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ABSORBING ELEMENTS IN LATTICE MODULES

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Abstract. In this paper we introduce and investigate 2-absorbing, n-absorbing, (n,k)-absorbing, weakly 2-absorbing, weakly n-absorbing and weakly (n,k)-absorbing elements in a lattice module M. Some characterizations of 2-absorbing and weakly 2-absorbing elements of M are obtained. By counter example it is shown that a weakly 2-absorbing element of M need not be 2-absorbing. Finally we show that if $N \in M$ is a 2-absorbing element, then $rad(N)$ is a 2-absorbing element of M.

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1. Introduction

The concept of 2-absorbing and weakly 2-absorbing ideals in commutative rings was introduced by A. Badawi in [4] and A. Badawi et. al. in [5] respectively as a generalization of prime and weakly prime ideals. D. F. Anderson et. al. in [3] generalized the concept of 2-absorbing ideals to n-absorbing ideals. A. Y. Darani et. al. in [8] generalized the concept of 2-absorbing and weakly 2-absorbing ideals to submodules of a module over a commutative ring. Further this concept was extended to n-absorbing submodules by A. Y. Darani et. al. in [9]. In multiplicative lattices, the study of 2-absorbing elements and weakly 2-absorbing elements was done by C. Jayaram et. al. in [11] while the study of n-absorbing elements and weakly n-absorbing elements was done by S. Ballal et. al. in [6]. Our aim is to extend the notion of absorbing elements in a multiplicative lattice to a notion of absorbing elements in lattice modules and study its properties.

A *multiplicative lattice* L is a complete lattice provided with commutative, associative and join distributive multiplication in which the largest element 1 acts as a multiplicative identity. An element $e \in L$ is called *meet principal* if $a \wedge be = ((a : e) \wedge b)e$ for all $a, b \in L$. An element $e \in L$ is called *join principal* if $(ae \vee b) : e = (b : e) \vee a$ for all $a, b \in L$. An element $e \in L$ is called *principal* if e is both meet principal and join principal. An element $a \in L$ is called *compact* if for $X \subseteq L$, $a \leqslant \vee X$ implies the existence of a finite number of elements $a_1, a_2, \cdots, a_n$ in X such that $a \leqslant a_1 \vee a_2 \vee \cdots \vee a_n$. The set of compact elements of L will be denoted by L_*. If each element of L is a join of compact elements of L, then L is called a *compactly generated lattice* or simply a *CG-lattice*. L is said to be a *principally generated lattice* or simply a *PG-lattice* if each element of L is the join of principal elements of L. Throughout this paper L denotes a compactly generated multiplicative lattice with 1 compact in which every finite product of compact elements is compact.

An element $a \in L$ is said to be *proper* if $a < 1$. A proper element $p \in L$ is called *a prime element* if $ab \leqslant p$ implies $a \leqslant p$ or $b \leqslant p$ where $a, b \in L$ and is called *a primary element* if $ab \leqslant p$ implies $a \leqslant p$ or $b^n \leqslant p$ for some $n \in \mathbb{Z}_+$ where $a, b \in L_*$. A proper element $p \in L$ is called *a weakly prime element* if $0 \neq ab \leqslant p$ implies $a \leqslant p$ or $b \leqslant p$ where $a, b \in L$ and is called *a weakly primary element* if $0 \neq ab \leqslant p$ implies $a \leqslant p$ or $b^n \leqslant p$ for some $n \in \mathbb{Z}_+$ where $a, b \in L_*$. A proper element $q \in L$ is called *p-primary* if q is primary and $p = \sqrt{q}$ is prime. A proper element $q \in L$ is called *p-weakly primary* if q is weakly primary and $p = \sqrt{q}$ is weakly prime. For $a, b \in L$, $(a : b) = \vee\{x \in L \mid xb \leqslant a\}$. The radical of $a \in L$ is denoted by $\sqrt{a}$ and is defined as $\vee\{x \in L_* \mid x^n \leqslant a,\ for\ some\ n \in \mathbb{Z}_+\}$. An element $a \in L$ is said to be *nilpotent* if $a^n = 0$ for some $n \in \mathbb{Z}_+$. A multiplicative lattice L is said to be *a reduced lattice* if $0 \in L$ is the only nilpotent element of L. The reader is referred to [2] for general background and terminology in multiplicative lattices.

