star(\lambda_1) \vee \tau^\star(\lambda_2)$ for each $\lambda_1, \lambda_2 \in I^X$.*
3. *$\tau(\bigvee_{i\in\Gamma} \lambda_i) \geq \bigwedge_{i\in\Gamma} \tau(\lambda_i)$ and $\tau^\star(\bigvee_{i\in\Gamma} \lambda_i) \leq \bigvee_{i\in\Gamma} \tau^\star(\lambda_i)$ for each $\lambda_i \in I^X, i \in \Gamma$.*

*The triplet $(X,\tau,\tau^\star)$ is called a double fuzzy topological space.*

**Definition 2.2.** *[4, 9] A fuzzy set $\lambda$ is called an $(r,s)$-fuzzy open if $\tau(\lambda) \geq r$ and $\tau^\star(\lambda) \leq s$, $\lambda$ is called an $(r,s)$-fuzzy closed if, and only if $\underline{1} - \lambda$ is an $(r,s)$-fuzzy open set.*

**Definition 2.3.** *[4, 9] A function $f : (X,\tau_1,\tau_1^\star) \to (Y,\tau_2,\tau_2^\star)$ is said to be a double fuzzy continuous if, and only if $\tau_1(f^{-1}(\nu)) \geq \tau_2(\nu)$ and $\tau_1^\star(f^{-1}(\nu)) \leq \tau_2^\star(\nu)$ for each $\nu \in I^Y$.*

**Theorem 2.4.** *[8, 5] Let $(X,\tau,\tau^\star)$ be a double fuzzy topological space. Then double fuzzy closure operator and double fuzzy interior operator of $\lambda \in I^X$ are defined by $C_{\tau,\tau^\star}(\lambda,r,s) = \bigwedge\{\mu \in I^X \mid \lambda \leq \mu, \tau(\underline{1}-\mu) \geq r, \tau^\star(\underline{1}-\mu) \leq s\}$, $I_{\tau,\tau^\star}(\lambda,r,s) = \bigvee\{\mu \in I^X \mid \mu \leq \lambda, \tau(\mu) \geq r, \tau^\star(\mu) \leq s\}$, where $r \in I_0$ and $s \in I_1$ such that $r+s \leq 1$.*

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**Definition 2.5.** *[6] Let $(X,\tau,\tau^\star)$ be a double fuzzy topological space. For each $\lambda,\mu \in I^X$, $r \in I_0$ and $s \in I_1$,*

1. *$\lambda$ is called an $(r,s)$-fuzzy b-open set if $\lambda \leq I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda,r,s),r,s) \vee C_{\tau,\tau^\star}(I_{\tau,\tau^\star}(\lambda,r,s),r,s)$.*
2. *$\lambda$ is called an $(r,s)$-fuzzy b-closed set if $\underline{1}-\lambda$ is an $(r,s)$-fuzzy b-open set.*
3. *An $(r,s)$-fuzzy b-closure of $\lambda$ is defined by $BC_{\tau,\tau^\star}(\lambda,r,s) = \bigwedge\{\mu \in I^X \mid \lambda \leq \mu$ and $\mu$ is $(r,s)$-fuzzy b-closed$\}$.*
4. *An $(r,s)$-fuzzy b-interior of $\lambda$ is defined by $BI_{\tau,\tau^\star}(\lambda,r,s) = \vee\{\mu \in I^X \mid \lambda \leq \mu$ and $\mu$ is $(r,s)$-fuzzy b-closed$\}$.*

## 3. $(r,s)$-fuzzy $b$-open sets

In this section, we study $(r,s)$-fuzzy $b$-border, $(r,s)$-fuzzy $b$-exterior and $(r,s)$-fuzzy $b$-frontier. Some of its interesting properties and characterizations are examined.

**Proposition 3.1.** *For any double fuzzy topological space $(X,\tau,\tau^\star)$, $\lambda,B \in I^X$, $r \in I_0$ and $s \in I_1$, we have*

1. *$BI_{\tau,\tau^\star}(\lambda,r,s)$ is the largest $(r,s)$-fuzzy b-open set with $BI_{\tau,\tau^\star}(\lambda,r,s) \leq \lambda$,*
2. *$\lambda = BI_{\tau,\tau^\star}(\lambda,r,s)$ if $\lambda$ is an $(r,s)$-fuzzy b-open set,*
3. *$BI_{\tau,\tau^\star}(BI_{\tau,\tau^\star}(\lambda,r,s),r,s) = BI_{\tau,\tau^\star}(\lambda,r,s)$, if $\lambda$ is an $(r,s)$-fuzzy b-open set,*
4. *$1-BI_{\tau,\tau^\star}(\lambda,r,s) = BC_{\tau,\tau^\star}(1-\lambda,r,s)$,*
5. *$1-BC_{\tau,\tau^\star}(\lambda,r,s) = BI_{\tau,\tau^\star}(1-\lambda,r,s)$,*
6. *If $\lambda \leq \mu$, then $BI_{\tau,\tau^\star}(\lambda,r,s) \leq BI_{\tau,\tau^\star}(\mu,r,s)$,*
7. *If $\lambda \leq \mu$, then $BC_{\tau,\tau^\star}(\lambda,r,s) \leq BC_{\tau,\tau^\star}(\mu,r,s)$,*
8. *$BI_{\tau,\tau^\star}(\lambda,r,s) \wedge BI_{\tau,\tau^\star}(\mu,r,s) = BI_{\tau,\tau^\star}(\lambda \wedge \mu,r,s)$,*
9. *$BI_{\tau,\tau^\star}(\lambda,r,s) \vee BI_{\tau,\tau^\star}(\mu,r,s) = BI_{\tau,\tau^\star}(\lambda \vee \mu,r,s)$.*

*Proof.* (1) and (2) follow from the definitions and (3) follows from (2).
(4). $BC_{\tau,\tau^\star}(1-\lambda,r,s) = \wedge\{\mu : \mu$ is $(r,s)$-fuzzy $b$-closed set, $\mu \geq 1-\lambda\} = 1-\vee\{1-\mu : 1-\mu$ is $(r,s)$ fuzzy $b$-open set, $1-\mu \leq \lambda\} = 1-BI_{\tau,\tau^\star}(\lambda,r,s)$.
(5). It is similar to (4).
(6). It is clear that $BI_{\tau,\tau^\star}(\lambda,r,s) = \{\mu : \mu$ is $(r,s)$-fuzzy $b$-open and, $\mu \leq \lambda\} = \vee\{\mu : \mu \leq \gamma$ and $\mu$ is an $(r,s)$-fuzzy $b$-open$\}$ = $BI_{\tau,\tau^\star}(\mu,r,s)$.
(7). If $\lambda \leq \mu$, it is similar to (6).
(8). Follows from (6).
(9). It is similar to (8). □

**Definition 3.2.** *For any double fuzzy topological space $(X,\tau,\tau^\star)$, $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$, we have the $(r,s)$-fuzzy b-border of $\lambda$, denoted by $BF_{\tau,\tau^\star}(\lambda,r,s)$, defined as $BB_{\tau,\tau^\star}(\lambda,r,s) = \lambda - BI_{\tau,\tau^\star}(\lambda,r,s)$.*

**Proposition 3.3.** *For any double fuzzy topological space $(X,\tau,\tau^\star)$, $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$, we have*

