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FURTHER ON THE COMPOSITION OF SAGBI BASES

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Abstract. Composition is the operation of replacing variables in a polynomial by other polynomials. In this paper, we show that composition commutes with SAGBI basis computation (possibly under different monomial orderings) if the leading monomials of the composition polynomials are a permuted powering.

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1. Introduction

Our interest in the subject of this paper is inspired by Hong ([6], [7]), where he addresses the problem of the behavior of Gröbner bases ([1], [2]) under composition of polynomials. Let $K[x_1, \ldots, x_n]$ denote the polynomial ring over the field K and G be a subset of $K[x_1, \ldots, x_n]$. Let F be a Gröbner basis (with respect to a monomial ordering $>$) of the ideal generated by G and Θ be a list of n polynomials. We say that composition by Θ commutes with the Gröbner basis computation if the composed set $F \circ \Theta$ is also a Gröbner basis (with respect to the monomial ordering $>^{'}$) of $G \circ \Theta$. We would like to investigate the conditions on Θ under which the composition commutes with the Gröbner basis computation. The case when $>$ and $>^{'}$ are the same monomial orderings is completely dealt with in [7]. There is a sequel of this paper also by Hong ([6]), which is devoted to the case when $F \circ \Theta$ is a Gröbner bases under $>^{'}$ (possibly different from $>$). He shows that this happens if the list of the leading monomials of Θ is a permuted powering (see Section 2.2 for terminology).

The concept of Gröbner basis for ideals of a polynomial ring over a field K can be adapted in a natural way to K-subalgebras of a polynomial ring. In [11] SAGBI (Subalgebra Analog to Gröbner Basis for Ideals) basis for the K-subalgebra of $K[x_1, \ldots, x_n]$ are defined, this concept was independently developed in [8]. Since many of the basic concepts of Gröbner bases transfer to the subalgebra case, it is

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natural to investigate the conditions under which composition by a set Θ commutes with SAGBI bases computation. The case of the same monomial ordering is treated in [9]. Let S be a SAGBI basis (with respect to monomial ordering $>$) of the subalgebra generated by G. The subject of this paper is to show that $S \circ \Theta$ is a SAGBI basis of $G \circ \Theta$ with respect to some monomial ordering $>^{'}$ (possibly different from $>$) if the list of the leading monomials of Θ is a permuted powering (Theorem 3.1).

The natural application of the results in this paper is the same as for Gröbner bases in [6]. Composed objects often occur in real-life mathematical models, and given a set G of polynomials in which the variables are defined in terms of other variables, it should be more efficient to compute a SAGBI basis of G before carrying out the composition. (Note, however, that in contrast to Gröbner bases, SAGBI bases computation may not terminate). We also mention that polynomial composition is a widely studied area. Since commutation with Gröbner bases computation implies commutation with SAGBI bases computation, all the compositions with leading monomial iterated powering in Section 5 of [6] clearly apply to the SAGBI case.

The paper is organized as follows. In Section 2, we briefly describe the underlying concept of SAGBI basis and composition of polynomials. After setting up the necessary notation, we present Lemma 2.12, which shows that the compositions with leading monomial iterated powering are compatible with non-equality. In Section 3, we present the main theorem (Theorem 3.1). The proof of this theorem is a consequence of Lemma 3.2 and Lemma 3.5. We have implemented a procedure based on our main theorem in a computer algebra system SINGULAR ([4]). In Section 4, we provide computational examples with associated C.P.U time of this procedure and also a current implementation of the SAGBI basis construction algorithm in SINGULAR.

2. Notation and definition

In this section, we will review some basic terminology and results of SAGBI basis theory and composition of polynomials that will be used in the subsequent sections. The reader who is already familiar with the theory is also encouraged to skim through this section in order to get familiar with the notational convention.

By a monomial in $K[x_1, \ldots, x_n]$ we mean an element of the form $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ with $\alpha_1, \ldots, \alpha_n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, we denote the set of all terms by Mon_n. Note that $1 = x_1^0 \ldots x_n^0 \in Mon_n$.