Let M be a complete lattice and L be a multiplicative lattice. Then M is called *L-module* or *module* over L if there is a multiplication between elements of L and M written as aB where $a \in L$ and $B \in M$ which satisfies the following properties: (1) $(\underset{\alpha}{\vee} a_\alpha)A = \underset{\alpha}{\vee} a_\alpha\ A$, (2) $a(\underset{\alpha}{\vee} A_\alpha) = \underset{\alpha}{\vee} a\ A_\alpha$, (3) $(ab)A = a(bA)$, (4) $1A = A$, (5) $0A = O_M$, for all $a, a_\alpha, b \in L$ and $A, A_\alpha \in M$ where 1 is the supremum of L and 0 is the infimum of L. We denote by O_M and I_M for the least element and the greatest element of M, respectively. Elements of L will generally be denoted by $a, b, c, \cdots$ and elements of M will generally be denoted by $A, B, C, \cdots$.

Let M be an L-module. For $N \in M$ and $a \in L$, $(N : a) = \vee\{X \in M \mid aX \leqslant N\}$. For $A, B \in M$, $(A : B) = \vee\{x \in L \mid xB \leqslant A\}$. An L-module M is called *a multiplication lattice module* if for every element $N \in M$ there exists an element $a \in L$ such that $N = aI_M$. An element $N \in M$ is said to be *proper* if $N < I_M$. A proper element $N \in M$ is said to be *prime* if for $a \in L$ and $X \in M$; $aX \leqslant N$ implies $X \leqslant N$ or $a \leqslant (N : I_M)$. A proper element $N \in M$ is said to be *weakly prime* if for $a \in L$ and $X \in M$; $O_M \neq aX \leqslant N$ implies $X \leqslant N$ or $a \leqslant (N : I_M)$.

If $N \in M$ is a prime element, then $(N : I_M)$ is a prime element in L. An element $N < I_M$ in M is said to be *primary* if for $a \in L$ and $X \in M$; $aX \leqslant N$ implies $X \leqslant N$ or $a^n \leqslant (N : I_M)$ for some $n \in \mathbb{Z}_+$. An element $N < I_M$ in M is said to be *weakly primary* if for $a \in L$ and $X \in M$; $O_M \neq aX \leqslant N$ implies $X \leqslant N$ or $a^n \leqslant (N : I_M)$ for some $n \in \mathbb{Z}_+$. A proper element $N \in M$ is said to be *p-primary* if N is primary and $p = \sqrt{N : I_M}$ is prime. A proper element $N \in M$ is said to be *p-weakly primary* if N is weakly primary and $p = \sqrt{N : I_M}$ is weakly prime. An element $N \in M$ is called a *radical element* if $(N : I_M) = \sqrt{(N : I_M)}$ where $\sqrt{(N : I_M)} = \vee\{x \in L_* \mid x^n \leqslant (N : I_M),\ for\ some\ n \in \mathbb{Z}_+\} = \vee\{x \in L_* \mid x^n I_M \leqslant N,\ for\ some\ n \in \mathbb{Z}_+\}$. An element $N \in M$ is called *meet principal* if $(b \wedge (B : N))N = bN \wedge B$ for all $b \in L, B \in M$. An element $N \in M$ is called *join principal* if $b \vee (B : N) = ((bN \vee B) : N)$ for all $b \in L, B \in M$. An element $N \in M$ is said to be *principal* if N is both meet principal and join principal. An element $N \in M$ is called *compact* if $N \leqslant \underset{\alpha}{\vee} A_\alpha$ implies $N \leqslant A_{\alpha_1} \vee A_{\alpha_2} \vee \cdots \vee A_{\alpha_n}$ for some finite subset $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. The set of compact elements of M is denoted by M_*. If each element of M is a join of compact elements of M, then M is called a *CG-lattice L-module*. If $(O_M : I_M) = 0$, then M is called a *faithful L-module*. M is said to be a *PG-lattice L-module* if each element of M is the join of principal elements of M. For all the definitions in a lattice module and some other definitions, one can refer [7].