1. *$BB_{\tau,\tau^\star}(\lambda,r,s) \leq BB_{\tau,\tau^\star}(\lambda,r,s)$,*
2. *If $\lambda$ is an $(r,s)$-fuzzy b-open, then $BB_{\tau,\tau^\star}(\lambda,r,s) = \bar{0}$,*
3. *$BB_{\tau,\tau^\star}(\lambda,r,s) \leq BC_{\tau,\tau^\star}(1-\lambda,r,s)$,*
4. *$BI_{\tau,\tau^\star}(BB_{\tau,\tau^\star}(\lambda,r,s),r,s) \leq \lambda$,*
5. *$BB\tau,\tau^\star(\lambda \vee \mu,r,s) \leq BB_{\tau,\tau^\star}(\lambda,r,s) \vee BB_{\tau,\tau^\star}(\mu,r,s)$,*
6. *$BB_{\tau,\tau^\star}(\lambda \wedge \mu,r,s) \geq BB_{\tau,\tau^\star}(\lambda,r,s) \wedge BB_{\tau,\tau^\star}(\mu,r,s)$.*

*Proof.* (1). For any $\lambda \in I^X$, since $BI_{\tau,\tau^\star}(\lambda,r,s) \leq BI_{\tau,\tau^\star}(\lambda,r,s)$, then $\lambda - BI_{\tau,\tau^\star}(\lambda,r,s) \leq \lambda - BI_{\tau,\tau^\star}(\lambda,r,s)$. Therefore $BB_{\tau,\tau^\star}(\lambda,r,s) \leq BB_{\tau,\tau^\star}(\lambda,r,s)$.
(2). For any an $(r,s)$-fuzzy $b$-open set $\lambda \in I^X$, we have $\lambda = BI_{\tau,\tau^\star}(\lambda,r,s)$. Thus $BB_{\tau,\tau^\star}(\lambda,r,s) = \bar{0}$.
(3). $BB_{\tau,\tau^\star}(\lambda,r,s) = \lambda - BI_{\tau,\tau^\star}(\lambda,r,s) = \lambda - (1-BC_{\tau,\tau^\star}(1-\lambda,r,s)) \leq 1-1+BB_{\tau,\tau^\star}(1-\lambda,r,s) = BB_{\tau,\tau^\star}(1-\lambda,r,s)$.
(4). $BI_{\tau,\tau^\star}(BB_{\tau,\tau^\star}(\lambda,r,s),r,s) = BI_{\tau,\tau^\star}(\lambda - BI_{\tau,\tau^\star}(\lambda,r,s),r,s) \leq \lambda - BI_{\tau,\tau^\star}(\lambda,r,s) \leq \lambda$ by (1) of Proposition 2.2.
Then $BI_{\tau,\tau^\star}(BB_{\tau,\tau^\star}(\lambda,r,s),r,s) \leq \lambda$.
(5). $BB_{\tau,\tau^\star}(\lambda \vee \mu,r,s) = (\lambda \vee \mu) - BI_{\tau,\tau^\star}(\lambda \vee \mu,r,s) = (\lambda \vee \mu) - (BI_{\tau,\tau^\star}(\lambda,r,s)) \vee (BI_{\tau,\tau^\star}(\mu,r,s)) \leq (\lambda - BI_{\tau,\tau^\star}(\lambda,r,s)) \vee (\mu - BI_{\tau,\tau^\star}(\mu,r,s))$ $= BB_{\tau,\tau^\star}(\lambda,r,s) \vee BB_{\tau,\tau^\star}(\mu,r,s)$. Therefore, $BB_{\tau,\tau^\star}(\lambda \vee \mu,r,s) \leq BE_{\tau,\tau^\star}(\lambda,r,s) \vee BE_{\tau,\tau^\star}(\mu,r,s)$.
(6) It is similar to (5). □

**Definition 3.4.** *For any double fuzzy topological space $(X,\tau,\tau^\star)$, $\mu \in I^X$, $r \in I_0$ and $s \in I_1$, we have the $(r,s)$-fuzzy b-frontier of $\lambda$, denoted by $BF_{\tau,\tau^\star}(\lambda,r,s)$ is defined as $BF_{\tau,\tau^\star}(\lambda,r,s) = BC_{\tau,\tau^\star}(\lambda,r,s) - BI_{\tau,\tau^\star}(\lambda,r,s)$.*

**Proposition 3.5.** *For any double fuzzy topological space $(X,\tau,\tau^\star)$, $\mu \in I^X$, $r \in I_0$ and $s \in I_1$, we have*

1. *$BF_{\tau,\tau^\star}(\lambda,r,s) \leq F_{\tau,\tau^\star}(\lambda,r,s)$,*
2. *$BB_{\tau,\tau^\star}(\lambda,r,s) \leq BF_{\tau,\tau^\star}(\lambda,r,s)$,*
3. *$BF_{\tau,\tau^\star}(1-\lambda,r,s) = BI_{\tau,\tau^\star}(\lambda,r,s)$,*
4. *$BF_{\tau,\tau^\star}(BI_{\tau,\tau^\star}(\lambda,r,s),r,s) \leq BF_{\tau,\tau^\star}(\lambda,r,s)$,*
5. *$BF_{\tau,\tau^\star}(BC_{\tau,\tau^\star}(\lambda,r,s),r,s) \leq BF_{\tau,\tau^\star}(\lambda,r,s)$,*
6. *$\lambda - BF_{\tau,\tau^\star}(\lambda,r,s) \leq BI_{\tau,\tau^\star}(\lambda,r,s)$,*
7. *$BF_{\tau,\tau^\star}(\lambda \vee \mu,r,s) \leq (BF_{\tau,\tau^\star}(\lambda,r,s)) \vee (BF_{\tau,\tau^\star}\mu,r,s))$,*
8. *$BF_{\tau,\tau^\star}(\lambda \wedge \mu,r,s) \geq (BF_{\tau,\tau^\star}(\lambda,r,s)) \wedge (BF_{\tau,\tau^\star}(\mu,r,s))$.*