If G is a subset of $K[x_1, \ldots, x_n]$ (not necessarily finite), then the subalgebra of $K[x_1, \ldots, x_n]$ generated by G is usually denoted $K[G]$. This notion is natural since the elements of $K[G]$ are precisely the polynomials in the set of formal variables G, viewed as elements of $K[G]$.

In this paper we always assume that monomial orderings are global monomial orderings (i.e. well orderings). Let $>$ be a monomial ordering on Mon_n(see for example [3]). We can associate to every non-zero polynomial $f \in K[x_1, \ldots, x_n]$ its leading monomial, denoted by $LM_>(f)(\in Mon_n)$. We call the coefficient of $LM_>(f)$ the leading coefficient of f denoted by $LC_>(f)$, $LC_>(f)LM_>(f)$ is called leading term denoted by $LT_>(f)$. We also define, for a subset $G \subset K[x_1, \ldots, x_n], LM_>(G) = \{LM_>(f) \mid f \in G\}$.

2.1. Review of SAGBI basis theory. Here we gather the theory concerning SAGBI bases that we will need. For a more complete exposition we refer to [11].

Definition 2.1. A G-monomial is a finite power product of the form $m(G) = g_1^{\alpha_1} \ldots g_m^{\alpha_m}$ where $g_i \in G$ for $i = 1, \ldots, m$, and $\alpha_1, \ldots, \alpha_m \in \mathbb{N}$.

Definition 2.2. A subset S (may be infinite) of $K[G]$ is called SAGBI basis of $K[G]$ with respect to $>$ if

$$K[LM_>(K[G])] = K[LM_>(S)].$$

We can show that if S is a SAGBI basis for $K[G]$, then S generates $K[G]$, i.e. $K[G] = K[S]$. When we say that S is a SAGBI basis, we simply mean that S is a SAGBI basis of $K[S]$.

Definition 2.3. We say that two G-monomials $m(G), m^{'}(G)$ form a critical pair $(m(G), m^{'}(G))$ of G if $LM_>(m(G)) = LM_>(m^{'}(G))$. If $c \in K$ such that $m(G)$ and $cm^{'}(G)$ have the same leading coefficient, then we define the T-polynomial of $(m(G), m^{'}(G))$ as $T(m(G), m'(G)) = m(G) - cm^{'}(G)$.

The next theorem gives the criterion for a set to be a SAGBI basis of $K[G]$.

Theorem 2.4. (c.f. [11]) *A subset S of $K[x_1, \ldots, x_n]$ is a SAGBI basis with respect to $>$ if and only if the T-polynomial of every critical pair $(m(S), m^{'}(S))$ of S either equal to zero, or can be written as*

$$T(m, m^{'}) = \sum_{i=1}^{t} c_i m_i(S), \qquad LM_>(m(S)) = LM_>(m^{'}(S)) > LM_>(m_i(S)) \; \forall i.$$

2.2. Composition of polynomials. We fix first some notations and notions. These we will be used throughout the paper.

- f and g are two non-zero polynomials and p, q are two monomials in $K[x_1, \ldots, x_n]$.
- Θ is a list $(\theta_1, \ldots, \theta_n)$ of n non-zero polynomials in $K[x_1, \ldots, x_n]$.
- $LM_>(\Theta)$ is the list $(LM_>(\theta_1), \ldots, LM_>(\theta_n))$.
- $Mat(LM_>(\Theta))$ is the exponent matrix of $LM_>(\Theta)$, that is, the $n \times n$ matrix whose (i, j)-th entry is given by $deg_{x_i}(LM_>(\theta_j))$. In other words, the j-th column of the matrix consists of the exponents of the leading monomial of θ_j.

Now we define the process of composition of polynomials.