According to [11], a proper element $q \in L$ is said to be a 2-*absorbing* element if for every $a, b, c \in L$; $abc \leqslant q$ implies either $ab \leqslant q$ or $bc \leqslant q$ or $ca \leqslant q$ and a proper element $q \in L$ is said to be a *weakly* 2-*absorbing* element if for every $a, b, c \in L$; $0 \neq abc \leqslant q$ implies either $ab \leqslant q$ or $bc \leqslant q$ or $ca \leqslant q$. Obviously a prime element of L is a 1-absorbing element and a weakly prime element of L is a weakly 1-absorbing element. According to [6], a proper element of $q \in L$ is said to be a *n-absorbing* element if for every $a_1, a_2, \cdots, a_n, a_{n+1} \in L$; $a_1 a_2 \cdots a_n a_{n+1} \leqslant q$ implies there are n of $a_i's$ whose product is less than or equal to q, that is, $\widehat{a}_i \leqslant q$ for some i $(1 \leqslant i \leqslant (n+1))$ where $\widehat{a}_i$ is the element $a_1 \cdots a_{i-1} a_{i+1} \cdots a_n a_{n+1}$ and a proper element of $q \in L$ is said to be a *weakly n-absorbing* element if for every $a_1, a_2, \cdots, a_n, a_{n+1} \in L$; $0 \neq a_1 a_2 \cdots a_n a_{n+1} \leqslant q$ implies there are n of $a_i's$ whose product is less than or equal to q. In this paper we introduce and investigate 2-absorbing, n-absorbing, (n, k)-absorbing, weakly 2-absorbing, weakly n-absorbing and weakly (n, k)-absorbing elements in a lattice module M. We give characterization for 2-absorbing and weakly 2-absorbing elements of M. By counter example we show that a weakly 2-absorbing element of M need not be 2-absorbing. We establish a condition for a weakly 2-absorbing element of M to be a 2-absorbing element. Finally we show if $N \in M$ is a 2-absorbing element then $rad(N)$ is a 2-absorbing element of M.

This paper is motivated by [8] and [9]. Many of the results obtained in this paper are versions of results in [8] and [9]. It should be mentioned that there is a significant difference between our results and the already existing ones presented in [8] and [9], as principal elements of M are used in these proofs. We have generalized the important results of a multiplication module over a commutative ring obtained in [10] to a multiplication lattice module M over a multiplicative lattice L, using the principal elements so as to establish the results of $rad(N)$.

2. Absorbing elements in M

In this section we introduce and study absorbing elements of an L-module M. We begin with the following definitions.

Definition 2.1. A proper element N of an L-module M is said to be *2-absorbing* if for every $a, b \in L$ and $Q \in M$; $abQ \leqslant N$ implies either $ab \leqslant (N : I_M)$ or $aQ \leqslant N$ or $bQ \leqslant N$.

Obviously a prime element of an L-module M is a 2-absorbing element. Also a prime element of M can be thought of as a 1-absorbing element.

Definition 2.2. A proper element N of an L-module M is said to be *weakly 2-absorbing* if for every $a, b \in L$ and $Q \in M$; $O_M \neq abQ \leqslant N$ implies either $ab \leqslant (N : I_M)$ or $aQ \leqslant N$ or $bQ \leqslant N$.