*Proof.* (1). We have $I_{\tau,\tau^*}(\lambda,r,s) \leq BI_{\tau,\tau^*}(\lambda,r,s)$. It follows that $C_{\tau,\tau^*}(\lambda,r,s) - I_{\tau,\tau^*}(\lambda,r,s) \leq BC_{\tau,\tau^*}(\lambda,r,s) - BI_{\tau,\tau^*}(\lambda,r,s)$. Hence $BF_{\tau,\tau^*}(\lambda,r,s) \leq F_{\tau,\tau^*}(\lambda,r,s)$.
(2). $BI_{\tau,\tau^*}(\lambda,r,s) = \lambda - BI_{\tau,\tau^*}(\lambda,r,s) \leq BC_{\tau,\tau^*}(\lambda,r,s) - BI_{\tau,\tau^*}(\lambda,r,s)$; since $\lambda \leq BC_{\tau,\tau^*}(\lambda,r,s) = BF_{\tau,\tau^*}(\lambda,r,s)$. Therefore, $BI_{\tau,\tau^*}(\lambda,r,s) \leq BI_{\tau,\tau^*}(\lambda,r,s)$.
(3). $BF_{\tau,\tau^*}(\lambda,r,s) = BC_{\tau,\tau^*}(\lambda,r,s) - BI_{\tau,\tau^*}(\lambda,r,s) = BC_{\tau,\tau^*}(\lambda,r,s) - (1 - BC_{\tau,\tau^*}(1-\lambda,r,s)) = BC_{\tau,\tau^*}(\lambda,r,s) - 1 + BC_{\tau,\tau^*}(1-\lambda,r,s) = BI_{\tau,\tau^*}(1-\lambda,r,s) + BC_{\tau,\tau^*}(1-\lambda,r,s) = BF_{\tau,\tau^*}(1-\lambda,r,s)$.
(4). $BF_{\tau,\tau^*}(BI_{\tau,\tau^*}(\lambda,r,s),r,s) = BC_{\tau,\tau^*}(BC_{\tau,\tau^*}(\lambda,r,s),r,s) - BI_{\tau,\tau^*}(BI_{\tau,\tau^*}(\lambda,r,s),r,s) \leq BC_{\tau,\tau^*}(\lambda,r,s) - BI_{\tau,\tau^*}(\lambda,r,s) = BF_{\tau,\tau^*}(\lambda,r,s)$.
(5). $BF_{\tau,\tau^*}(BC_{\tau,\tau^*}(\lambda,r,s),r,s) = BC_{\tau,\tau^*}(BC_{\tau,\tau^*}(\lambda,r,s),r,s) - BI_{\tau,\tau^*}(BC_{\tau,\tau^*}(\lambda,r,s),r,s) = BC_{\tau,\tau^*}(\lambda,r,s) - BI_{\tau,\tau^*}(BC_{\tau,\tau^*}(\lambda,r,s),r,s)$
$\geq BC_{\tau,\tau^*}(\lambda,r,s) - BI_{\tau,\tau^*}(\lambda,r,s) = BF_{\tau,\tau^*}(\lambda,r,s)$.
(6). Now $\lambda - BF_{\tau,\tau^*}(\lambda,r,s) = \lambda - (BC_{\tau,\tau^*}(\lambda,r,s) - BI_{\tau,\tau^*}(\lambda,r,s)) \leq BC_{\tau,\tau^*}(\lambda,r,s) - BC_{\tau,\tau^*}(\lambda,r,s) + BI_{\tau,\tau^*}(\lambda,r,s) = BI_{\tau,\tau^*}(\lambda,r,s)$.
(7). $BF_{\tau,\tau^*}(\lambda \vee \mu,r,s) = BC_{\tau,\tau^*}(A \vee \mu,r,s) - BI_{\tau,\tau^*}(\lambda \vee \mu,r,s) = BC_{\tau,\tau^*}(\lambda \vee \mu,r,s) - (BI_{\tau,\tau^*}(\lambda,r,s) \vee BI_{\tau,\tau^*}(\mu,r,s)) = (BC_{\tau,\tau^*}(\lambda,r,s) \vee BC_{\tau,\tau^*}(\mu,r,s)) - (BI_{\tau,\tau^*}(\lambda,r,s) \vee BI_{\tau,\tau^*}(\mu,r,s)) \leq (BC_{\tau,\tau^*}(\lambda,r,s) - BI_{\tau,\tau^*}(\lambda,r,s)) \vee (BC_{\tau,\tau^*}(\lambda,r,s) - BI_{\tau,\tau^*}(\mu,r,s))$
$= BI_{\tau,\tau^*}(\lambda,r,s) \vee BI_{\tau,\tau^*}(\mu,r,s)$.
(8). It is similar to (7). □

**Definition 3.6.** *For any double fuzzy topological space $(X,\tau,\tau^\star)$, $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$, the $(r,s)$-fuzzy b-exterior of $\lambda$, denoted by $BE_{\tau,\tau^*}(\lambda,r,s)$ is defined as $BE_{\tau,\tau^*}(\lambda,r,s) = BI_{\tau,\tau^*}(1-\lambda,r,s)$.*

**Proposition 3.7.** *For any double fuzzy topological space $(X,\tau,\tau^\star)$, $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$, we have*

1. $E_{\tau,\tau^*}(\lambda,r,s) \leq BE_{\tau,\tau^*}(\lambda,r,s)$,
2. $BE_{\tau,\tau^*}(\lambda,r,s) = 1 - BC_{\tau,\tau^*}(\lambda,r,s)$,
3. $BE_{\tau,\tau^*}(BE_{\tau,\tau^*}(\lambda,r,s),r,s) = BI_{\tau,\tau^*}(BC_{\tau,\tau^*}(\lambda,r,s),r,s)$,
4. *If* $\lambda \leq \mu$, *then* $BE_{\tau,\tau^*}(\lambda,r,s) \geq BE_{\tau,\tau^*}(\mu,r,s)$,
5. $BE_{\tau,\tau^*}(1,r,s) = 0$,
6. $BE_{\tau,\tau^*}(0,r,s) = 1$,
7. $BI_{\tau,\tau^*}(\lambda,r,s) \leq BE_{\tau,\tau^*}(BE_{\tau,\tau^*}(\lambda,r,s),r,s)$,
8. $BE_{\tau,\tau^*}(\lambda \vee \mu,r,s) \leq BE_{\tau,\tau^*}(\lambda,r,s) \wedge BE_{\tau,\tau^*}(\mu,r,s)$,
9. $BE_{\tau,\tau^*}(\lambda \wedge \mu,r,s) \geq BE_{\tau,\tau^*}(\lambda,r,s) \vee BE_{\tau,\tau^*}(\mu,r,s)$.

