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Research Article

Meromorphic functions sharing small functions with their difference polynomial in several variables

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Abstract

The aim of this paper is to deal with the uniqueness problem on meromorphic functions in $\mathbb{C}^m$ sharing small functions with their difference polynomial, and the results obtained can be seen as some extensions of previous results from one complex variable to several complex variables.

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1. Introduction and main results

Let f be a non-constant meromorphic function in the complex domain. In this paper, we assume that the reader is familiar with the standard notions in Nevanlinna's value distribution theory, such as the proximation function $m(r, f)$, the (integrated) counting function $N(r, f)$, and the Nevanlinna characteristic function $T(r, f)$ (see e.g., [2, 14, 26]). A meromorphic function $\alpha(z)$ is called a small function with respect to f if $T(r, \alpha) = o(T(r, f))$ as $r \to \infty$ outside a possible exceptional set of finite logarithmic measure. The family of all small function of f is denoted by $S(f)$, and let $\widehat{S}(f) = S(f) \cup \{\infty\}$. We recall the following definition.

Definition 1.1 ([12,26])**.** For $\alpha \in S(f) \cap S(g)$, we say that f and g share α CM (IM) provided that $f - \alpha$ and $g - \alpha$ have the same zeros counting multiplicity (ignoring multiplicity). If $1/f$ and $1/g$ share 0 CM (IM), then we say that f and g share ∞ CM (IM).

The well-known five-value theorem [20] says that two non-constant meromorphic functions f and g on the complex plane $\mathbb{C}$ must be equal if they have the same inverse images (ignoring multiplicities) for five distinct values in $\mathbb{P}^1(\mathbb{C})$. Li and Qiao [16] proved that the five-value theorem remains valid if five values are replaced with five small functions. Rubel and Yang [22] considered the uniqueness problem that for an entire function f, if f shares two finite values CM with f', then $f = f'$. The result was considered into the case of meromorphic functions by Mues and Steinmetz [19] and Gundersen [8]. We restate it as follows:

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Theorem 1.2 ([8,19]). *If a meromorphic function f and its derivative f' share two distinct finite values a_1, a_2 CM, then $f = f'$.*

Variations and generations for Theorem 1.2 have been studied through out the last decades(see e.g., [15,23]). In 1993, Rüssmann [23] considered the case of the linear differential polynomial and proved the following result:

Theorem 1.3 ([23]). *Let f be a meromorphic function and*

$$L(f) := f^{(n)} + a_{n-1}f^{(n-1)} + \cdots + a_0 f,\ n \geq 2,$$

where $a_j (j = 0, \cdots, n-1)$ are polynomials. If f and $L(f)$ share two distinct finite values CM, then $L(f) = f$ up to some exceptional cases which were also given.

Owing to the difference analogue of the logarithmic derivative lemma verified independently by Halburd-Korhonen [9, 10] and Chiang-Feng [4], many authors have paid more attention to the study of uniqueness problems for meromorphic functions sharing values or small functions with their shifts or difference operators(see [3, 5, 7, 11]). Heittokangas et. al. [11] firstly considered a shift analogue of Theorem 1.2 and proved the following result:

Theorem 1.4 ([11]). *Let $f(z)$ be a meromorphic function of finite order, and let $\eta \in \mathbb{C}$. If $f(z)$ and $f(z+\eta)$ share three distinct functions $a, b, c \in \hat{S}(f)$ with period η CM, then $f(z) = f(z+\eta)$ for all $z \in \mathbb{C}$.*

Cui and Chen [5] considered that a meromorphic function and its difference operator share three distinct values CM, and proved the following result:

Theorem 1.5 ([5]). *Let $f(z)$ be a nonconstant meromorphic function of finite order, and η be a nonzero finite complex constant. Let a, b be two distinct finite complex constants and n be a positive integer. If $\Delta_\eta^n f(z)$ and $f(z)$ share a, b, ∞ CM, then $\Delta_\eta^n f(z) \equiv f(z)$.*

For the case of $n = 1$ in Theorem 1.5, one can also refer to [18]. And the version that sharing small functions of Theorem 1.5 was obtained by Gao et. al. [7]. As we mentioned above, a large number of research works on uniqueness problem have been studied in complex plane(see e.g., [3, 5, 7, 8, 11, 16, 18, 22, 25, 26]). One may ask whether there exist some corresponding uniqueness results for meromorphic functions sharing values with their shifts or difference operators in the case of higher dimension? In 2018, Liu and Zhang [17] gave an affirmative answer and showed some uniqueness results on meromorphic functions f in several complex variables as follows in which f shared three small functions with the general difference polynomial in f.

Define the difference polynomial in f as

$$P(f) = a_0(z)f(z) + a_1(z)f(z+\eta) + ... + a_n(z)f(z+n\eta)\ (n \in \mathbb{N}),$$

where $z \in \mathbb{C}^m, \eta \in \mathbb{C}^m \backslash \{\mathbf{0}\}$ and $a_k(z)(0 \leq k \leq n)$ are small functions of f which are not all zero and such that $\sum_{k=0}^{n} a_k(z) = 0$. Obviously, $P(f)$ denotes the more general difference polynomial. Especially, if $a_k(z) = C_n^k(-1)^{n-k}(0 \leq k \leq n)$, then $P(f) = \Delta_\eta^n f$.

For the difference polynomial $P(f)$ in f with constant coefficients, Liu and Zhang [17] proved the following result:

Theorem 1.6 ([17]). *Let $f : \mathbb{C}^m \to \mathbb{P}^1$ be a nonconstant meromorphic function of finite order and let $a(z), b(z) (\not\equiv 0) \in S(f)$ be two periodic meromorphic functions with period η, where $z, \eta \in \mathbb{C}^m$. If $f(z) - a(z)$, $P(f) - b(z)$, and $\Delta_\eta \circ P(f) - b(z)$ share $0, \infty$ CM, then $P(f) = \Delta_\eta \circ P(f)$.*

In this paper, we make further study on uniqueness problems of meromorphic functions sharing small functions with their difference polynomial $P(f)$ in several complex variables. One first shows a difference analogue of Theorem 1.3 for meromorphic functions in several complex variables as follows.

Theorem 1.7. *Let $f : \mathbb{C}^m \to \mathbb{P}^1$ be a nonconstant meromorphic function of finite order such that $P(f) \not\equiv 0$ and let $a(z), b(z) \in S(f)$ be two distinct periodic meromorphic functions with period η, where $z \in \mathbb{C}^m$. If $f(z)$ and $P(f)$ share $a(z), b(z), \infty$ CM, then $P(f) \equiv f(z)$.*

The following corollary can be derived immediately from Theorem 1.7, which can be seen as an extension of Theorem 1.5 from one complex variable to several complex variables.

Corollary 1.8. *Let $f : \mathbb{C}^m \to \mathbb{P}^1$ be a nonconstant meromorphic function of finite order such that $\Delta_\eta^n f(z) \not\equiv 0$ and let $a(z), b(z) \in S(f)$ be two distinct periodic meromorphic functions with period η, where $z \in \mathbb{C}^m$. If $f(z)$ and $\Delta_\eta^n f(z)$ share $a(z), b(z), \infty$ CM, then $\Delta_\eta^n f(z) \equiv f(z)$.*

Furthermore, if ∞ is replaced by a small function $c(z) \in S(f)$ in Theorem 1.7, we obtain the following theorem:

Theorem 1.9. *Let $f : \mathbb{C}^m \to \mathbb{P}^1$ be a nonconstant meromorphic function of finite order such that $P(f) \not\equiv 0$ and let $a(z), b(z), c(z) \in S(f)$ be three distinct periodic meromorphic functions with period η, where $z \in \mathbb{C}^m$. If $f(z)$ and $P(f)$ share $a(z), b(z), c(z)$ CM, then $P(f) \equiv f(z)$.*

For the case $a_k(z) = C_n^k(-1)^{n-k}(0 \leq k \leq n)$, Theorem 1.9 can be written as follows.

Corollary 1.10. *Let $f : \mathbb{C}^m \to \mathbb{P}^1$ be a nonconstant meromorphic function of finite order such that $P(f) \not\equiv 0$ and let $a(z), b(z), c(z) \in S(f)$ be three distinct periodic meromorphic functions with period η, where $z \in \mathbb{C}^m$. If $f(z)$ and $\Delta_\eta^n f(z)$ share $a(z), b(z), c(z)$ CM, then $\Delta_\eta^n f(z) \equiv f(z)$.*

By Theorem 1.7 and Theorem 1.9, one gets directly the following result. We omit the details of the proof.

Theorem 1.11. *Let $f : \mathbb{C}^m \to \mathbb{P}^1$ be a nonconstant meromorphic function of finite order such that $P(f) \not\equiv 0$ and let $a(z), b(z) \in S(f), c(z) \in \widehat{S}(f)$ be three distinct periodic meromorphic functions with period η, where $z \in \mathbb{C}^m$. If $f(z)$ and $P(f)$ share $a(z), b(z), c(z)$ CM, then $P(f) \equiv f(z)$.*

The following examples show that the conditions and the conclusion in Theorem 1.11 can be satisfied and that some conditions are necessary.