Definition 2.3. Let $n \in \mathbb{Z}_+$. A proper element N of an L-module M is said to be *n-absorbing* if for every $a_1, a_2, \cdots, a_n \in L$ and $Q \in M$; $a_1 a_2 \cdots a_n Q \leqslant N$ implies either $a_1 a_2 \cdots a_n \leqslant (N : I_M)$ or there are (n-1) of $a_i^{\prime} s$ whose product with Q is less than or equal to N, that is, either $a_1 a_2 \cdots a_n \leqslant (N : I_M)$ or $\widehat{a}_i Q \leqslant N$ for some i $(1 \leqslant i \leqslant n)$ where $\widehat{a}_i$ is the element $a_1 \cdots a_{i-1} a_{i+1} \cdots a_n$.

Definition 2.4. Let $n \in \mathbb{Z}_+$. A proper element N of an L-module M is said to be *weakly n-absorbing* if for every $a_1, a_2, \cdots, a_n \in L$ and $Q \in M$; $O_M \neq a_1 a_2 \cdots a_n Q \leqslant N$ implies either $a_1 a_2 \cdots a_n \leqslant (N : I_M)$ or there are (n-1) of $a_i^{\prime} s$ whose product with Q is less than or equal to N, that is, either $a_1 a_2 \cdots a_n \leqslant (N : I_M)$ or $\widehat{a}_i Q \leqslant N$ for some i $(1 \leqslant i \leqslant n)$ where $\widehat{a}_i$ is the element $a_1 \cdots a_{i-1} a_{i+1} \cdots a_n$.

Definition 2.5. Let $n, k \in \mathbb{Z}_+$ where $n > k$. A proper element N of an L-module M is said to be *(n, k)-absorbing* if for every $a_1, a_2, \cdots, a_n \in L$ and $Q \in M$; $a_1 a_2 \cdots a_n Q \leqslant N$ implies either there are k of the $a_i^{\prime} s$ whose product is less than or equal to $(N : I_M)$ or there are $(k-1)$ of the $a_i's$ whose product with Q is less than or equal to N.

Definition 2.6. Let $n, k \in \mathbb{Z}_+$ where $n > k$. A proper element N of an L-module M is said to be *weakly* (n,k)*-absorbing* if for every $a_1, a_2, \cdots, a_n \in L$ and $Q \in M$; $O_M \neq a_1 a_2 \cdots a_n Q \leqslant N$ implies either there are k of the $a_i's$ whose product is less than or equal to $(N : I_M)$ or there are $(k-1)$ of the $a_i's$ whose product with Q is less than or equal to N.

Now we give the characterization of a 2-absorbing element of M.

Theorem 2.7. *Let M be a CG-lattice L-module and N be a proper element M. Then the following statements are equivalent:*

(1) *N is a 2-absorbing element of M.*
(2) *for every $a, b \in L$ and $Q \in M$ such that $N \leqslant Q$; $abQ \leqslant N$ implies either $ab \leqslant (N : I_M)$ or $aQ \leqslant N$ or $bQ \leqslant N$.*
(3) *for every $a, b \in L$ such that $ab \nleqslant (N : I_M)$; either $(N : ab) = (N : a)$ or $(N : ab) = (N : b)$.*
(4) *for any elements $r, s \in L_*, K \in M_*$; if $rsK \leqslant N$ then either $rs \leqslant (N : I_M)$ or $rK \leqslant N$ or $sK \leqslant N$.*

Proof. (1) $\Rightarrow$ (2) It is obvious.