*Proof.* (1). Since $BC_{\tau,\tau^*}(\lambda,r,s) \leq BC_{\tau,\tau^*}(\lambda,r,s)$, $1 - BC_{\tau,\tau^*}(\lambda,r,s) \geq 1 - BC_{\tau,\tau^*}(\lambda,r,s))$. Then $BI_{\tau,\tau^*}(1-\lambda,r,s) \geq BC_{\tau,\tau^*}(1-\lambda,r,s)$. Therefore, by definition, $BE_{\tau,\tau^*}(\lambda,r,s) \geq BC_{\tau,\tau^*}(\lambda,r,s)$.
(2). It follows from the definitions.
(3). $BE_{\tau,\tau^*}(BE_{\tau,\tau^*}(\lambda,r,s),r,s) = BE_{\tau,\tau^*}(BI_{\tau,\tau^*}(1-\lambda,r,s),r,s) = BE_{\tau,\tau^*}(1 - BC_{\tau,\tau^*}(\lambda,r,s),r,s) = BI_{\tau,\tau^*}(1-(1-BI_{\tau,\tau^*}(\lambda,r,s),r,s)) = BI_{\tau,\tau^*}(1-1+BC_{\tau,\tau^*}(\lambda,r,s),r,s) = BI_{\tau,\tau^*}(BC_{\tau,\tau^*}(\lambda,r,s),r,s)$.
(4). Let $\lambda \leq \mu$. By using Proposition 2.2(1), $BC_{\tau,\tau^*}(\lambda,r,s) \leq BC_{\tau,\tau^*}(\mu,r,s)$. Therefore $1 - BC_{\tau,\tau^*}(\lambda,r,s) \geq 1 - BC_{\tau,\tau^*}(\mu,r,s)$. But $BI_{\tau,\tau^*}(1-\lambda,r,s) \geq BI_{\tau,\tau^*}(1-\mu,r,s)$. Hence, $BE_{\tau,\tau^*}(\lambda,r,s) \geq BE_{\tau,\tau^*}(\mu,r,s)$.
(5). By (2) $BE_{\tau,\tau^*}(1,r,s) = \bar{1} - BC_{\tau,\tau^*}(\bar{1},r,s) = \bar{1} - \bar{1} = \bar{0}$.
(6). It is similar to (5).
(7). $BE_{\tau,\tau^*}(BE_{\tau,\tau^*,r,s},r,s) = BE_{\tau,\tau^*}(BI_{\tau,\tau^*}(1-\lambda,r,s),r,s) = BE_{\tau,\tau^*}(1 - BC_{\tau,\tau^*},r,s),r,s) = BI_{\tau,\tau^*}(BC_{\tau,\tau^*}(\lambda,r,s),r,s) \geq BC_{\tau,\tau^*}(\lambda,r,s)$. Hence $BI_{\tau,\tau^*}(\lambda,r,s) \leq BE_{\tau,\tau^*}(BE_{\tau,\tau^*}(\lambda,r,s),r,s)$.
(8). $BE_{\tau,\tau^*}(\lambda \vee \mu,r,s) = BI_{\tau,\tau^*}(1-(\lambda - \vee \mu),r,s) = BI_{\tau,\tau^*}(1-\lambda) \wedge (1-\mu),r,s) \leq BI_{\tau,\tau^*}((1-\lambda),r,s) \wedge BI_{\tau,\tau^*}((1-\mu),r,s)) = BE_{\tau,\tau^*}(\lambda,r,s)$
$\wedge BE_{\tau,\tau^*}(\mu,r,s)$.
(9). $BE_{\tau,\tau^*}(\lambda \wedge \mu,r,s) = BI_{\tau,\tau^*}(\bar{1} - (\lambda \wedge \mu),r,s) = BI_{\tau,\tau^*}((\bar{1} - \lambda \vee \bar{1} - \mu),r,s) \geq BI_{\tau,\tau^*}(\bar{1} - \lambda,r,s) \vee BI_{\tau,\tau^*}(\bar{1} - \mu,r,s) = BE_{\tau,\tau^*}(\lambda,r,s) \vee BE_{\tau,\tau^*}(\mu,r,s)$. □

# 4. Some functions via $(r,s)$-fuzzy $b$-open sets

In this section, some characterizations of double fuzzy $b$-continuous, double fuzzy $b$-open, double fuzzy $b$-closed and double fuzzy $b$-irresolute functions are studied.

**Definition 4.1.** *A function $f : (X,\tau_1,\tau_1^\star) \to (Y,\tau_2,\tau_2^\star)$ is called*

1. *double fuzzy b-open if for every $(r,s)$-fuzzy b-open set $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$, $f(\lambda)$ is an $(r,s)$-fuzzy b-open in $I^Y$.*
2. *double fuzzy b-closed if for every $(r,s)$-fuzzy b-closed set $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$, $f(\lambda)$ is an $(r,s)$-fuzzy b-closed in $I^Y$.*
3. *double fuzzy b-continuous if for every $\lambda \in I^Y$ with $\tau_2(\lambda) \geq r$, $\tau_2^\star(\lambda) \leq s$, $r \in I_0$ and $s \in I_1$, $f^{-1}(\lambda)$ is an $(r,s)$-fuzzy b-open in $I^X$.*
4. *double fuzzy b-irresolute if $f^{-1}(\lambda)$ is an $(r,s)$-fuzzy b-closed set for every $(r,s)$-fuzzy b-closed set $\lambda \in I^Y$, $r \in I_0$, $s \in I_1$.*

**Theorem 4.2.** *For a bijective function $f : (X,\tau_1,\tau_1^\star) \to (Y,\tau_2,\tau_2^\star)$, the following are equivalent:*

1. *$f$ is double fuzzy b-irresolute function.*
2. *For every fuzzy set $\lambda \in I^X$, $f(BC_{\tau,\tau^*}(\lambda,r,s)) \leq BC_{\tau,\tau^*}(f(\lambda),r,s)$,*
3. *For every fuzzy set $\mu \in I^Y$, $BC_{\tau,\tau^*}(f^{-1}(\mu),r,s) \leq f^{-1}(BC_{\tau,\tau^*}(\mu,r,s)$.*

*Proof.* (1) $\Rightarrow$ (2): Suppose $\lambda \in I^X$ and $BC_{\tau,\tau^*}(f(\lambda),r,s) \in I^Y$ is an $(r,s)$-fuzzy $b$-closed, then by (1), $f^{-1}(BC_{\tau_2,\tau_2^*}(f(\lambda),r,s) \in I^X$ is an $(r,s)$-fuzzy $b$-closed set, $r \in I_0$ and $s \in I_1$. Therefore, $BC_{\tau_1,\tau_1^*}(f^{-1}(BC_{\tau_2,\tau_2^*}(f(\lambda),r,s),r,s) = f^{-1}(BC_{\tau_2,\tau_2^*}(f(\lambda),r,s))$. Since $\lambda \leq f^{-1}(f(\lambda))$ and $BC_{\tau_1,\tau_1^*}(\lambda,r,s) \leq BC_{\tau_1,\tau_1^*}(f^{-1}(f(\lambda),r,s))$. Also, $f(\lambda) \leq BC_{\tau_2,\tau_2^*}(f(\lambda),r,s)$. Then $BC_{\tau_1,\tau_1^*}(\lambda,r,s) \leq BC_{\tau_1,\tau_1^*}(f^{-1}(BC_{\tau_2,\tau_2^*}(f(\lambda),r,s),r,s)) = f^{-1}(BC_{\tau_2,\tau_2^*}(f(\lambda),r,s))$.
(2) $\Rightarrow$ (3): Suppose $\mu \in I^Y$, by (2) $f(BC_{\tau_1,\tau_1^*}(f^{-1}(\mu),r,s) \leq BC_{\tau_2,\tau_2^*}(f(f^{-1}(\mu)),r,s) \leq BC_{\tau_2,\tau_2^*}(\mu,r,s)$. That is, $f(BC_{\tau_1,\tau_1^*}(f^{-1}(\mu),r,s)) \leq$

$BC_{\tau_2,\tau_2^\star}(\mu,r,s)$. Therefore, $f^{-1}(f(BC_{\tau_1,\tau_1^\star}(f^{-1}(\mu),r,s))) \leq f^{-1}(BC_{\tau_2,\tau_2^\star}(\mu,r,s))$. Hence, $BC_{\tau_1,\tau_1^\star}(f^{-1}(\mu),r,s) \leq f^{-1}(BC_{\tau_2,\tau_2^\star}(\mu,r,s))$.
(3) $\Rightarrow$ (1): Suppose $\mu \in I^Y$ is an $(r,s)$-fuzzy $b$-closed set. Then $BC_{\tau_2,\tau_2^\star}(\mu,r,s) = \mu$. By (3) $BC_{\tau_1,\tau_1^\star}(f^{-1}(\mu),r,s) \leq f^{-1}(BC_{\tau_2,\tau_2^\star}(\mu,r,s)) = f^{-1}(\mu)$. But $f^{-1}(\mu) \leq BC_{\tau_1,\tau_1^\star}(f^{-1}(\mu),r,s)$. Therefore, $f^{-1}(\mu) = BC_{\tau_1,\tau_1^\star}(\mu,r,s)$. That is, $f^{-1}(\mu) \in I^X$ is $(r,s)$-fuzzy $b$-closed. Thus, $f$ is a double fuzzy $b$-irresolute function. □