Definition 2.5. Let $f \in K[x_1, \ldots, x_n]$. We define the composition of f by Θ, written as $f \circ \Theta$, as the polynomial obtained from f by replacing each x_i by θ_i. We also define, for a subset $G \subset K[x_1, \ldots, x_n]$, $G \circ \Theta = \{g \circ \Theta \mid g \in G\}$.

Now we state some basic properties and facts about the composition and leading monomials. These will be used throughout the paper.

Proposition 2.6. (c.f. [6])

a) $(f + g) \circ \Theta = f \circ \Theta + g \circ \Theta$.
b) $(fg) \circ \Theta = (f \circ \Theta)(g \circ \Theta)$.
c) $LM_>(fg) = LM_>(f)LM_>(g)$.
d) $LM_>(p \circ \Theta) = p \circ LM_>(\Theta)$.

Remark 2.7. *We have a natural correspondence between the set* $G = \{g_1, g_2, \ldots\}$ *and* $G \circ \Theta = \{g_1 \circ \Theta, g_2 \circ \Theta, \ldots\}$ *therefore for any* G*-monomial* $m(G)$*, its composition with* Θ *satisfies*

$$m(G) \circ \Theta = m(G \circ \Theta)$$

Also all the critical pairs of $G \circ \Theta$ *are of the form* $(m(G \circ \Theta), m^{'}(G \circ \Theta))$*, for some* G*-monomials* $m(G)$*,* $m^{'}(G)$*.*

Definition 2.8. The composition of $>$ by Θ, written as $>\circ\Theta$ is the binary relation over Mon_n defined such that, for all monomials p, q

$$p >\circ\Theta\, q \Longleftrightarrow p \circ LM_>(\Theta) > q \circ LM_>(\Theta).$$

The relation $>\circ\Theta$ is not necessarily a monomial ordering. For a counter example see [6]. However, under some condition on Θ it becomes a monomial ordering.

Definition 2.9. The list $LM_>(\Theta)$ is called a permuted powering, if and only if ,

$$LM_>(\Theta) = (x_{\pi(1)}^{\lambda_1}, \dots, x_{\pi(n)}^{\lambda_n})$$

for some permutation of π of $(1, \dots, n)$ and some $\lambda_1, \dots, \lambda_n > 0$.

Lemma 2.10. ([6]) *Let $LT_>(\Theta)$ be a permuted powering. Then*

(i) $>\circ\Theta$ *is a monomial ordering. In this case we denote it by* $>_\Theta$.

(ii) $LM_>(f \circ \Theta) = LM_{>_\Theta}(f) \circ LM_>(\Theta)$.

Definition 2.11. We say that composition by Θ is compatible with non-equality, if, for all monomials p, q we have

$$p \neq q \Longrightarrow p \circ LM_>(\Theta) \neq q \circ LM_>(\Theta).$$

Lemma 2.12. *If $LM_>(\Theta)$ is a permuted powering then composition by Θ is compatible with non-equality.*

Proof. Let $p = x^{\alpha_1} \dots x^{\alpha_n}$ and $q = x^{\beta_1} \dots x^{\beta_n}$. As $LM_>(\Theta)$ is permuted powering, therefore we have $p \circ LM_>(\Theta) = x_{\pi(1)}^{\alpha_1\lambda_1} \dots x_{\pi(n)}^{\alpha_n\lambda_n}$ and $q \circ LM_>(\Theta) = x_{\pi(1)}^{\beta_1\lambda_1} \dots x_{\pi(n)}^{\beta_n\lambda_n}$. If $p \neq q$ then $\alpha_i \neq \beta_i$ for some i, it implies $\alpha_i\lambda_i \neq \beta_i\lambda_i$. This shows that $p \circ LM_>(\Theta) \neq q \circ LM_>(\Theta)$. □

3. Main result

Theorem 3.1. (Main theorem) *If the list $LM_>(\Theta)$ is a permuted powering and S is a SAGBI basis of $K[G]$ with respect to $>_\Theta$ then $S \circ \Theta$ is a SAGBI basis of $K[G \circ \Theta]$ with respect to $>$.*

The proof of main theorem is based on the following lemma.