(2) $\Rightarrow$ (3) Suppose (2) holds. Let $K \in M$ be such that $K \leqslant (N : ab)$ and $ab \nleqslant (N : I_M)$ for $a, b \in L$. Then $abK \leqslant N$. Clearly $ab(K \vee N) = (abK) \vee (abN) \leqslant N$. Let $U = K \vee N$. Now as $N \leqslant U$, $abU \leqslant N$ and $ab \nleqslant (N : I_M)$; by (2) it follows that either $aU \leqslant N$ or $bU \leqslant N$ which implies either $aK \leqslant N$ or $bK \leqslant N$ and so either $K \leqslant (N : a)$ or $K \leqslant (N : b)$. Hence we have either $(N : ab) \leqslant (N : a)$ or $(N : ab) \leqslant (N : b)$. Obviously $(N : a) \leqslant (N : ab)$ and $(N : b) \leqslant (N : ab)$. Thus either $(N : ab) = (N : a)$ or $(N : ab) = (N : b)$.

(3) $\Rightarrow$ (4) Suppose (3) holds. Let $rsK \leqslant N$ and $rs \nleqslant (N : I_M)$ for $r, s \in L_*$, $K \in M_*$. Then by (3) we have either $(N : rs) = (N : r)$ or $(N : rs) = (N : s)$. So as $K \leqslant (N : rs)$ we have either $K \leqslant (N : r)$ or $K \leqslant (N : s)$. Thus either $rK \leqslant N$ or $sK \leqslant N$.

(4) $\Rightarrow$ (1) Suppose (4) holds. Let $abX \leqslant N$, $aX \nleqslant N$ and $bX \nleqslant N$ for $a, b \in L, X \in M$. As L and M are compactly generated, there exist $r, s \in L_*$ and $Y, Y' \in M_*$ such that $r \leqslant a, s \leqslant b$, $Y \leqslant X$, $Y' \leqslant X$, $rY' \nleqslant N$ and $sY' \nleqslant N$. Then $rs \leqslant ab$. Now $r, s \in L_*, (Y \vee Y') \in M_*$ such that $rs(Y \vee Y') \leqslant abX \leqslant N, r(Y \vee Y') \nleqslant N$ and $s(Y \vee Y') \nleqslant N$. So by (4) $rs \leqslant (N : I_M)$ which implies $ab \leqslant (N : I_M)$. Therefore N is a 2-absorbing element of M. $\square$

A similar characterization of a weakly 2-absorbing element of M is as follows.

Theorem 2.8. *Let M be a CG-lattice L-module and N be a proper element of M. Then the following statements are equivalent:*

1. *N is a weakly 2-absorbing element of M.*
2. *For every $a, b \in L$ and $Q \in M$ such that $N \leqslant Q$; $O_M \neq abQ \leqslant N$ implies either $ab \leqslant (N : I_M)$ or $aQ \leqslant N$ or $bQ \leqslant N$.*
3. *For every $a, b \in L$ such that $ab \nleqslant (N : I_M)$; either $(N : ab) = (O_M : ab)$ or $(N : ab) = (N : a)$ or $(N : ab) = (N : b)$.*
4. *For every $r, s \in L_*, K \in M_*$; if $O_M \neq rsK \leqslant N$ then either $rs \leqslant (N : I_M)$ or $rK \leqslant N$ or $sK \leqslant N$.*

Proof. (1) $\Rightarrow$ (2) It is obvious.

(2) $\Rightarrow$ (3) Suppose (2) holds. Let $K \in M$ be such that $K \leqslant (N : ab)$ and $ab \nleqslant (N : I_M)$ for $a, b \in L$. Then $abK \leqslant N$. If $abK = O_M$, then $K \leqslant (O_M : ab)$. If $abK \neq O_M$, then $O_M \neq ab(K \vee N) = (abK) \vee (abN) \leqslant N$. Let $U = K \vee N$. Now as $N \leqslant U$, $O_M \neq abU \leqslant N$ and $ab \nleqslant (N : I_M)$; by (2) it follows that either $aU \leqslant N$ or $bU \leqslant N$ which implies either $aK \leqslant N$ or $bK \leqslant N$ and so either $K \leqslant (N : a)$ or $K \leqslant (N : b)$. Hence we have either $(N : ab) \leqslant (O_M : ab)$ or $(N : ab) \leqslant (N : a)$ or $(N : ab) \leqslant (N : b)$. Obviously $(O_M : ab) \leqslant (N : ab)$, $(N : a) \leqslant (N : ab)$ and $(N : b) \leqslant (N : ab)$. Thus either $(N : ab) = (O_M : ab)$ or $(N : ab) = (N : a)$ or $(N : ab) = (N : b)$.