**Proposition 4.3.** *A function $f:(X,\tau_1,\tau_1^\star) \to (Y,\tau_2,\tau_2^\star)$ is a double fuzzy b-closed if, and only if for each $\lambda \in I^X$, $BC_{\tau,\tau^\star}(f(\lambda),r,s) \leq f(BC_{\tau,\tau^\star}(\lambda,r,s))$.*

*Proof.* Suppose that $f$ is a double fuzzy $b$-closed function and $\lambda$ is any fuzzy set in $X$. Then $f(BC_{\tau,\tau^\star}(\lambda,r,s))$ is an $(r,s)$-fuzzy $b$-closed in $I^Y$. Therefore, $BC_{\tau,\tau^\star}(f(BC_{\tau,\tau^\star}(\lambda,r,s),r,s) = f(BC_{\tau,\tau^\star}(\lambda,r,s))$. Since $\lambda \leq BC_{\tau,\tau^\star}(\lambda,r,s))f(\lambda) \leq f(BC_{\tau,\tau^\star}(\lambda,r,s))$. Then $BC_{\tau,\tau^\star}(f(\lambda),r,s) \leq BC_{\tau,\tau^\star}(f(BC_{\tau,\tau^\star}(\lambda,r,s)),r,s) = f(BC_{\tau,\tau^\star}(\lambda,r,s))$. Hence for every fuzzy set $\lambda \in I^X$, $BC_{\tau,\tau^\star}(f(\lambda),r,s) \leq f(BC_{\tau,\tau^\star}(\lambda,r,s)$. Conversely, suppose that for every fuzzy set $\lambda \in I^X$, $BC_{\tau,\tau^\star}(f(\lambda),r,s) \leq f(BC_{\tau,\tau^\star}(\lambda,r,s))$. Since $\lambda$ is an $(r,s)$-fuzzy $b$-closed set, $BC_{\tau,\tau^\star}(\lambda,r,s) = \lambda$. Therefore, $f(BC_{\tau,\tau^\star}(\lambda,r,s)) = f(\lambda) \leq BC_{\tau,\tau^\star}(\lambda,r,s)$. Hence, $f(\lambda) = f(BC_{\tau,\tau^\star}(\lambda,r,s)) = BC_{\tau,\tau^\star}(f(\lambda),r,s)$, which implies that $f(\lambda) \in I^Y$ is an $(r,s)$-fuzzy $b$-closed set, that is, $f$ is double fuzzy $b$-closed function. □

**Proposition 4.4.** *If $f:(X,\tau_1,\tau_1^\star) \to (Y,\tau_2,\tau_2^\star)$ is a double fuzzy b-irresolute funtion, then $BC_{\tau,\tau^\star}(f^{-1}(\lambda),r,s)$ is zero for every $(r,s)$-fuzzy b-open set $A \in I^Y$.*

*Proof.* Let $\lambda$ be an $(r,s)$-fuzzy $b$-open set in $I^Y$. Then $f^{-1}(\lambda) \in I^X$ is $(r,s)$-fuzzy $b$-open. Therefore, $BC_{\tau,\tau^\star}(f^{-1}(\lambda),r,s) = f^{-1}(\lambda)$. By definition, $BC_{\tau,\tau^\star}(f^{-1}(\lambda),r,s) = f^{-1}(\lambda) - BC_{\tau,\tau^\star}(f^{-1}(\lambda),r,s)$. Hence, $BE_{\tau,\tau^\star}(f^{-1}(\lambda),r,s) = f^{-1}(\lambda) - f^{-1}(\lambda) = \bar{0}$. □

**Definition 4.5.** *A double fuzzy topological space $(X,\tau,\tau^\star)$ is said to be a double fuzzy b-$T_{\frac{1}{2}}$ space if each $(r,s)$-fuzzy b-closed set is $(r,s)$-fuzzy closed set in $X$.*

**Proposition 4.6.** *For any double fuzzy topological spaces $(X,\tau_1,\tau_1^\star)$ and $(Y,\tau_2,\tau_2^\star)$ if the map $f:(X,\tau,\tau^\star) \to (Y,\tau_2,\tau_2^\star)$ is a bijective, the following statements are equivalent:*

1. *$f$ and $f^{-1}$ are double fuzzy b-irresolute.*
2. *$f$ is double fuzzy b-continuous and double fuzzy b-open.*
3. *$f$ is double fuzzy b-continuous and double fuzzy b-closed.*
4. *$f(BC_{\tau,\tau^\star}(\lambda,r,s)) = BC_{\tau,\tau^\star}(f(\lambda),r,s))$ for every $\lambda \in I^X$:*

*Proof.* (1) $\Rightarrow$ (2): Suppose $\mu$ is an $(r,s)$-fuzzy $b$-open set in $X$. Since $f^{-1}$ is double fuzzy $b$-irresolute, $(f^{-1})^{-1}(\mu) \in I^Y$ is $(r,s)$-fuzzy $b$-open set, so $f$ is double fuzzy $b$-open. Now, let $\gamma \in I^Y$ be an $(r,s)$-fuzzy $b$-open set, then it is an $(r,s)$-fuzzy $b$-open. But by hypothesis, $f^{-1}$ are double fuzzy $b$-irresolute, then $f^{-1}(\gamma) \in I^X$ is an $(r,s)$-fuzzy $b$-open, that is $f$ is double fuzzy $b$-continuous.
(2) $\Rightarrow$ (3): Let $\lambda \in I^X$ is an $(r,s)$-fuzzy $b$-closed set, then $1-\lambda \in I^X$ is an $(r,s)$-fuzzy $b$-open set. By (2), $1-f(\lambda) = f(1-\lambda)$ is an $(r,s)$-fuzzy $b$-open set in $I^Y$, which implies that $f(\lambda)$ is an $(r,s)$-fuzzy $b$-closed set. Hence $f$ is a double fuzzy $b$-closed function.
(3) $\Rightarrow$ (4): Let $\lambda \in I^X$, we have $\lambda \leq f^{-1}(f(\lambda))$ and $f(\lambda) \leq BC_{\tau_2,\tau_2^\star}(f(\lambda),r,s))$. Then $\lambda \leq f^{-1}(BC_{\tau_2,\tau_2^\star}(f(\lambda),r,s)$. Now, $BC_{\tau_2,\tau_2^\star}(f(\lambda),r,s) \in I^Y$ is an $(r,s)$-fuzzy $b$-closed set. But $(Y,\tau_2,\tau_2^\star)$ is a double fuzzy $b$-$T_{\frac{1}{2}}$ space, and $BC_{\tau_2,\tau_2^\star}(f(\lambda),r,s)$ is an $(r,s)$-fuzzy closed set, then $BC_{\tau_2,\tau_2^\star}(f(\lambda),r,s) \in I^Y$ is an $(r,s)$-fuzzy $b$-closed set. Since $f$ is double fuzzy $b$-continuous, $f^{-1}(BC_{\tau_2,\tau_2^\star}(f(\lambda),r,s))$ is an $(r,s)$-fuzzy $b$-closed set, which implies $BC_{\tau,\tau^\star}(f^{-1}(BC_{\tau,\tau^\star}(f(\lambda),r,s),r,s) = f^{-1}(BC_{\tau,\tau^\star}(f(\lambda),r,s))$. But $BC_{\tau,\tau^\star}(f(\lambda),r,s) \leq BC_{\tau,\tau^\star}(f^{-1}(BC_{\tau,\tau^\star}(f(\lambda),r,s)),r,s)$ and $BC_{\tau,\tau^\star}(f(\lambda),r,s) \leq f^{-1}(BC_{\tau,\tau^\star}(f(\lambda),r,s))$. Then $f(BC_{\tau,\tau^\star}(\lambda,r,s)) \leq BC_{\tau,\tau^\star}(f(\lambda),r,s)$. Also, $BC_{\tau,\tau^\star}(f(\lambda),r,s) \leq f(BC_{\tau,\tau^\star}(\lambda,r,s))$. Hence $f(BC_{\tau,\tau^\star}(\lambda,r,s)) = BC_{\tau,\tau^\star}(f(\lambda),r,s)$.
(4) $\Rightarrow$ (1): $\lambda \in I^X$, by hypothesis of (4), $f(BC_{\tau,\tau^\star}(\lambda,r,s) = BC_{\tau,\tau^\star}(f(\lambda),r,s)$. Therefore, $f(BC_{\tau,\tau^\star}(\lambda,r,s) \leq BC_{\tau,\tau^\star}(f(\lambda),r,s)$. Then, $f$ is a double fuzzy $b$-irresolute function. Now, suppose $B \in I^Y$ is $(r,s)$-fuzzy $b$-closed. Then $BC_{\tau,\tau^\star}(\mu,r,s) = B)f(BC_{\tau,\tau^\star}(\mu,r,s)) = f(\mu)$. But by (4), $BC_{\tau,\tau^\star}(f(\mu),r,s) = f(BC_{\tau,\tau^\star}(\mu,r,s))$. Therefore, $BC_{\tau,\tau^\star}(f(\mu),r,s) = f(\mu)$. Then $f(\mu) \in I^Y$ is an $(r,s)$-fuzzy $b$-closed set. Therefore, $f^{-1}$ is double fuzzy $b$-irresolute. □