Example 1.12. Let $m = 2, \eta = (\ln 2, 0), z = (z^1, z^2)$, and $f(z) = e^{z^1+z^2}$. Let $n = 2$ and $P(f) = a_0(z)f(z) + a_1(z)f(z+\eta) + a_2(z)f(z+2\eta)$. Obviously, $f(z+k\eta) = 2^k e^{z^1+z^2}(k = 0, 1, 2)$ and $P(f) = (a_0(z) + 2a_1(z) + 4a_2(z))e^{z^1+z^2}$.

- Let $a_0(z) = 2(z^1+z^2) + 3, a_1(z) = -3(z^1+z^2) - 5, a_2(z) = z^1 + z^2 + 2$. One thus knows $f = P(f) = e^{z^1+z^2}$.
- Let $a_k(z) = C_n^k(-1)^{n-k}(k = 0, 1, 2)$, then $P(f) = \Delta_\eta^n f = e^{z^1+z^2} = f$.

Example 1.13. Let $m = 2, \eta = (\ln 3, 0), z = (z^1, z^2)$, and $f(z) = e^{z^1+z^2}$. Let $n = 2, a_0(z) = 3(z^1+z^2) + 2, a_1(z) = -4(z^1+z^2) - 3, a_2(z) = z^1 + z^2 + 1$. Then, $f(z+k\eta) = 3^k e^{z^1+z^2}(k = 0, 1, 2)$ and $P(f) = 2e^{z^1+z^2}$. Obviously, $f(z)$ and $P(f)$ share $0, \infty$ CM, but $P(f) \not\equiv f(z)$.

2. Basic notions and auxiliary lemmas

We firstly recall some basis notions in several complex variables (see [21,24]). For a point $z_0 \in \mathbb{C}^m$, the entire function $f(z)$ on $\mathbb{C}^m$ can be written as $f(z)=\sum_{i=0}^{\infty} Q_i(z-z_0)$, where the term $Q_i(z-z_0)$ is either identically zero or a homogeneous polynomial of degree i. For a divisor ν on $\mathbb{C}^m$, we denote the zero-multiplicity of f at z_0 by $\nu_f(z_0)=\min\{i \mid Q_i(z-z_0)\neq 0\}$. Therefore, we can define a divisor ν_f such that $\nu_f(z_0)$ equals the zero multiplicity of ν_f at z_0 in the sense of [6, Definition 2.1] whenever z_0 is a regular point of an analytic set $|\nu_f|=\overline{\{z\in\mathbb{C}^m|\ \nu_f(z)\neq 0\}}$. Let $f(z)$ be a nonzero meromorphic function on $\mathbb{C}^m$. For each $z_0\in\mathbb{C}^m$, we can choose non-zero holomorphic functions f_1 and f_2 on a neighborhood U of z_0 such that $f=\frac{f_1}{f_2}$ on U and $\dim\{z\in\mathbb{C}^m \mid f_1(z)=f_2(z)=0\}\leq m-2$. Define $\nu_f=\nu_{f_1}$ and $\nu_{\frac{1}{f}}=\nu_{f_2}$, which are independent of the choices of f_1 and f_2.

Let $z=(z^1,z^2,\cdots,z^m)\in\mathbb{C}^m$ and $r>0$, we set $\|z\|=\sqrt{|z^1|^2+\cdots+|z^m|^2}$, and

$$S_m(r)=\{z\in\mathbb{C}^m|\ \|z\|=r\},\quad B_m(r)=\{z\in\mathbb{C}^m|\ \|z\|<r\}.$$

Define the differential operators $\partial=\sum_{j=1}^m \frac{\partial}{\partial z_j}dz_j$ and $\overline{\partial}=\sum_{j=1}^m \frac{\partial}{\partial \overline{z}_j}d\overline{z}_j$. We set $d=\partial+\overline{\partial}$, $d^c=\frac{\sqrt{-1}}{4\pi}(\overline{\partial}-\partial)$, and write

$$\eta_m(z):=\left(dd^c\|z\|^2\right)^{m-1},\quad \sigma_m(z):=d^c\log\|z\|^2\wedge\left(dd^c\log\|z\|^2\right)^{m-1}$$

for $z\in\mathbb{C}^m\backslash\{\mathbf{0}\}$. Set

$$n(t,\nu_f)=\begin{cases}\sum\limits_{|z|\leq t}\nu_f(z), & \text{if}\quad m=1,\\ \displaystyle\int_{|\nu|\cap B_m(t)}\nu_f(z)\eta_m(z), & \text{if}\quad m\geq 2.\end{cases}$$

The counting functions of ν_f and the proximity function of f can be define respectively by

$$N(r,\nu_f)=\int_1^r\frac{n(t,\nu_f)}{t^{2m-1}}dt,\quad (1<r<\infty),$$
$$m(r,f)=\int_{S_m(r)}\log^+|f(z)|\sigma_m(z).$$

Then the Nevanlinna characteristic function of f is defined as $T(r,f)=m(r,f)+N(r,f)$.

The following lemmas can be used in the latter proofs of main results in this paper frequently.

Lemma 2.1. ([13, Theorem 3.1]). *Let $f:\mathbb{C}^m\to\mathbb{P}^1$ be a non-constant meromorphic function such that $f(0)\neq 0,\infty$, and let $c\in\mathbb{C}^m,\epsilon>0$. If the hyper-order $\varsigma(f)=\varsigma<2/3$, then*

$$\int_{\partial B_m(r)}\log^+\left|\frac{f(z+c)}{f(z)}\right|\sigma_m(z)=o\left(\frac{T(r,f)}{r^{1-\frac{3}{2}\varsigma-\epsilon}}\right)$$

where $r\to\infty$ outside of a possible exceptional set $E\subset[1,+\infty)$ of finite logarithmic measure $\int_E 1/dt<\infty$.

Lemma 2.2. ([13, Theorem 4.3]). *Let $f:\mathbb{C}^m\to\mathbb{P}^1$ be a meromorphic function, let $c\in\mathbb{C}^m,\epsilon>0$. If hyper-order $\varsigma(f)=\varsigma<2/3$, then*

$$T(r,f(z+c))=T(r,f)+o\left(\frac{T(r,f)}{r^{1-\frac{3}{2}\varsigma-\epsilon}}\right)$$

where $r\to\infty$ outside of an exceptional set of finite logarithmic measure.

Lemma 2.3. *Let f be a non-constant meromorphic function of finite order on $\mathbb{C}^m$, $\eta \in \mathbb{C}^m \setminus \{\mathbf{0}\}$. Then for any periodic small function $a(z)$ of f with period η,*

$$m\left(r, \frac{P(f)}{f(z)-a(z)}\right) = o(T(r,f))$$

where $r \to \infty$ outside of a possible exceptional set $E \subset [1, +\infty)$ of finite logarithmic measure $\int_E 1/dt < \infty$.

Proof. By $\sum_{k=0}^{n} a_k = 0$ and Lemma 2.1 we have

$$\begin{aligned} m\left(r, \frac{P(f)}{f(z)-a(z)}\right) &= m\left(r, \frac{\sum_{k=0}^{n} a_k(z)(f(z+kc)-a(z))}{f(z)-a(z)}\right) \\ &\leq \sum_{k=0}^{n} m\left(r, \frac{f(z+kc)-a(z))}{f(z)-a(z)}\right) + o(T(r,f)) \\ &= o(T(r, f-a)) + o(T(r,f)) = o(T(r,f)). \end{aligned}$$

□

Lemma 2.4. ([1, Corollary 4.5]). *Let $a_1(z), a_2(z), \cdots, a_n(z)$ be n meromorphic functions in $\mathbb{C}^m$ and $g_1(z), g_2(z), \cdots, g_n(z)$ be n entire functions in $\mathbb{C}^m$ satisfying*

$$\sum_{i=1}^{n} a_i(z) e^{g_i(z)} \equiv 0.$$

If for all $1 \leq i \leq n$

$$T(r, a_i) = o(T(r, e^{g_j - g_k})), j \neq k,$$

then $a_i(z) \equiv 0$ for $1 \leq i \leq n$.

Lemma 2.5. ([17]). *Let $\alpha(z)$ be a polynomial in z, $z = (z^1, z^2, \cdots, z^m) \in \mathbb{C}^m$. If $\alpha(z)$ is of degree $n (\geq 1)$, then $\deg(\alpha(z+c) - \alpha(z)) < n$ holds for any $c = (c^1, c^2, \cdots, c^m) \in \mathbb{C}^m$.*

Lemma 2.6. ([12, Theorem 1.101]). *Suppose that $f_1(z), f_2(z), \cdots, f_n(z)$ are linearly independent meromorphic functions in $\mathbb{C}^m$ such that*

$$f_1 + f_2 + \cdots + f_n \equiv 1.$$

Then for $1 \leq j \leq n$, $R > \rho > r > r_0$,

$$\begin{aligned} T(r, f_j) \leq N(r, f_j) + \sum_{k=1}^{n} \left\{ N(r, \frac{1}{f_k}) - N(r, f_k) \right\} + N(r, W) \\ - N(r, \frac{1}{W}) + l \log \left\{ (\frac{\rho}{r})^{2m-1} \frac{T(R)}{\rho - r} \right\} + O(1), \end{aligned}$$

where $W = W_{\nu_1 \cdots \nu_{n-1}}(f_1, f_2, \cdots, f_n) \not\equiv 0$ is a Wronskian determinant,

$$n - 1 \leq l = |\nu_1| + \cdots + |\nu_{n-1}| \leq \frac{n(n-1)}{2},$$

and where

$$T(r) = \max_{1 \leq k \leq n} \{T(r, f_k)\}.$$

3. Proof of Theorem 1.7

Since $f(z)$ and $P(f)$ share a, b, ∞, there exist two polynomials $\alpha(z), \beta(z)$, $z \in \mathbb{C}^m$ such that

$$\frac{P(f)-a(z)}{f(z)-a(z)} = e^{\alpha(z)}, \frac{P(f)-b(z)}{f(z)-b(z)} = e^{\beta(z)}. \tag{3.1}$$