Lemma 3.2. *If $LM_>(\Theta)$ is a permuted powering and S is a SAGBI basis with respect to $>_\Theta$ then $S \circ \Theta$ is a SAGBI basis with respect to $>$.*

Before proving of Lemma 3.2 we will give Lemma 3.3.

Lemma 3.3. *Assume that $LM_>(\Theta)$ is a permuted powering. For some S-monomial $m(S), m^{'}(S)$, if $(m(S \circ \Theta), m^{'}(S \circ \Theta))$ is a critical pair of $S \circ \Theta$ with respect to $>$ then $(m(S), m^{'}(S))$ is a critical pair of S with respect to $>_\Theta$.*

Proof. Assume that $(m(S \circ \Theta), m^{'}(S \circ \Theta))$ is a critical pair of $S \circ \Theta$ with respect to $>$ then $LM_>(m(S \circ \Theta)) = LM_>(m^{'}(S \circ \Theta))$. We know from Remark 2.7 that $m(S \circ \Theta) = m(S) \circ \Theta$ and $m^{'}(S \circ \Theta) = m^{'}(S) \circ \Theta$ so by Lemma 2.10

we get $LM_{>_\Theta}(m(S)) \circ LM_>(\Theta) = LM_{>_\Theta}(m^{'}(S)) \circ LM_>(\Theta)$. Since our composition is compatible with non-equality with respect to $>$ (Lemma 2.12), we get $LM_{>_\Theta}(m(S)) = LM_{>_\Theta}(m^{'}(S))$ i.e $(m(S), m^{'}(S))$ is a critical pair of S with respect to $>_\Theta$. □

Now we give the proof of Lemma 3.2

Proof. Assume that $LM_>(\Theta)$ is a permuted powering. Let S is a SAGBI basis with respect to $>_\Theta$. We need to show $S \circ \Theta$ is a SAGBI basis with respect to $>$. We will use Theorem 2.4, so let $(m(S \circ \Theta), m^{'}(S \circ \Theta))$ be arbitrary critical pair of $S \circ \Theta$ with respect to $>$. From Lemma 3.3, we know that $(m(S), m^{'}(S))$ is critical pair of S with respect to $>_\Theta$. Since S is a SAGBI basis with respect to $>_\Theta$, by Theorem 2.4 we can write

$$m(S) - cm^{'}(S) = \sum_i c_i m_i(S) \text{ (or zero) where } c, c_i \in K \textit{ and} \tag{1}$$

$$LM_{>_\Theta}(m(S)) = LM_{>_\Theta}(m^{'}(S)) >_\Theta LM_{>_\Theta}(m_i(S)) \ \forall i. \tag{2}$$

Composing the equation (1) with Θ and using proposition 2.6 we get

$$m(S \circ \Theta) - cm^{'}(S \circ \Theta) = \sum_i c_i m_i(S \circ \Theta) \text{ (or zero)}. \tag{3}$$

From the definition of the relation $>_\Theta$, we get the inequality in (2) as

$$LM_{>_\Theta}(m(S)) \circ LM_>(\Theta) = LM_{>_\Theta}(m^{'}(S)) \circ LM_>(\Theta) > LM_{>_\Theta}(m_i(S)) \circ LM_>(\Theta) \ \forall i.$$

Using Lemma 2.10, this becomes

$$LM_>(m(S \circ \Theta)) = LM_>(m^{'}(S \circ \Theta)) > LM_>(m_i(S \circ \Theta)) \ \forall i. \tag{4}$$

The leading terms of the left-hand side of (3) cancel. Thus (3) and (4) together give a representation as in Theorem 2.4, and since the critical pair $(m(S \circ \Theta), m^{'}(S \circ \Theta))$ of $S \circ \Theta$ with respect to $>$ was arbitrary, we conclude that $S \circ \Theta$ is a SAGBI basis with respect to $>$. □

The following lemma is not difficult to prove.