## 5. Interrelations

In this section, we present the relationship among the concepts introduced in Sections 3 and 4.

**Proposition 5.1.** *If $A$ is an $(r,s)$-fuzzy b-closed set in a double fuzzy topological space $(X,\tau,\tau^\star)$, then*

1. *$BE_{\tau,\tau^\star}(\lambda,r,s) = BE_{\tau,\tau^\star}(\lambda,r,s)$.*
2. *$BE_{\tau,\tau^\star}(\lambda,r,s) = 1-A$.*

*Proof.* (1). Let $A \in I^X$ be an $(r,s)$-fuzzy $b$-closed, we have $BC_{\tau,\tau^\star}(\lambda,r,s) = A$. But by definiton $BB_{\tau,\tau^\star}(\lambda,r,s) = A - BI_{\tau,\tau^\star}(\lambda,r,s) = BC_{\tau,\tau^\star}(\lambda,r,s) - BI_{\tau,\tau^\star}(\lambda,r,s) = BF_{\tau,\tau^\star}(\lambda,r,s)$. Therefore $BB_{\tau,\tau^\star}(\lambda,r,s) = BF_{\tau,\tau^\star}(\lambda,r,s)$.
(2). Let $A$ be an $(r,s)$-fuzzy $b$-closed set, we get $BC_{\tau,\tau^\star}(\lambda,r,s) = A$. Then $BI_{\tau,\tau^\star}(1-\lambda,r,s) = 1-A$. Therefore by definition, $BE_{\tau,\tau^\star}(\lambda,r,s) = 1-A$. □

**Proposition 5.2.** *For a function $f:(X,\tau_1,\tau_2^\star) \to (Y,\tau_2,\tau_2^\star)$, the following hold:*

1. *If $f$ is any function, then $BE_{\tau,\tau^\star}(f^{-1}(\lambda),r,s) \leq BC_{\tau,\tau^\star}(1-f^{-1}(\lambda),r,s)$ for each fuzzy set $A \in I^Y$.*
2. *If $f$ is a double fuzzy b-continuous function, then for every $(r,s)$-fuzzy b-closed set $A \in I^Y$, we have $BE_{\tau,\tau^\star}(f^{-1}(\lambda),r,s) = BE_{\tau,\tau^\star}(f^{-1}(\lambda),r,s)$.*

*Proof.* (1). Let $A \in I^Y$. Then by definition $BE_{\tau_1,\tau_1^*}(f^{-1}(\lambda),r,s) = BI_{\tau_1,\tau_1^*}(1-f^{-1}(\lambda),r,s) \leq 1-f^{-1}(\lambda)$. Also, $BE_{\tau_1,\tau_1^*}(f^{-1}(\lambda),r,s) \leq 1-BI_{\tau_1,\tau_1^*}(f^{-1}(\lambda),r,s) = BC_{\tau_1,\tau_1^*}(1-f^{-1}(\lambda),r,s)$. Therefore, $BE_{\tau_1,\tau_1^*}(f^{-1}(\lambda),r,s) \leq BC_{\tau_1,\tau_1^*}(1-f^{-1}(\lambda),r,s)$.
(2). Let $A$ be an $(r,s)$-fuzzy $b$-closed set in $Y$. Then, $f^{-1}(\lambda)$ is an $(r,s)$-fuzzy $b$-closed set in $X$. Therefore, $BC_{\tau_1,\tau_1^*}(f^{-1}(\lambda),r,s) = f^{-1}(\lambda)$. Hence, $BF_{\tau_1,\tau_1^*}(f^{-1}(\lambda),r,s) = BC_{\tau_1,\tau_1^*}(f^{-1}(\lambda),r,s) - BI_{\tau_1,\tau_1^*}(f^{-1}(\lambda),r,s) = f^{-1}(\lambda) - BI_{\tau_1,\tau_1^*}(f^{-1}(\lambda),r,s) = BE_{\tau_1,\tau_1^*}(f^{-1}(\lambda),r,s)$. Therefore, $BF_{\tau_1,\tau_1^*}(f^{-1}(\lambda),r,s) = BE_{\tau,\tau_1^*}(f^{-1}(\lambda),r,s)$. □

**Definition 5.3.** *A double fuzzy topological space $(X,\tau,\tau^\star)$ is said to be a double fuzzy b-$T_{\frac{1}{2}}$ space if each $(r,s)$-fuzzy b-closed set is $(r,s)$-fuzzy closed set in $X$.*

**Proposition 5.4.** *If $(X,\tau,\tau^\star)$ is a double fuzzy b-$T_{\frac{1}{2}}$ space and A is an $(r,s)$-fuzzy b-closed set in X, then the following statements hold:*

1. $B_{\tau,\tau^\star}(\lambda,r,s) = BF_{\tau,\tau^\star}(\lambda,r,s)$,
2. $BE_{\tau,\tau^\star}(\lambda,r,s) = \bar{1}-\lambda$.