If $e^{\alpha(z)} \equiv 1$ or $e^{\beta(z)} \equiv 1$ or $e^{\alpha(z)} \equiv e^{\beta(z)}$, then by (3.1), we get $P(f) \equiv f(z)$. We now suppose to the contrary that $P(f) \not\equiv f(z)$, which means

$$e^{\alpha(z)} \not\equiv 1, e^{\beta(z)} \not\equiv 1, e^{\alpha(z)} \not\equiv e^{\beta(z)}.$$

By Lemma 2.3, by the first equation in (3.1) one has

$$\begin{aligned} T(r, e^{\alpha}) = m\left(r, e^{\alpha}\right) &\leq m\left(r, \frac{P(f)}{f(z)-a(z)}\right) + m\left(r, \frac{a(z)}{f(z)-a(z)}\right) + O(1) \\ &\leq T(r,f) + S(r,f). \end{aligned} \tag{3.2}$$

Similarity,

$$T(r, e^{\beta}) \leq T(r,f) + S(r,f), \tag{3.3}$$

where $r \to \infty$ outside of a possible exceptional set $E \subset [1, +\infty)$ of finite logarithmic measure $\int_E 1/dt < \infty$. On the other hand, we get from (3.1) that,

$$f = \frac{be^{\beta} - ae^{\alpha} + a - b}{e^{\beta} - e^{\alpha}} = \frac{(b-a)(e^{\beta}-1)}{e^{\beta}-e^{\alpha}} + a, \tag{3.4}$$

$$P(f) = \frac{e^{\alpha}(b-a)(e^{\beta}-1)}{e^{\beta}-e^{\alpha}} + a. \tag{3.5}$$

It follows from (3.2)-(3.4) that

$$\begin{aligned} T(r,f) &\leq T(r,e^{\beta}) + T(r, e^{\beta} - e^{\alpha}) + S(r,f) \\ &\leq T(r, e^{\alpha}) + 2T(r, e^{\beta}) + S(r,f) \\ &\leq 3T(r,f) + S(r,f), \end{aligned}$$

which implies that

$$T(r,f) = O(T(r,e^{\alpha}) + T(r,e^{\beta})) \tag{3.6}$$

For $\eta \in \mathbb{C}^m$, $f : \mathbb{C}^m \to \mathbb{P}^1$, $\alpha : \mathbb{C}^m \to \mathbb{P}^1$, $\beta : \mathbb{C}^m \to \mathbb{P}^1$ and $k \in \{0, 1, 2, \cdots, n\}$, we use the short notations for brevity:

$$\begin{aligned} &\bar{f}^k = f(z+k\eta),\ \bar{\alpha}^k = \alpha(z+k\eta),\ \bar{\beta}^k = \beta(z+k\eta), \\ &\bar{A}^k = \bar{\alpha}^k - \alpha,\ \bar{B}^k = \bar{\beta}^k - \beta. \end{aligned}$$

By (3.4) and the definition of $P(f)$, we have

$$P(f) = \sum_{k=0}^{n} a_k \bar{f}^k = \sum_{k=0}^{n} a_k \left(\bar{f}^k - a\right) = \sum_{k=0}^{n} a_k \frac{(b-a)\left(e^{\bar{\beta}^k} - 1\right)}{e^{\bar{\beta}^k} - e^{\bar{\alpha}^k}}.$$

Together with (3.5),

$$\sum_{k=0}^{n} a_k \frac{(b-a)\left(e^{\bar{\beta}^k} - 1\right)}{e^{\bar{\beta}^k} - e^{\bar{\alpha}^k}} = \frac{e^{\alpha}(b-a)(e^{\beta}-1)}{e^{\beta}-e^{\alpha}} + a.$$

Multiplying by $\prod_{k=0}^{n}\left(e^{\bar{\beta}^k}-e^{\bar{\alpha}^k}\right)$ on both sides of the above equation yields

$$\sum_{k=0}^{n} a_k(b-a)\left(e^{\bar{\beta}^k}-1\right)\prod_{j=0,j\neq k}^{n}\left(e^{\bar{\beta}^j}-e^{\bar{\alpha}^j}\right)$$
$$=e^{\alpha}(b-a)(e^{\beta}-1)\prod_{k=1}^{n}\left(e^{\bar{\beta}^k}-e^{\bar{\alpha}^k}\right)+a\prod_{k=0}^{n}\left(e^{\bar{\beta}^k}-e^{\bar{\alpha}^k}\right). \tag{3.7}$$

Let's divide into three cases as follows.

Case 1. $\deg(\alpha)>\deg(\beta)\geq 0$. In this case, $\deg(\alpha)\geq 1$, $T(r,e^{\beta})=o(T(r,e^{\alpha}))$. By (3.2) and (3.6), one further has

$$T(r,e^{\beta})=S(r,f),\quad T(r,f)=O(T(r,e^{\alpha})).$$

Then for $k\in\{0,1,\cdots,n\}$, $a,b,a_k\in S(e^{\alpha})$. In view of Lemma 2.5, $\deg(\bar{A}^k)<\deg(\alpha)$ and thus $T(r,e^{\bar{A}^k})=o(T(r,e^{\alpha}))$.

For (3.7), we rewrite it as the following form:

$$\sum_{k=0}^{n} a_k(b-a)\left(e^{\bar{\beta}^k}-1\right)\prod_{j=0,j\neq k}^{n}\left(e^{\bar{\beta}^j}-e^{\bar{A}^j}e^{\alpha}\right)$$
$$=e^{\alpha}(b-a)(e^{\beta}-1)\prod_{k=1}^{n}\left(e^{\bar{\beta}^k}-e^{\bar{A}^k}e^{\alpha}\right)+a\prod_{k=0}^{n}\left(e^{\bar{\beta}^k}-e^{\bar{A}^k}e^{\alpha}\right). \tag{3.8}$$

The above equality can be seen as the polynomial in e^{α} with small function coefficients as follows:

$$D_{n+1}e^{(n+1)\alpha}+D_ne^{n\alpha}+\cdots+D_1e^{\alpha}+D_0=0, \tag{3.9}$$

where D_s are polynomials in $a,b,e^{\bar{A}_k},e^{\bar{\beta}_k},a_k(k=0,1,\cdots,n)$ and are also small functions of e^{α}, namely,

$$T(r,D_s)=o\left(T(r,e^{\alpha})\right)(s=0,1,\cdots,n+1).$$

Applying Lemma 2.4 to (3.9), one deduce that $D_s=0(s=0,1,\cdots,n+1)$. On the other hand,

$$D_{n+1}=(a-b)(e^{\beta}-1)\prod_{k=1}^{n}\left(-e^{\bar{A}^k}\right)-a\prod_{k=0}^{n}\left(-e^{\bar{A}^k}\right)$$
$$=\Big((b-a)(e^{\beta}-1)-a\Big)\prod_{k=0}^{n}\left(-e^{\bar{A}^k}\right).$$

Thus by $D_{n+1}=0$, one has $(b-a)(e^{\beta}-1)-a=0$. If $a=0$, then $e^{\beta}=1$ which contradicts the assumption that $e^{\beta(z)}\not\equiv 1$. If $b=0$, $e^{\beta}=0$ which is impossible. So, $ab\neq 0$ and $e^{\beta}=\dfrac{b}{b-a}$. From (3.8), the term D_0 can be written as

$$D_0=\sum_{k=0}^{n} a_k(b-a)\left(e^{\bar{\beta}^k}-1\right)\prod_{j=0,j\neq k}^{n}e^{\bar{\beta}^j}-a\prod_{k=0}^{n}e^{\bar{\beta}^k}.$$

Substituting $e^{\beta}=\dfrac{b}{b-a}$ into the above equality, one deduces

$$D_0=-\frac{ab^{n+1}}{(b-a)^{n+1}}\neq 0,$$

which is a contradiction.