Lemma 3.4. $K[S] = K[G] \Rightarrow K[S \circ \Theta] = K[G \circ \Theta]$.

Lemma 3.5. *Consider the following statements.*

(A) *S is a SAGBI basis of $K[G]$ with respect $>_\Theta$ then $S \circ \Theta$ is SAGBI basis of $K[G \circ \Theta]$ with respect to $>$.*

(B) *S is SAGBI basis with respect $>_\Theta$ then $S \circ \Theta$ is SAGBI basis with respect to $>$.*

Then (B) $\Rightarrow$ (A).

Proof. Let S be a SAGBI basis of $K[G]$ with respect $>_\Theta$ then we trivially have S is SAGBI basis with respect $>_\Theta$. Then from (B), we have, $S \circ \Theta$ is SAGBI basis with respect to $>$. Since S is SAGBI basis of $K[G]$ with respect to $>$ therefore $K[S] = K[G]$. Then by Lemma 3.4, we have $K[S \circ \Theta] = K[G \circ \Theta]$. Therefore, we conclude that $S \circ \Theta$ is SAGBI basis of $K[G \circ \Theta]$ with respect to $>$. □

The proof of the Main Theorem 3.1 is an immediate consequence of Lemma 3.2 and Lemma 3.5.

4. Examples

In this section we illustrate the use of the main theorem by several examples (see [7] for examples of compositions Θ such that $LM_>(\Theta)$ is a permuted powering). The main theorem can be immediately applied to the case of finitely generated monomial subalgebras and symmetric subalgebra.

Example 4.1. *Let G be set of monomials. Let Θ be such that $LM_>(\Theta)$ is a permuted powering. Then $G \circ \Theta$ is a SAGBI basis with respect to $>$. This follows immediately from the main theorem and the fact that G is already a SAGBI basis of $K[G]$ under every monomial ordering, in particular under $>_\Theta$.*

For example, let $G = \{x^3y, xy^2\}$ and $\Theta = (x^2 - 2xy + 7, 2y^5 - y^2 + 1)$. Let $>$ be the lexicographical ordering with $x > y$. Then clearly $LM_>(\Theta) = (x^2, y^5)$ is permuted powering. Thus we conclude that

$$G \circ \Theta = \{(x^2 - 2xy + 7)^3(2y^5 - y^2 + 1), (x^2 - 2xy + 7)(2y^5 - y^2 + 1)^2\}$$

is a SAGBI basis with respect to $>$.

The case of the symmetric algebra is similar because the elementary symmetric polynomials form a SAGBI basis with respect to all monomial orderings ([11]). Therefore for a set G of elementary symmetric polynomials in $K[x_1, \ldots, x_n]$ and for any Θ such that $LM_>(\Theta)$ is a permuted powering, $G \circ \Theta$ is a SAGBI basis with respect to $>$.

Now we give more examples illustrating the use of the main theorem of this paper.

Example 4.2. *Considered ring $\mathbb{Q}[x, y, z]$. Let*
$G = \{x^2z, y^2, xy + y, 2xy^2 + y^3\}$
$\Theta = ((x^3 + y + z)^3, (x + y^2 + z)^3, (x + y + z)^3)$

We would like to compute the SAGBI basis of $G \circ \Theta$ with respect to w-degree reverse lexicographical monomial ordering with $w = (2, 3, 6)$, where $x > y > z$. Following the notation of [10], *the monomial ordering responds to the matrix M*

$$\begin{pmatrix} 2 & 3 & 6 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

First we check that $LM_>(\Theta) = (x^9, y^6, z^3)$ is a permuted powering. Thus the main theorem of this paper applies. Next we determine the monomial ordering $>_\Theta$ by multiplying the matrix M with the exponent matrix of $LM_>(\Theta)$

$$\begin{pmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

obtaining a matrix for $>_\Theta$.

$$\begin{pmatrix} 18 & 18 & 18 \\ 0 & 0 & -3 \\ 0 & -6 & 0 \end{pmatrix}.$$

This corresponds to the degree reverse lexicographical ordering. Thus we compute the SAGBI basis S of G with respect to the degree reverse lexicographical ordering, obtaining

$$S = \{x^2z, y^2, xy + y, y^3 + 2xy^2, 2xy^2 + y^2\}.$$

From this we obtain a SAGBI basis of $G \circ \Theta$ with respect to the w-degree reverse lexicographical ordering, simply by composing of S with Θ.