*Proof.* (1). Let $\lambda \in I^X$ be an $(r,s)$-fuzzy $b$-closed set. Then $\lambda$ is an $(r,s)$-fuzzy closed set in $X$, which implies $B_{\tau,\tau^\star}(\lambda,r,s) = \lambda$. But by definition, $B_{\tau,\tau^\star}(\lambda,r,s) = \lambda - BI_{\tau,\tau^\star}(\lambda,r,s) = BC_{\tau,\tau^\star}(\lambda,r,s) - BI_{\tau,\tau^\star}(\lambda,r,s) = BF_{\tau,\tau^\star}(\lambda,r,s)$.
(2). By definition, $BE_{\tau,\tau^\star}(\lambda,r,s) = BI_{\tau,\tau^\star}(\lambda,r,s) = \bar{1}-\lambda$. □

**Proposition 5.5.** *If $(X,\tau,\tau^\star)$ is a double fuzzy b-$T_{\frac{1}{2}}$ space and $f:(X,\tau_1,\tau_1^\star) \to (Y,\tau_2,\tau_2^\star)$ is a fuzzy b-irresolute function. Then for an $(r,s)$-fuzzy b-closed set $\lambda$ in Y, then the following statements hold:*

1. $B_{\tau_1,\tau_1^*}(f^{-1}(\lambda),r,s) = BF_{\tau_1,\tau_1^*}(f^{-1}(\lambda),r,s)$,
2. $BE_{\tau_1,\tau_1^*}(f^{-1}(\lambda),r,s) = \bar{1}-f^{-1}(\lambda)$.

*Proof.* (1). Suppose $\lambda \in I^Y$ is an $(r,s)$-fuzzy $b$-closed set. Then $f^{-1}(\lambda)$ is an $(r,s)$-fuzzy $b$-closed set in $X$. Since by hypothesis, $(X,\tau,\tau^\star)$ is double fuzzy $b$-$T_{\frac{1}{2}}$ space, $f^{-1}(\lambda)$ is an $(r,s)$-fuzzy closed set in $X$. Then $B_{\tau_1,\tau_1^*}(f^{-1}(\lambda),r,s) = f^{-1}(\lambda)$. Also $B_{\tau_1,\tau_1^*}(f^{-1}(\lambda),r,s) = f^{-1}(\lambda) - B_{\tau_1,\tau_1^*}(f^{-1}(\lambda),r,s) = B_{\tau_1,\tau_1^*}(f^{-1}(\lambda),r,s) - B_{\tau_1,\tau_1^*}(f^{-1}(\lambda),r,s) = B_{\tau_1,\tau_1^*}(f^{-1}(\lambda),r,s)$. Hence $B_{\tau_1,\tau_1^*}(f^{-1}(\lambda),r,s) = B_{\tau_1,\tau_1^*}(f^{-1}(\lambda),r,s)$.
(2). By definition $B_{\tau_1,\tau_1^*}(f^{-1}(\lambda),r,s) = B_{\tau_1,\tau_1^*}(1-f^{-1}(\lambda),r,s) = 1-f^{-1}(\lambda)$. □

**Proposition 5.6.** *If $(X,\tau,\tau^\star)$ is a double fuzzy b-$T_{\frac{1}{2}}$ space and $f:(X,\tau_1,\tau_1^\star) \to (Y,\tau_2,\tau_2^\star)$ is a fuzzy b-closed function. The for an $(r,s)$-fuzzy b-closed set $\lambda$ in X, then the following statements hold:*

1. $B_{\tau_2,\tau_2^*}(f(\lambda),r,s) = BF_{\tau_2,\tau_2^*}(f(\lambda),r,s)$,
2. $BE_{\tau_2,\tau_2^*}(f(\lambda),r,s) = \bar{1}-f(\lambda)$.

**Proposition 5.7.** *If $(X,\tau_1,\tau_1^\star)$ is a double fuzzy b-$T_{\frac{1}{2}}$ space and $f:(X,\tau_1,\tau_1^\star) \to (Y,\tau_2,\tau_2^\star)$ and $g:(Y,\tau_2,\tau_2^\star) \to (Z,\tau_3,\tau_3^\star)$ are fuzzy b-irresolute function. Then for an $(r,s)$-fuzzy b-closed set $\lambda \in I^Z$, then the following statements hold:*

1. $B_{\tau_1,\tau_1^*}((g\circ f)^{-1}(\lambda),r,s) = BF_{\tau_1,\tau_1^*}((g\circ f)^{-1}(\lambda),r,s)$,
2. $BE_{\tau_1,\tau_1^*}((g\circ f)^{-1}(\lambda),r,s) = \bar{1}-(g\circ f)^{-1}(\lambda)$.

*Proof.* (1). Let $\lambda$ be a $(r,s)$-fuzzy $b$-closed set in $Z$. Then by hypothesis of $g$ is fuzzy $b$-irresolute, $g^{-1}(\lambda) \in I^Y$ is a $(r,s)$-fuzzy $b$-closed set. Also, $f$ is fuzzy $b$-irresolute, $f^{-1}(g^{-1}1(\lambda)) \in I^X$ is an $(r,s)$-fuzzy $b$-closed set. Thus $(g\circ f)^{-1}(\lambda) \in I^X$ is a $(r,s)$-fuzzy $b$-closed. Since $(X,\tau_1,\tau_1^\star)$is double fuzzy $b$-$T_{\frac{1}{2}}$ space, $(g\circ f)^{-1}(\lambda)$ is a $(r,s)$-fuzzy closed in $X$. So by definition, we have $B_{\tau_1,\tau_1^*}((g\circ f)^{-1}(\lambda),r,s) = (g\circ f)^{-1}(\lambda) - GI_{\tau_1,\tau_1^*}((g\circ f)^{-1}(\lambda);r;s) = BC_{\tau_1,\tau_1^*}((g\circ f)^{-1}(\lambda),r,s) - BI_{\tau_1,\tau_1^*}((g\circ f)^{-1}(\lambda);r;s) = BF_{\tau_1,\tau_1^*}((g\circ f)^{-1}(\lambda),r,s)$.
(2). By definition, $GE_{\tau_1,\tau_1^*}((g\circ f)^{-1}(\lambda),r,s) = BI_{\tau_1,\tau_1^*}(\bar{1}-(g\circ f)^{-1}(\lambda),r,s) = \bar{1}-BC_{\tau_1,\tau_1^*}((g\circ f)^{-1}(\lambda),r,s) = \bar{1}-(g\circ f)^{-1}(\lambda)$, since $(g\circ f)^{-1}(\lambda)$ is a $(r,s)$-fuzzy closed set. Therefore, $B_{\tau_1,\tau_1^*}((g\circ f)^{-1}(\lambda),r,s) = \bar{1}-(g\circ f)^{-1}(\lambda)$. □