Case 2. $\deg(\beta) > \deg(\alpha) \geq 0$. By the similar method as in Case 1, from (3.3) and (3.6) one deduces that

$$T(r, e^{\alpha}) = S(r, f), \quad T(r, f) = O(T(r, e^{\beta})),$$

and $a, b, a_k (k = 0, 1, \cdots, n)$ are all small functions with respect to e^{β}. By Lemma 2.5, one also has $\deg(\bar{B}^k) < \deg(\beta)$ and $T(r, e^{\bar{B}^k}) = o(T(r, e^{\beta}))$ for $k \in \{0, 1, 2, \cdots, n\}$. For (3.7), one has the following form:

$$\begin{aligned}\sum_{k=0}^{n} a_k(b-a)\left(e^{\bar{B}^k}e^{\beta}-1\right)\prod_{j=0,j\neq k}^{n}\left(e^{\bar{B}^j}e^{\beta}-e^{\bar{\alpha}^j}\right)\\ = e^{\alpha}(b-a)(e^{\beta}-1)\prod_{k=1}^{n}\left(e^{\bar{B}^k}e^{\beta}-e^{\bar{\alpha}^k}\right)+a\prod_{k=0}^{n}\left(e^{\bar{B}^k}e^{\beta}-e^{\bar{\alpha}^k}\right).\end{aligned}$$

The above equality can be further rewritten as

$$D_{n+1}e^{(n+1)\beta} + D_n e^{n\beta} + \cdots + D_1 e^{\beta} + D_0 = 0,$$

where $D_s \in S(e^{\beta})$. It thus follows from Lemma 2.4 that $D_s = 0$ for $s = 0, 1, \cdots, n+1$. By utilizing $\sum_{k=0}^{n} a_k = 0$, one has for the term D_{n+1},

$$\begin{aligned}D_{n+1} &= \sum_{k=0}^{n} a_k(b-a)e^{\bar{B}^k}\prod_{j=0,j\neq k}^{n} e^{\bar{B}^j} - e^{\alpha}(b-a)\prod_{k=1}^{n} e^{\bar{B}^k} - a\prod_{k=0}^{n} e^{\bar{B}^k}\\ &= -\left(a+(b-a)e^{\alpha}\right)\prod_{k=0}^{n} e^{\bar{B}^k}.\end{aligned}$$

Then $D_{n+1} = 0$ yields $a + (b-a)e^{\alpha} = 0$. By some simple discussion, one gets the contradiction for the case that $a = 0$ or $b = 0$. Hence, $ab \neq 0$ and $e^{\alpha} = \dfrac{a}{a-b}$. Furthermore, D_0 turns into

$$\begin{aligned}D_0 &= -\sum_{k=0}^{n} a_k(b-a)\prod_{j=0,j\neq k}^{n}\left(-e^{\bar{\alpha}^j}\right) + e^{\alpha}(b-a)\prod_{k=1}^{n}\left(-e^{\bar{\alpha}^k}\right) - a\prod_{k=0}^{n}\left(-e^{\bar{\alpha}^k}\right)\\ &= -\frac{ba^{n+1}}{(b-a)^{n+1}}.\end{aligned}$$

Since a, b are two distinct functions, $D_0 \neq 0$, which contradicts the fact that $D_s = 0$ for $s = 0, 1, \cdots, n+1$.

Case 3. $\deg(\beta) = \deg(\alpha) \geq 0$.

Subcase 3.1. $\deg(\beta) = \deg(\alpha) = 0$. In this case, e^{α} and e^{β} are nonzero constants. Together with (3.4), $f(z)$ is a periodic function of period η. By the definition of $P(f)$, we have

$$P(f) = \sum_{k=0}^{n} a_k \bar{f}^k = \sum_{k=0}^{n} a_k f = 0,$$

which contradicts to the assumption that $P(f) \not\equiv 0$.

Subcase 3.2. $\deg(\beta) = \deg(\alpha) \geq 1$. Obviously, $T(r, e^{\beta}) = O(T(r, e^{\alpha}))$. By (3.6), one has

$$T(r, f) = O(T(r, e^{\alpha})),\ T(r, f) = O(T(r, e^{\beta})),$$

and $a, b, a_k, e^{\bar{A}^k},\ e^{\bar{B}^k} (k = 0, 1, \cdots, n)$ are all small functions of e^{α} and e^{β}. Set $\gamma(z) = \beta(z) - \alpha(z)$, then $\gamma(z)$ is a polynomial in $z \in \mathbb{C}^m$ such that $\deg(\gamma) \leq \deg(\alpha) = \deg(\beta)$.

If $\deg(\gamma) < \deg(\alpha)$, $T(r, e^{\gamma}) = o(T(r, e^{\alpha}))$. Then (3.7) becomes

$$\sum_{k=0}^{n} a_k(b-a)\left(e^{\bar{B}^k} e^{\gamma} e^{\alpha} - 1\right) \prod_{j=0, j\neq k}^{n} \left(e^{\bar{B}^j} e^{\gamma} e^{\alpha} - e^{\bar{A}^j} e^{\alpha}\right)$$
$$= e^{\alpha}(b-a)(e^{\gamma}e^{\alpha} - 1) \prod_{k=1}^{n} \left(e^{\bar{B}^k} e^{\gamma} e^{\alpha} - e^{\bar{A}^k} e^{\alpha}\right) + a \prod_{k=0}^{n} \left(e^{\bar{B}^k} e^{\gamma} e^{\alpha} - e^{\bar{A}^k} e^{\alpha}\right).$$

The above equality can be rewritten as the following form:

$$D_{n+2} e^{(n+2)\alpha} + D_{n+1} e^{(n+1)\alpha} + D_n e^{n\alpha} = 0,$$

where D_s satisfies $T(r, D_s) = o(T(r, e^{\alpha}))(s = n, n+1, n+2)$. One thus gets $D_s \equiv 0(s = n, n+1, n+2)$ by Lemma 2.4. On the other hand,

$$D_{n+2} = (b-a)e^{\gamma} \prod_{k=1}^{n} \left(e^{\bar{B}^k} e^{\gamma} - e^{\bar{A}^k}\right) = (b-a)e^{\gamma - n\alpha} \prod_{k=1}^{n} \left(e^{\bar{\beta}^k} - e^{\bar{\alpha}^k}\right),$$

which yields $D_{n+2} \neq 0$ for $e^{\alpha} \neq e^{\beta}$, a contradiction.

If $\deg(\gamma) = \deg(\alpha)$, one rewrite (3.7) as

$$\sum_{k=0}^{n} a_k(b-a)\left(e^{\bar{B}^k} e^{\beta} - 1\right) \prod_{j=0, j\neq k}^{n} \left(e^{\bar{B}^j} e^{\beta} - e^{\bar{A}^j} e^{\alpha}\right)$$
$$= e^{\alpha}(b-a)(e^{\beta} - 1) \prod_{k=1}^{n} \left(e^{\bar{B}^k} e^{\beta} - e^{\bar{A}^k} e^{\alpha}\right) + a \prod_{k=0}^{n} \left(e^{\bar{B}^k} e^{\beta} - e^{\bar{A}^k} e^{\alpha}\right),$$

and we get

$$\sum_{k=0}^{\chi} D_k e^{l_k \alpha + s_k \beta} = 0, \tag{3.10}$$

where $0 \leq l_k \leq n+1, 0 \leq s_k \leq n+1$ and $D_k(k = 0, 1, \cdots, \chi)$ are all small functions of e^{α} and e^{β}. Notice that D_k are not all zeros. For the term $e^{(n+1)\alpha + \beta}$, its coefficient $(b-a)\prod_{k=1}^{n}(-e^{\bar{A}^k})$ is not zero since $b - a \neq 0$, and the coefficient$(b-a)\prod_{k=1}^{n} e^{\bar{B}^k}$ of the term $e^{\alpha + (n+1)\beta}$ is not zero.

Assume that $\deg(l_i\alpha + s_i\beta - l_j\alpha - s_j\beta) = \deg(\alpha) = \deg(\beta)$ for all $0 \leq i \leq \chi, 0 \leq j \leq \chi$. Set

$$\varphi_k = D_k e^{l_k \alpha + s_k \beta}, k = 0, 1, \cdots, \chi.$$

From (3.10), $\sum_{k=0}^{\chi} \varphi_k = 0$. We deduce from the basic linear algebra that there exist $j \in \{0, 1, 2, \cdots, \chi\}$ and some nonzero complex numbers λ_k such that

$$\varphi_j = \sum_{k\in\kappa} \lambda_k \varphi_k, \quad \kappa \subset \{0, 1, \cdots, j-1, j+1, \cdots, \chi\},$$

where $\{\varphi_k | k \in \kappa\}$ are linearly independent. Divide both of the two sides of the above equality by φ_j, we have

$$1 = \sum_{k\in\kappa} \lambda_k \frac{\varphi_k}{\varphi_j} = \sum_{k\in\kappa} \lambda_k \frac{D_k}{D_j} e^{(l_k - l_j)\alpha + (s_k - s_j)\beta}.$$

Note that the zeros and poles of $\left\{\lambda_k \frac{D_k}{D_j} e^{(l_k - l_j)\alpha + (s_k - s_j)\beta}\right\}$ and their Wronskian determinant come only from the zeros and poles of $D_k(k \in \kappa)$ and D_j. Then by Lemma 2.6,

$$T\left(r, \lambda_k \frac{D_k}{D_j} e^{(l_k - l_j)\alpha + (s_k - s_j)\beta}\right) \leq O\left(\sum_{k\in\kappa} T(r, D_k) + T(r, D_j)\right) = o(T(r, e^{\alpha})).$$

This is impossible since $\deg(l_i\alpha + s_i\beta - l_j\alpha - s_j\beta) = \deg(\alpha) = \deg(\beta)$. Hence there exist two distinct integers $i, j \in \{0, 1, 2, \cdots, \chi\}$ such that $\deg(l_i\alpha + s_i\beta - l_j\alpha - s_j\beta) < \deg(\alpha) = \deg(\beta)$. Noting that $(l_i, s_i) \neq (l_j, s_j)$. If $l_i - l_j = 0, s_i - s_j \neq 0$ or $l_i - l_j \neq 0, s_i - s_j = 0$, then $\deg(l_i\alpha + s_i\beta - l_j\alpha - s_j\beta) = \deg(\alpha) = \deg(\beta)$. Hence, $l_i - l_j \neq 0$ and $s_i - s_j \neq 0$.