The next examples show that the above illustration of the main theorem can be used to compute SAGBI basis efficiently. We have performed experiments in the computer algebra system SINGULAR ([4]). CPU time was evaluated on an Intel Xeon X7560 2.3 GHz system with 96 GB memory under Ubuntu Server 10.04 . In this regard we took Example 4.2 as the first example. Further we have the following examples, in which we consider the ring $\mathbb{Q}[x, y, z]$.

Example 4.3.
$G = \{x^2, y^2, xy + y, 2xy^2 + y^9\}$
$\Theta = ((x^2 + y), (y^2 + z), (x + z^2))$
$>$ is a degree reverse lexicographical ordering.

Example 4.4.
$G = \{x^2, x^4 + x^5 + x^6, x^7, y^2, y^3 + x^8\}$
$\Theta = ((x^2 + yz)^2, (y^2 + xz)^2, (x + z^2)^2)$
$>$ *is a degree reverse lexicographical ordering.*

Example 4.5.
$G = \{x^2z^2, y^2z^2, xyz^2 + yz, 2xy^2z^3 + y^7z^7\}$
$\Theta = ((x + z^3)^3, (x^2 + y)^2, (y^2 + z))$
$>$ *is a degree reverse lexicographical ordering.*

Example 4.6.
$G = \{x^2y^4, y^4z^2, xy^4z + y^2z, 2xy^6z^2 + y^{10}z^5\}$
$\Theta = ((z + x^2), (y), (x + y^3))$
$>$ *is a lexicographical ordering such that* $z > x > y$.

In Examples 4.3, and 4.4 $>$, we obviously see that $LM_>(\Theta)$ is a permuted powering, thus the main theorem of this paper applies. Next one easily finds that $>_\Theta$ is same as $>$. (In general, whenever $Mat(LM_>(\Theta))$ is a diagonal matrix with the same diagonal entries, we have $>_\Theta = >$). Hence first we compute a SAGBI basis S of G with respect to $>_\Theta$ and compose it by Θ obtaining a SAGBI basis of $S \circ \Theta$ with respect to $>$.

In Example 4.5, we have $LM_>(\Theta) = (z^9, x^4, y^2)$, therefore the $LM_>(\Theta)$ is a permuted powering. By the same process as in example 4.1 we obtain the matrix of $>_\Theta$

$$\begin{pmatrix} 4 & 2 & 9 \\ 0 & -2 & 0 \\ -4 & 0 & 0 \end{pmatrix}.$$

In Example 4.6, we have $LM_>(\Theta) = (z, y, x)$, therefore the $LM_>(\Theta)$ is a permuted powering. It is easy to see that $>_\Theta$ is also a lexicographical ordering such that $x > z > y$.

Table 1 compares the the time (in seconds) for the computation of the SAGBI basis of $G \circ \Theta$ using main theorem and the current implementations of SAGBI basis algorithm in SINGULAR.

4.1. Conclusion. The examples show that it is very efficient to use the procedure based on our main theorem to compute a SAGBI basis $K[G \circ \Theta]$ in case of Θ is a permuted powering. It shows that composition problem deserves more careful consideration. One may ask about the generalization of the composition problem to

Example	comp	SAGBI
4.2	0	47
4.3	27	157
4.4	18	311
4.5	142	too expensive
4.6	12	85

TABLE 1

the reduced case. The conditions under which composition commutes with *reduced* Gröbner basis computation is discussed in [5]. Nevertheless, it still is an open problem as to when composition commutes with *reduced* SAGBI basis computation.

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