**Example 5.8.** *Let $X=\{a,b\}$ and $f:(X,\tau_1,\tau_1^\star) \to (Y,\tau_2,\tau_2^\star)$ be the identity function. Define the fuzzy subsets $\lambda_1,\lambda_2,\lambda_3,\mu_1$ and $\mu_3$ as follows:*
$\lambda_1(a)=0.67,\quad \lambda_1(b)=0.64: \quad \lambda_2(a)=0.67,\quad \lambda_2(b)=0.35; \lambda_3(a)=0.33,\quad \lambda_3(b)=0.34;\quad \mu_1(a)=0.75,\quad \mu_1(b)=0.67;$
$\mu_2(a)=0.67,\quad \mu_2(b)=0.49.$
*Let $\tau_1,\tau_1^\star: I^X \to I$ and $\tau_2,\tau_2^\star: I^X \to I$ defined as follows:*

$$\tau_1(\lambda)=\begin{cases}1 & \text{if } \lambda=\bar{0} \text{ or } \bar{1}\\ \frac{1}{4} & \text{if } \lambda=\lambda_1\\ \frac{1}{2} & \text{if } \lambda=\lambda_2\\ \frac{3}{4} & \text{if } \lambda=\lambda_3\\ 0 & \text{otherwise,}\end{cases} \quad \tau_1^\star(\lambda)=\begin{cases}0 & \text{if } \lambda=\bar{0} \text{ or } \bar{1}\\ \frac{3}{4} & \text{if } \lambda=\lambda_1\\ \frac{1}{2} & \text{if } \lambda=\lambda_2\\ \frac{1}{4} & \text{if } \lambda=\lambda_3\\ 1 & \text{otherwise,}\end{cases} \quad \tau_2(\lambda)=\begin{cases}1 & \text{if } \lambda=\bar{0} \text{ or } \bar{1}\\ \frac{1}{4} & \text{if } \lambda=\mu_1\\ \frac{1}{8} & \text{if } \lambda=\mu_2\\ 0 & \text{otherwise,}\end{cases} \quad \tau_2^\star(\lambda)=\begin{cases}0 & \text{if } \lambda=\bar{0} \text{ or } \bar{1}\\ \frac{1}{8} & \text{if } \lambda=\mu_1\\ \frac{1}{4} & \text{if } \lambda=\mu_2\\ 1 & \text{otherwise.}\end{cases}$$

*Then the identity function $f:(X,\tau_1,\tau_1^\star) \to (X,\tau_2,\tau_2^\star)$ is double fuzzy b-irresolute but $(X,\tau_1,\tau_1^\star)$ is not a double fuzzy b-$T_{\frac{1}{2}}$ space.*

**Example 5.9.** *Let $X=\{a,b\}$ and $f:(X,\tau_1,\tau_1^\star)\to(Y,\tau_2,\tau_2^\star)$ be the identity function. Define the fuzzy subsets $\lambda_1,\lambda_2,\lambda_3,\mu_1$ and $\mu_3$ are as in the example 5.9: Let $\tau_1,\tau_1^\star:I^X\to I$ and $\tau_2,\tau_2^\star:I^X\to I$ defined as follows:*

$$\tau_1(\lambda)=\begin{cases}1 & if\ \lambda=\bar{0}\ or\ \bar{1}\\ \frac{1}{4} & if\ \lambda=\mu_1\\ \frac{1}{8} & if\ \lambda=\mu_2\\ 0 & otherwise,\end{cases}\quad \tau_1^\star(\lambda)=\begin{cases}0 & if\ \lambda=\bar{0}\ or\ \bar{1}\\ \frac{1}{8} & if\ \lambda=\mu_1\\ \frac{1}{4} & if\ \lambda=\mu_2\\ 1 & otherwise,\end{cases}\quad \tau_2(\lambda)=\begin{cases}1 & if\ \lambda=\bar{0}\ or\ \bar{1}\\ \frac{1}{4} & if\ \lambda=\lambda_1\\ \frac{1}{2} & if\ \lambda=\lambda_2\\ \frac{3}{4} & if\ \lambda=\lambda_3\\ 0 & otherwise,\end{cases}\quad \tau_2^\star(\lambda)=\begin{cases}0 & if\ \lambda=\bar{0}\ or\ \bar{1}\\ \frac{3}{4} & if\ \lambda=\lambda_1\\ \frac{1}{2} & if\ \lambda=\lambda_2\\ \frac{1}{4} & if\ \lambda=\lambda_3\\ 1 & otherwise.\end{cases}$$

*Then the identity function $f:(X,\tau_1,\tau_1^\star)\to(Y,\tau_2,\tau_2^\star)$ is double fuzzy b-closed but $(X,\tau_1,\tau_1^\star)$ is not a double fuzzy b-$T_{\frac{1}{2}}$ space.*

**Example 5.10.** *Let $X=\{a,b\}$ and $f:(X,\tau_1,\tau_1^\star)\to(X,\tau_2,\tau_2^\star)$ be the identity function. The fuzzy subsets $\lambda_1,\lambda_2,\lambda_3,\mu_1$ and $\mu_3$ as defined are as in Example 5.9 and $\gamma_1$ and $\gamma_2$ as follows:*
$\gamma_1(a)=0.75,\quad \gamma_1(b)=0.75:\quad \gamma_2(a)=0.67,\quad \gamma_2(b)=0.40.$

*Let $\tau_1,\tau_1^\star:I^X\to I$, $\tau_2,\tau_2^\star:I^X\to I$ and $\tau_3,\tau_3^\star:I^X\to I$ defined as follows:*

$$\tau_1(\lambda)=\begin{cases}1 & if\ \lambda=\bar{0}\ or\ \bar{1}\\ \frac{1}{4} & if\ \lambda=\lambda_1\\ \frac{1}{2} & if\ \lambda=\lambda_2\\ \frac{3}{4} & if\ \lambda=\lambda_3\\ 0 & otherwise,\end{cases}\quad \tau_1^\star(\lambda)=\begin{cases}0 & if\ \lambda=\bar{0}\ or\ \bar{1}\\ \frac{3}{4} & if\ \lambda=\lambda_1\\ \frac{1}{2} & if\ \lambda=\lambda_2\\ \frac{1}{4} & if\ \lambda=\lambda_3\\ 1 & otherwise,\end{cases}\quad \tau_2(\lambda)=\begin{cases}1 & if\ \lambda=\bar{0}\ or\ \bar{1}\\ \frac{1}{4} & if\ \lambda=\mu_1\\ \frac{1}{8} & if\ \lambda=\mu_2\\ 0 & otherwise,\end{cases}\quad \tau_2^\star(\lambda)=\begin{cases}0 & if\ \lambda=\bar{0}\ or\ \bar{1}\\ \frac{1}{8} & if\ \lambda=\mu_1\\ \frac{1}{4} & if\ \lambda=\mu_2\\ 1 & otherwise,\end{cases}$$

$$\tau_3(\lambda)=\begin{cases}1 & if\ \lambda=\bar{0}\ or\ \bar{1}\\ \frac{1}{4} & if\ \lambda=\gamma_1\\ \frac{1}{8} & if\ \lambda=\gamma_2\\ 0 & otherwise,\end{cases}\quad \tau_3^\star(\lambda)=\begin{cases}0 & if\ \lambda=\bar{0}\ or\ \bar{1}\\ \frac{1}{8} & if\ \lambda=\gamma_1\\ \frac{1}{4} & if\ \lambda=\gamma_2\\ 1 & otherwise.\end{cases}$$

*Then the identity functions $f:(X,\tau_1,\tau_1^\star)\to(X,\tau_2,\tau_2^\star)$ and $g:(X,\tau_2,\tau_2^\star)\to(X,\tau_3,\tau_3^\star)$ are double fuzzy b-irresolute but $(X,\tau_1,\tau_1^\star)$ is not a double fuzzy b-$T_{\frac{1}{2}}$ space.*

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