By simple computation, we have

$$\begin{aligned} e^{\beta} &= \left(e^{(s_i-s_j)\beta}\right)^{1/(s_i-s_j)} = \left(e^{(l_j-l_i)\alpha} e^{(s_i-s_j)\beta-(l_j-l_i)\alpha}\right)^{1/(s_i-s_j)} \\ &= e^{((s_i-s_j)\beta-(l_j-l_i)\alpha)/(s_i-s_j)} e^{((l_j-l_i)\alpha)/(s_i-s_j)} = He^{t\alpha}, \end{aligned}$$

where $H = e^{((s_i-s_j)\beta-(l_j-l_i)\alpha)/(s_i-s_j)}$ and $t = (l_j - l_i) / (s_i - s_j)$. It follows from $\deg(l_i\alpha + s_i\beta - l_j\alpha - s_j\beta) < \deg(\alpha) = \deg(\beta)$ that $H \in S(e^{\alpha})$, $H \in S(e^{\beta})$. On the other hand, $\deg(\gamma) = \deg(\alpha) = \deg(\beta)$, we know that $t \neq 1$. Without loss of generality, we assume that $|t| \leq 1$. Otherwise, we may consider $e^{\alpha} = \left(e^{\beta}/H\right)^{1/t}$.

Suppose that $t = q/p > 0$, where p, q are positive co-prime integers and $q < p$. Denote $e^{\widetilde{\alpha}} = e^{\alpha/p}$, $T(r, e^{\alpha}) = pT(r, e^{\widetilde{\alpha}})$ and we know $a, b, H, e^{\bar{A}^k}, e^{\bar{B}^k}, a_k (k = 0, 1, \cdots, n)$ are all small functions with respect to $e^{\widetilde{\alpha}}$. Note that $e^{\beta} = He^{t\alpha} = He^{q\widetilde{\alpha}}$, then (3.7) can be rewritten as

$$\begin{aligned} &\sum_{k=0}^{n} a_k(b-a)\left(e^{\bar{B}^k} He^{q\widetilde{\alpha}} - 1\right) \prod_{j=0, j\neq k}^{n} \left(e^{\bar{B}^j} He^{q\widetilde{\alpha}} - e^{\bar{A}^j} e^{p\widetilde{\alpha}}\right) \\ &= e^{p\widetilde{\alpha}}(b-a)\left(He^{q\widetilde{\alpha}} - 1\right) \prod_{k=1}^{n} \left(e^{\bar{B}^k} He^{q\widetilde{\alpha}} - e^{\bar{A}^k} e^{p\widetilde{\alpha}}\right) + a \prod_{k=0}^{n} \left(e^{\bar{B}^k} He^{q\widetilde{\alpha}} - e^{\bar{A}^k} e^{p\widetilde{\alpha}}\right). \end{aligned}$$

For the above equality, one also gets

$$Ee^{(n+1)p\widetilde{\alpha}+q\widetilde{\alpha}} + P_0(\widetilde{\alpha}) = 0, \tag{3.11}$$

where $E = (a-b)H \prod_{k=1}^{n} \left(-e^{\bar{A}^k}\right)$ is a nonzero polynomial for $a \neq b$ and $P_0(e^{\widetilde{\alpha}})$ is a polynomial in $e^{\widetilde{\alpha}}$ of degree at most $(n+1)p$ with small coefficients. Obviously, $T(r, E) = o\left(T(r, e^{\widetilde{\alpha}})\right)$. Applying Lemma 2.4 to (3.11) again, we deduce that $E \equiv 0$, a contradiction.

Suppose now that $t = -q/p < 0$, where p, q are positive co-prime integers and $q \leq p$. Denote $e^{\widetilde{\alpha}} = e^{\alpha/p}$, then $e^{\beta} = He^{t\alpha} = He^{-q\widetilde{\alpha}}$ and $T(r, e^{\alpha}) = pT(r, e^{\widetilde{\alpha}})$. Therefrore, $a, b, H, e^{\bar{A}^k}, e^{\bar{B}^k}, a_k (k = 0, 1, \cdots, n)$ are all small functions with respect to $e^{\widetilde{\alpha}}$. From (3.7), one gets

$$\begin{aligned} &\sum_{k=0}^{n} a_k(b-a)\left(e^{\bar{B}^k} He^{-q\widetilde{\alpha}} - 1\right) \prod_{j=0, j\neq k}^{n} \left(e^{\bar{B}^j} He^{-q\widetilde{\alpha}} - e^{\bar{A}^j} e^{p\widetilde{\alpha}}\right) \\ &= e^{p\widetilde{\alpha}}(b-a)\left(He^{-q\widetilde{\alpha}} - 1\right) \prod_{k=1}^{n} \left(e^{\bar{B}^k} He^{-q\widetilde{\alpha}} - e^{\bar{A}^k} e^{p\widetilde{\alpha}}\right) + a \prod_{k=0}^{n} \left(e^{\bar{B}^k} He^{-q\widetilde{\alpha}} - e^{\bar{A}^k} e^{p\widetilde{\alpha}}\right). \end{aligned}$$

Multiplying by $e^{(n+1)q\widetilde{\alpha}}$ on both sides of the above equation,

$$\begin{aligned} &\sum_{k=0}^{n} a_k(b-a)\left(e^{\bar{B}^k} H - e^{q\widetilde{\alpha}}\right) \prod_{j=0, j\neq k}^{n} \left(e^{\bar{B}^j} H - e^{\bar{A}^j} e^{(p+q)\widetilde{\alpha}}\right) \\ &= e^{p\widetilde{\alpha}}(b-a)\left(H - e^{q\widetilde{\alpha}}\right) \prod_{k=1}^{n} \left(e^{\bar{B}^k} H - e^{\bar{A}^k} e^{(p+q)\widetilde{\alpha}}\right) + a \prod_{k=0}^{n} \left(e^{\bar{B}^k} H - e^{\bar{A}^k} e^{(p+q)\widetilde{\alpha}}\right). \end{aligned}$$

Furthermore, we obtain

$$Ee^{(n+1)(p+q)\widetilde{\alpha}} + P_0(e^{\widetilde{\alpha}}) + E_0 = 0, \tag{3.12}$$

where

$$E = -b\prod_{k=0}^{n}\left(-e^{\bar{A}^k}\right), E_0 = -a\prod_{k=0}^{n}\left(e^{\bar{B}^k}H\right)$$

and $P_0(\widetilde{e^{\alpha}})$ is a polynomial in $\widetilde{e^{\alpha}}$ with small function coefficients such that $1 \le \deg(P_0(\widetilde{e^{\alpha}})) \le (n+1)p + nq$. Obviously, $E \in S(\widetilde{e^{\alpha}})$, $E_0 \in S(\widetilde{e^{\alpha}})$ Applying Lemma 2.4 to (3.12), $E = E_0 = 0$. One thus gets $a = b = 0$, a contradiction.

4. Proof of Theorem 1.9

Since $a(z), b(z), c(z)$ are distinct, we may assume that $a(z)c(z) \not\equiv 0$. Let

$$g_1(z) = \frac{f(z)-a(z)}{f(z)-b(z)}\cdot\frac{c(z)-b(z)}{c(z)-a(z)}, \quad g_2(z) = \frac{P(f)-a(z)}{P(f)-b(z)}\cdot\frac{c(z)-b(z)}{c(z)-a(z)}. \tag{4.1}$$

Then $g_1(z)$ and $g_2(z)$ share the values $0, 1, \infty$ CM. Hence, there exist two polynomials $\alpha(z)$ and $\beta(z)$, $z \in \mathbb{C}^m$ such that

$$\frac{g_1(z)}{g_2(z)} = e^{\alpha(z)}, \quad \frac{g_1(z)-1}{g_2(z)-1} = e^{\beta(z)}. \tag{4.2}$$

Suppose that $P(f) \not\equiv f(z)$. Then

$$e^{\alpha(z)} \not\equiv 1, e^{\beta(z)} \not\equiv 1, e^{\alpha(z)} \not\equiv e^{\beta(z)}.$$

By Lemma 2.2, we deduce from (4.1) and (4.2) that

$$T(r, e^{\alpha}) \le (n+2)T(r,f) + S(r,f), \tag{4.3}$$

$$T(r, e^{\beta}) \le (n+2)T(r,f) + S(r,f), \tag{4.4}$$

where $r \to \infty$ outside of an exceptional set of finite logarithmic measure. It follows from (4.2) that

$$g_1(z) = \frac{1-e^{\beta(z)}}{1-e^{\beta(z)-\alpha(z)}}, \quad g_2(z) = \frac{1-e^{\beta(z)}}{e^{\alpha(z)}-e^{\beta(z)}}. \tag{4.5}$$

By (4.1), (4.3), (4.4), and (4.5) we have

$$\begin{aligned} T(r,f) &= T(r,g_1) + S(r,f) \le T(r,e^{\alpha}) + 2T(r,e^{\beta}) + S(r,f) \\ &\le (3n+6)T(r,f) + S(r,f). \end{aligned} \tag{4.6}$$

Let $d = \dfrac{c-b}{c-a}$, $d \ne 0, 1$. We thus conclude from (4.1) and (4.5) that

$$f(z) = \frac{(b-a)d(e^{\alpha}-e^{\beta})}{(1-d)e^{\alpha}-e^{\alpha+\beta}+de^{\beta}} + b, \tag{4.7}$$

$$P(f) = \frac{b-ade^{\alpha}+(ad-b)e^{\beta}}{1-(1-d)e^{\beta}-de^{\alpha}}. \tag{4.8}$$

As similar discussion as showed in Theorem 1.7, we use the short notations for brevity:

$$\bar{f}^k = f(z+k\eta), \bar{\alpha}^k = \alpha(z+k\eta), \bar{\beta}^k = \beta(z+k\eta), \bar{A}^k = \bar{\alpha}^k - \alpha, \bar{B}^k = \bar{\beta}^k - \beta.$$

By (4.7) and the definition of $P(f)$, we have

$$P(f) = \sum_{k=0}^{n} a_k\bar{f}^k = \sum_{k=0}^{n} a_k\left(\bar{f}^k - b\right) = \sum_{k=0}^{n} a_k\frac{(b-a)d(e^{\bar{\alpha}^k}-e^{\bar{\beta}^k})}{(1-d)e^{\bar{\alpha}^k}-e^{\bar{\alpha}^k+\bar{\beta}^k}+de^{\bar{\beta}^k}}.$$

Together with (4.8),

$$\sum_{k=0}^{n} a_k\frac{(b-a)d(e^{\bar{\alpha}^k}-e^{\bar{\beta}^k})}{(1-d)e^{\bar{\alpha}^k}-e^{\bar{\alpha}^k+\bar{\beta}^k}+de^{\bar{\beta}^k}} = \frac{b-ade^{\alpha}+(ad-b)e^{\beta}}{1-(1-d)e^{\beta}-de^{\alpha}}. \tag{4.9}$$

Multiplying by $\prod_{k=0}^{n}\left((1-d)e^{\bar{\alpha}^k}-e^{\bar{\alpha}^k+\bar{\beta}^k}+de^{\bar{\beta}^k}\right)$ and $\left(1-(1-d)e^{\beta}-de^{\alpha}\right)$ on both sides of (4.9), it yields

$$\sum_{k=0}^{n}a_k(b-a)d(e^{\bar{\alpha}^k}-e^{\bar{\beta}^k})\left(1-(1-d)e^{\beta}-de^{\alpha}\right)\cdot\prod_{j=0,j\neq k}^{n}\left[(1-d)e^{\bar{\alpha}^j}-e^{\bar{\alpha}^j+\bar{\beta}^j}+de^{\bar{\beta}^j}\right]$$
$$=\left(b-ade^{\alpha}+(ad-b)e^{\beta}\right)\prod_{k=0}^{n}\left((1-d)e^{\bar{\alpha}^k}-e^{\bar{\alpha}^k+\bar{\beta}^k}+de^{\bar{\beta}^k}\right). \tag{4.10}$$

We consider it in three cases.

Case 1. $\deg(\alpha)>\deg(\beta)\geq 0$. Obviously, $\deg(\alpha)\geq 1$, $T(r,e^{\beta})=o(T(r,e^{\alpha}))$. It follows from (4.6) that

$$T(r,f)\leq T(r,e^{\alpha})+S(r,f)\leq(3n+6)T(r,f)+S(r,f),$$

which implies that $T(r,f)=O(T(r,e^{\alpha}))$. Thus, $a,b,d,a_k,\bar{A}^k(k=0,1,\cdots,n)$ are all small functions of e^{α}. From (4.10), we have

$$\sum_{k=0}^{n}a_k(b-a)d\left(e^{\bar{A}^k}e^{\alpha}-e^{\bar{\beta}^k}\right)\left(1-(1-d)e^{\beta}-de^{\alpha}\right)$$
$$\cdot\prod_{j=0,j\neq k}^{n}\left(\left(1-d-e^{\bar{\beta}^j}\right)e^{\bar{A}^j}e^{\alpha}+de^{\bar{\beta}^j}\right)$$
$$=\left(b-ade^{\alpha}+(ad-b)e^{\beta}\right)\prod_{k=0}^{n}\left(\left(1-d-e^{\bar{\beta}^k}\right)e^{\bar{A}^k}e^{\alpha}+de^{\bar{\beta}^k}\right).$$

For the above equality, we further get

$$D_{n+2}e^{(n+2)\alpha}+D_{n+1}e^{(n+1)\alpha}+\cdots+D_1e^{\alpha}+D_0=0, \tag{4.11}$$

where $D_s(s=0,1,\cdots,n+2)$ are polynomials in $a,b,d,e^{\bar{A}_k},e^{\bar{\beta}_k},a_k(k=0,1,\cdots,n)$. Applying Lemma 2.4 to (4.11), $D_s\equiv 0(s=0,1,\cdots,n+2)$. In particular,

$$D_0=\sum_{k=0}^{n}a_k(b-a)d(-e^{\bar{\beta}^k})\left(1-(1-d)e^{\beta}\right)\prod_{j=0,j\neq k}de^{\bar{\beta}^j}-\left(b+(ad-b)e^{\beta}\right)\prod_{k=0}^{n}de^{\bar{\beta}^k}$$
$$=\sum_{k=0}^{n}a_k(a-b)\left(1-(1-d)e^{\beta}\right)\prod_{k=0}^{n}de^{\bar{\beta}^k}-\left(b+(ad-b)e^{\beta}\right)\prod_{k=0}^{n}de^{\bar{\beta}^k}$$
$$=-\left(b+(ad-b)e^{\beta}\right)\prod_{k=0}^{n}de^{\bar{\beta}^k},$$

which ensures that $b+(ad-b)e^{\beta}=0$. Note that $b-ad=\dfrac{c(b-a)}{c-a}\neq 0$, we have that

$$e^{\beta}=\frac{b}{b-ad}. \tag{4.12}$$

Substituting it into D_{n+2}, it gives

$$D_{n+2}=\sum_{k=0}^{n}a_k(b-a)de^{\bar{A}^k}(-d)\prod_{j=0,j\neq k}^{n}\left(1-d-e^{\bar{\beta}^j}\right)e^{\bar{A}^j}+ad\prod_{k=0}^{n}\left(1-d-e^{\bar{\beta}^k}\right)e^{\bar{A}^k}$$
$$=\left(\sum_{k=0}^{n}a_k(b-a)d(-d)\left(1-d-\frac{b}{b-ad}\right)^{n}+ad\left(1-d-\frac{b}{b-ad}\right)^{n+1}\right)\prod_{k=0}^{n}e^{\bar{A}^k}$$
$$=ad\left(1-d-\frac{b}{b-ad}\right)^{n+1}\prod_{k=0}^{n}e^{\bar{A}^k}=0.$$

By $ad \neq 0$, $1 - d - \dfrac{b}{b - ad} = 0$. Together with (4.12), we have

$$e^{\beta} = 1 - d.$$

Substituting it into (4.9), we get

$$\frac{b-a}{1-d}\sum_{k=0}^{n} a_k e^{\bar{A}^k} e^{\alpha} = \frac{a+b-ad-ae^{\alpha}}{2-d-e^{\alpha}}. \tag{4.13}$$

For simplicity, we denote $U = \dfrac{b-a}{1-d}\sum_{k=0}^{n} a_k e^{\bar{A}^k}$, it is easy to check that $T(r, U) = S(r, e^{\alpha})$. By the assumption that $P(f) \not\equiv 0$, $U \neq 0$. (4.13) can be rewritten as the following equality.

$$Ue^{2\alpha} = ((2-d)U + a)\, e^{\alpha} + ad - b - a,$$

which yields $2T(r, e^{\alpha}) = T(r, e^{\alpha}) + S(r, e^{\alpha})$, a contradiction.

Case 2. $\deg(\beta) > \deg(\alpha) \geq 0$. In this case, $\deg(\beta) \geq 1$, $T(r, e^{\alpha}) = o(T(r, e^{\beta}))$. By the similar discussion as showed in Case 1, one has $T(r, f) = O(T(r, e^{\beta}))$. And (4.10) can be rewritten as

$$D_{n+2}e^{(n+2)\beta} + D_{n+1}e^{(n+1)\beta} + \cdots + D_1 e^{\beta} + D_0 = 0, \tag{4.14}$$

where $D_s \in S(e^{\beta})$. Applying Lemma 2.4 to (4.14), $D_s \equiv 0$ for $s = 0, 1, \cdots, n+2$. For the term D_0, we know

$$\begin{aligned}
D_0 &= \sum_{k=0}^{n} a_k(b-a)de^{\bar{\alpha}^k}(1 - de^{\alpha}) \prod_{j=0, j\neq k}^{n} (1-d)e^{\bar{\alpha}^j} - (b - ade^{\alpha}) \prod_{k=0}^{n} (1-d)e^{\bar{\alpha}^k} \\
&= \sum_{k=0}^{n} a_k(b-a)d(1-de^{\alpha})(1-d)^n \prod_{k=0}^{n} e^{\bar{\alpha}^k} - (b - ade^{\alpha})\,(1-d)^{n+1} \prod_{k=0}^{n} e^{\bar{\alpha}^k} \\
&= (ade^{\alpha} - b)\,(1-d)^{n+1} \prod_{k=0}^{n} e^{\bar{\alpha}^k}.
\end{aligned}$$

Since $ad \neq 0$ and $d \neq 1$, $D_0 = 0$ implies

$$e^{\alpha} = \frac{b}{ad}. \tag{4.15}$$

Substituting it into D_{n+2}, it gives

$$\begin{aligned}
D_{n+2} &= \sum_{k=0}^{n} a_k(b-a)de^{\bar{B}^k}(1-d) \prod_{j=0, j\neq k}^{n} (d - e^{\bar{\alpha}^j})e^{\bar{B}^j} + (b-ad) \prod_{k=0}^{n} (d - e^{\alpha})e^{\bar{B}^k} \\
&= \left(\sum_{k=0}^{n} a_k(b-a)d(1-d) + (b-ad)(d-e^{\alpha}) \right) \prod_{k=0}^{n} e^{\bar{B}^k}(d-e^{\alpha})^n \\
&= (b-ad)\left(d - \frac{b}{ad}\right) \prod_{k=0}^{n} e^{\bar{B}^k} \left(d - \frac{b}{ad}\right)^n.
\end{aligned}$$

Owing to $(b - ad) \neq 0$, we deduce from $D_{n+2} = 0$ that $d = \dfrac{b}{ad}$. Together with (4.15), $e^{\alpha} = d$. Substituting it into (4.9), we get

$$(a-b)\sum_{k=0}^{n} a_k e^{\bar{B}^k} e^{\beta} \left(e^{\beta} - d - 1\right) = (b-ad)e^{\beta} - ad^2 + b. \tag{4.16}$$

Set $V = (a-b)\sum_{k=0}^{n} a_k e^{\bar{B}^k}$. As one knows $b - ad \not\equiv 0$, from the above equality, $V \not\equiv 0$ and $T(r, V) = S(r, e^{\beta})$. Then (4.16) has the following form:

$$Ve^{2\beta} = ((d+1)V + b - ad)\, e^{\beta} - ad^2 + b,$$

which yields $2T(r, e^{\beta}) = T(r, e^{\beta}) + S(r, e^{\beta})$, a contradiction.

Case 3. $\deg(\beta) = \deg(\alpha) \geq 0$.

Subcase 3.1. $\deg(\beta) = \deg(\alpha) = 0$. In this case, e^{α} and e^{β} are nonzero constants. Together with (4.7), $f(z)$ is a periodic function of period η. By the definition of $P(f)$, we have

$$P(f) = \sum_{k=0}^{n} a_k \bar{f}^k = \sum_{k=0}^{n} a_k f = 0,$$

which contradicts to the assumption that $P(f) \not\equiv 0$.

Subcase 3.2. $\deg(\beta) = \deg(\alpha) \geq 1$. Obviously, $T(r, e^{\beta}) = O(T(r, e^{\alpha}))$. (4.6) gives

$$T(r, f) = O(T(r, e^{\alpha})).$$

Set $\gamma(z) = \beta(z) - \alpha(z)$, $z \in \mathbb{C}^m$ where $\gamma(z)$ satisfies $\deg(\gamma) \leq \deg(\alpha) = \deg(\beta)$. If $\deg(\gamma) < \deg(\alpha)$, obviously, $T(r, e^{\gamma}) = o\left(T(r, e^{\alpha})\right)$. By (4.10),

$$\begin{aligned}
&\sum_{k=0}^{n} a_k (b-a) d \left(e^{\bar{A}^k} e^{\alpha} - e^{\bar{B}^k + \gamma} e^{\alpha}\right) \left(1 - (1-d) e^{\gamma} e^{\alpha} - d e^{\alpha}\right) \\
&\qquad \cdot \prod_{j=0, j\neq k}^{n} \left((1-d) e^{\bar{A}^j} e^{\alpha} - e^{\bar{A}^j + \bar{B}^j + \gamma} e^{2\alpha} + d e^{\bar{B}^j + \gamma} e^{\alpha}\right) \\
&= \left(b - ade^{\alpha} + (ad-b) e^{\gamma} e^{\alpha}\right) \prod_{k=0}^{n} \left((1-d) e^{\bar{A}^k} e^{\alpha} - e^{\bar{A}^k + \bar{B}^k + \gamma} e^{2\alpha} + d e^{\bar{B}^k + \gamma} e^{\alpha}\right).
\end{aligned}$$

Furthermore, we get

$$D_{2n+3} e^{(2n+3)\alpha} + D_{2n+2} e^{(2n+2)\alpha} + \cdots + D_{n+1} e^{(n+1)\alpha} = 0, \tag{4.17}$$

where $D_s (s = n+1, \cdots, 2n+3)$ are small functions of e^{α}. Applying Lemma 2.4 to (4.17), we obtain $D_s \equiv 0 (s = n+1, \cdots, 2n+3)$. In particular, for the term D_{2n+3} one has

$$D_{2n+3} = \left((ad-b) e^{\gamma} - ad\right) \prod_{k=0}^{n} \left(-e^{\bar{A}^k + \bar{B}^k + \gamma}\right) = 0.$$

Owing to $(ad - b) = \dfrac{c(a-b)}{c-a} \neq 0$, $e^{\gamma} = \dfrac{ad}{ad-b}$, namely, $e^{\alpha} = \dfrac{ad-b}{ad} e^{\beta}$. Thus, (4.9) gives

$$\sum_{k=0}^{n} a_k \frac{1}{H_0 e^{\bar{B}^k} e^{\beta} - H_1} = \frac{H_2^2}{e^{\beta} - H_2}, \tag{4.18}$$

where

$$H_0 = 1 - \frac{b}{ad}, \quad H_1 = 1 - \frac{b - bd}{ad}, \quad H_2 = \frac{a}{a-b}.$$

Note that H_0, H_1, H_2 are small functions of e^{β} and

$$H_0 = 1 - \frac{b}{ad} = \frac{c(a-b)}{a(c-b)} \neq 0, \quad H_2 = \frac{a}{a-b} \neq 0.$$

Multiplying by $\left(e^{\beta} - H_2\right) \prod_{k=0}^{n} \left(H_0 e^{\bar{B}^k} e^{\beta} - H_1\right)$ on both sides of (4.18), we get

$$\sum_{k=0}^{n} a_k \left(e^{\beta} - H_2\right) \prod_{j=0, j\neq k}^{n} \left(H_0 e^{\bar{B}^j} e^{\beta} - H_1\right) = H_2^2 \prod_{k=0}^{n} \left(H_0 e^{\bar{B}^k} e^{\beta} - H_1\right), \tag{4.19}$$

which equivalents to

$$E_{n+1} e^{(n+1)\beta} + E_n e^{n\beta} + \cdots + E_1 e^{\beta} + E_0 = 0, \tag{4.20}$$

where $E_s(s=0,1,\cdots,n+1)$ are polynomials in $H_0,H_1,H_2,e^{\bar{B}^k},a_k(k=0,1,\cdots,n)$. Applying Lemma 2.4 to (4.20), $E_s\equiv 0(s=0,1,\cdots,n+1)$. On the other hand, we see that

$$E_0=\sum_{k=0}^{n}a_k(-H_1)^n(-H_2)-H_2^2(-H_1)^{n+1}=-H_2^2(-H_1)^{n+1}=0,$$

which ensures $H_1=0$ since $H_2\neq 0$. By (4.19), we get

$$(H_4-H_0H_2^2)e^{\beta}=H_2H_4,$$

where $H_4=\sum_{k=0}^{n}a_ke^{-\bar{B}^k}$. Together with $H_0H_2\neq 0$, $H_4\neq 0$ and $(H_4-H_0H_2^2)\neq 0$, a contradiction.

If $\deg(\gamma)=\deg(\alpha)=\deg(\beta)$, then (4.10) can be rewritten as

$$\begin{aligned}\sum_{k=0}^{n}&a_k(b-a)d\left(e^{\bar{A}^k}e^{\alpha}-e^{\bar{B}^k}e^{\beta}\right)\left(1-(1-d)e^{\beta}-de^{\alpha}\right)\\&\cdot\prod_{j=0,j\neq k}^{n}\left[\left(1-d-e^{\bar{B}^j}e^{\beta}\right)e^{\bar{A}^j}e^{\alpha}+de^{\bar{B}^j}e^{\beta}\right]\\&=\left[b-ade^{\alpha}+(ad-b)e^{\beta}\right]\prod_{k=0}^{n}\left[\left(1-d-e^{\bar{B}^k}e^{\beta}\right)e^{\bar{A}^k}e^{\alpha}+de^{\bar{B}^k}e^{\beta}\right].\end{aligned}$$

From the above equality, we get

$$\sum_{k=0}^{\chi}D_ke^{l_k\alpha+s_k\beta}=0, \tag{4.21}$$

where $0\leq l_k\leq n+2, 0\leq s_k\leq n+2(k=0,1,\cdots,\chi)$ and $D_k(k=0,1,\cdots,\chi)$ are polynomials in $a,b,d,e^{\bar{A}^k},e^{\bar{B}^k},a_k(k=0,1,\cdots,n)$. We thus have

$$T(r,D_k)=o(T(r,e^{\alpha}))=o(T(r,e^{\beta}))(k=0,1,\cdots,\chi).$$

We know that the coefficients of the equation(4.21) are not all zeros. For the term $e^{(n+2)\alpha+(n+1)\beta}$, its coefficient $-ad\prod_{k=0}^{n}\left(-e^{\bar{B}^k}e^{\bar{A}^k}\right)$ is not zero since $ad\neq 0$. For the term $e^{(n+1)\alpha+(n+2)\beta}$, its coefficient$(ad-b)\prod_{k=0}^{n}\left(-e^{\bar{B}^k}e^{\bar{A}^k}\right)$ is not zero since $ad-b\neq 0$.

Next, we assume that $\deg(l_i\alpha+s_i\beta-l_j\alpha-s_j\beta)=\deg(\alpha)=\deg(\beta)$ for all $0\leq i\leq\chi, 0\leq j\leq\chi$. Let $\varphi_k=D_ke^{l_k\alpha+s_k\beta}(k=0,1,\cdots,\chi)$. Thus, $\sum_{k=0}^{\chi}\varphi_k=0$. From basic linear algebra, we deduce that there exist $j\in\{0,1,2,\cdots,\chi\}$ and a set $\kappa\subset\{0,1,\cdots,j-1,j+1,\cdots,\chi\}$ such that

$$\varphi_j=\sum_{k\in\kappa}\lambda_k\varphi_k, \tag{4.22}$$

where $\lambda_k(k\in\kappa)$ are some nonzero complex numbers and $\{\varphi_k|k\in\kappa\}$ is linearly independent. Dividing both of the two sides of (4.22) by φ_j, we have

$$1=\sum_{k\in\kappa}\lambda_k\frac{\varphi_k}{\varphi_j}=\sum_{k\in\kappa}\lambda_k\frac{D_k}{D_j}e^{(l_k-l_j)\alpha+(s_k-s_j)\beta}.$$

It is not difficult to verify that the zeros and poles of $\left\{\lambda_k\frac{D_k}{D_j}e^{(l_k-l_j)\alpha+(s_k-s_j)\beta}\right\}$ and their Wronskian determinant come only from the zeros and poles of $D_k(k\in\kappa)$ and D_j. Then by Lemma 2.6 we have that

$$T(r,\lambda_k\frac{D_k}{D_j}e^{(l_k-l_j)\alpha+(s_k-s_j)\beta})\leq O\left(\sum_{k\in\kappa}T(r,D_k)+T(r,D_j)\right)=o(T(r,e^{\alpha})).$$

This is a contradiction for $\deg(l_i\alpha + s_i\beta - l_j\alpha - s_j\beta) = \deg(\alpha) = \deg(\beta)$. Hence there exist two distinct integers $i, j \in \{0, 1, 2, \cdots, m\}$ such that $\deg(l_i\alpha + s_i\beta - l_j\alpha - s_j\beta) < \deg(\alpha) = \deg(\beta)$. Noting that $l_i - l_j \neq 0$ and $s_i - s_j \neq 0$.

By the similar method as in Theorem 1.7,

$$e^{\beta} = He^{t\alpha},$$

where $H = e^{((s_i-s_j)\beta-(l_j-l_i)\alpha)/(s_i-s_j)} \in S(e^{\alpha}) \cap S(e^{\alpha})$ and $t = \frac{l_j-l_i}{s_i-s_j}(t \neq 1)$. Without loss of generality, we may suppose $|t| \leq 1$ for otherwise we may consider $e^{\alpha} = \left(e^{\beta}/H\right)^{1/t}$.

Suppose that $t = q/p > 0$, where p, q are positive co-prime integers and $q < p$. We denote $e^{\widetilde{\alpha}} = e^{\alpha/p}$, then $e^{\beta} = He^{t\alpha} = He^{q\widetilde{\alpha}}$ and $T(r, e^{\alpha}) = pT(r, e^{\widetilde{\alpha}})$. Therefore $a, b, d, H, e^{\bar{A}^k}, e^{\bar{B}^k}, a_k(k = 0, 1, \cdots, n)$ are all small functions with respect to $e^{\widetilde{\alpha}}$. From that we rewrite (4.10) as

$$\begin{aligned}\sum_{k=0}^{n} a_k(b-a)d\left(e^{\bar{A}^k}e^{p\widetilde{\alpha}} - He^{\bar{B}^k}e^{q\widetilde{\alpha}}\right)&\left(1-(1-d)He^{q\widetilde{\alpha}} - de^{p\widetilde{\alpha}}\right)\\ &\cdot \prod_{j=0, j\neq k}^{n}\left((1-d)e^{\bar{A}^j}e^{p\widetilde{\alpha}} - e^{\bar{A}^j+\bar{B}^j}He^{(p+q)\widetilde{\alpha}} + dHe^{\bar{B}^j}e^{q\widetilde{\alpha}}\right)\\ &= \left(b - ade^{p\widetilde{\alpha}} + (ad-b)He^{q\widetilde{\alpha}}\right)\\ &\cdot \prod_{k=0}^{n}\left((1-d)e^{\bar{A}^k}e^{p\widetilde{\alpha}} - e^{\bar{A}^k+\bar{B}^k}He^{(p+q)\widetilde{\alpha}} + dHe^{\bar{B}^k}e^{q\widetilde{\alpha}}\right).\end{aligned}$$

Furthermore, we have

$$Ee^{(n+2)p\widetilde{\alpha}+(n+1)q\widetilde{\alpha}} + P_0(e^{\widetilde{\alpha}}) = 0, \tag{4.23}$$

where $E \in S(e^{\widetilde{\alpha}})$ is a polynomial in $a, b, d, H, e^{\bar{A}^k}, e^{\bar{B}^k}, a_k(k = 0, 1, \cdots, n)$ and $P_0(e^{\widetilde{\alpha}})$ is a polynomial in $e^{\widetilde{\alpha}}$ of degree at most $(n+1)p + (n+2)q$ with coefficients being small with respect to $e^{\widetilde{\alpha}}$. Applying Lemma 2.4 to (4.23), $E \equiv 0$. Since $ad \neq 0$, we have

$$E = adH^{n+1}\prod_{k=0}^{n} e^{\bar{A}^K}e^{\bar{B}^k} \neq 0$$

a contradiction.

Suppose that $t = -q/p < 0$, where p, q are positive co-prime integers and $q \leq p$. Set $e^{\widetilde{\alpha}} = e^{\alpha/p}$, then $e^{\beta} = He^{t\alpha} = He^{-q\widetilde{\alpha}}$. From (4.10), we have

$$\begin{aligned}\sum_{k=0}^{n} a_k(b-a)d\left(e^{\bar{A}^k}e^{p\widetilde{\alpha}} - He^{\bar{B}^k}e^{-q\widetilde{\alpha}}\right)&\left(1-(1-d)He^{-q\widetilde{\alpha}} - de^{p\widetilde{\alpha}}\right)\\ &\cdot \prod_{j=0, j\neq k}^{n}\left((1-d)e^{\bar{A}^j}e^{p\widetilde{\alpha}} - e^{\bar{A}^j+\bar{B}^j}He^{(p-q)\widetilde{\alpha}} + dHe^{\bar{B}^j}e^{-q\widetilde{\alpha}}\right)\\ &= \left(b - ade^{p\widetilde{\alpha}} + (ad-b)He^{-q\widetilde{\alpha}}\right)\\ &\cdot \prod_{k=0}^{n}\left((1-d)e^{\bar{A}^k}e^{p\widetilde{\alpha}} - e^{\bar{A}^k+\bar{B}^k}He^{(p-q)\widetilde{\alpha}} + dHe^{\bar{B}^k}e^{-q\widetilde{\alpha}}\right).\end{aligned}$$

Multiplying by $e^{(n+2)q\widetilde{\alpha}}$ on both sides of the above equation, we obtain

$$\sum_{k=0}^{n} a_k(b-a)d\left(e^{\bar{A}^k}e^{(p+q)\widetilde{\alpha}} - He^{\bar{B}^k}\right)\left(e^{q\widetilde{\alpha}} - (1-d)H - de^{(p+q)\widetilde{\alpha}}\right)$$
$$\cdot \prod_{j=0,j\neq k}^{n}\left((1-d)e^{\bar{A}^j}e^{(p+q)\widetilde{\alpha}} - e^{\bar{A}^j+\bar{B}^j}He^{p\widetilde{\alpha}} + dHe^{\bar{B}^j}\right)$$
$$= \left(be^{q\widetilde{\alpha}} - ade^{(p+q)\widetilde{\alpha}} + (ad-b)H\right)$$
$$\cdot \prod_{k=0}^{n}\left((1-d)e^{\bar{A}^k}e^{(p+q)\widetilde{\alpha}} - e^{\bar{A}^k+\bar{B}^k}He^{p\widetilde{\alpha}} + dHe^{\bar{B}^k}\right).$$

We conclude that

$$Ee^{(n+2)(p+q)\widetilde{\alpha}} + P_0(e^{\widetilde{\alpha}}) = 0. \tag{4.24}$$

where $E \in S(e^{\widetilde{\alpha}})$ and $P_0(e^{\widetilde{\alpha}})$ is a polynomial in $e^{\widetilde{\alpha}}$ of degree at most $(n+1)p + (n+2)q$ with coefficients being small with respect to $e^{\widetilde{\alpha}}$. Applying Lemma 2.4 to (4.24),$E \equiv 0$. On the other hand,

$$E = \sum_{k=0}^{n} a_k(b-a)de^{\bar{A}^k}(-d)\prod_{j=0,j\neq k}^{n}(1-d)e^{\bar{A}^j} + ad\sum_{k=0}^{n}(1-d)e^{\bar{A}^k}$$
$$= (1-d)^n\prod_{k=0}^{n}e^{\bar{A}^k}\left(\sum_{k=0}^{n}a_k(a-b)d^2 + ad\right)$$
$$= (1-d)^n\prod_{k=0}^{n}e^{\bar{A}^k}ad \neq 0,$$

a contradiction. That completes the proof of Theorem 1.9.

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