(3) $\Rightarrow$ (4) Suppose (3) holds. Let $O_M \neq rsK \leqslant N$ and $rs \nleqslant (N : I_M)$ for $r, s \in L_*$, $K \in M_*$. Then by (3) we have either $(N : rs) = (N : r)$ or $(N : rs) = (N : s)$ or $(N : rs) = (O_M : rs)$. Since $K \leqslant (N : rs)$ it follows that either $K \leqslant (O_M : rs)$ or $K \leqslant (N : r)$ or $K \leqslant (N : s)$. As $K \leqslant (O_M : rs)$ gives $rsK = O_M$, a contradiction, we must have either $K \leqslant (N : r)$ or $K \leqslant (N : s)$ which implies either $rK \leqslant N$ or $sK \leqslant N$.

(4) $\Rightarrow$ (1) Suppose (4) holds. Let $O_M \neq abX \leqslant N$, $aX \nleqslant N$ and $bX \nleqslant N$ for $a, b \in L, X \in M$. As L and M are compactly generated, there exist $r, s \in L_*$ and $Y, Y' \in M_*$ such that $r \leqslant a, s \leqslant b$, $Y \leqslant X$, $Y' \leqslant X$, $rY' \nleqslant N$, $sY' \nleqslant N$ and $O_M \neq rsY'$. Then $rs \leqslant ab$. Now $r, s \in L_*, (Y \vee Y') \in M_*$ such that $O_M \neq rs(Y \vee Y') \leqslant abX \leqslant N, r(Y \vee Y') \nleqslant N$ and $s(Y \vee Y') \nleqslant N$. So by (4) $rs \leqslant (N : I_M)$ which implies $ab \leqslant (N : I_M)$. Therefore N is a weakly 2-absorbing element of M. □

In the next theorem, we show that the meet and join of a family of ascending chain of 2-absorbing elements of M are again 2-absorbing.

Theorem 2.9. *Let $\{N_i \mid i \in \mathbb{Z}_+\}$ be a (ascending or descending) chain of 2-absorbing elements of an L-module M. Then*

(1) $\bigwedge_{i\in\mathbb{Z}_+} N_i$ *is a 2-absorbing element of M.*

(2) $\bigvee_{i\in\mathbb{Z}_+} N_i$ *is a 2-absorbing element of M if I_M is compact.*

Proof. Let $N_1 \leqslant N_2 \leqslant \cdots \leqslant N_i \leqslant \cdots$ be an ascending chain of 2-absorbing elements of M.

(1) Clearly, $(\bigwedge_{i\in\mathbb{Z}_+} N_i) \neq I_M$. Let $abQ \leqslant (\bigwedge_{i\in\mathbb{Z}_+} N_i)$ and $aQ \nleqslant (\bigwedge_{i\in\mathbb{Z}_+} N_i)$ for $a, b \in L$, $Q \in M$. Then $aQ \nleqslant N_j$ for some $j \in \mathbb{Z}_+$ but $abQ \leqslant N_j$ which implies $ab \leqslant (N_j : I_M)$ or $bQ \leqslant N_j$ as N_j is a 2-absorbing element. Now let $N_i \neq N_j$. Then as $\{N_i\}$ is a chain we have either $N_i < N_j$ or $N_j < N_i$. If $N_i < N_j$ then as N_i is a 2-absorbing element, $abQ \leqslant N_i$ and $aQ \nleqslant N_i$ we have either $ab \leqslant (N_i : I_M)$ or $bQ \leqslant N_i$. If $N_j < N_i$ then either $ab \leqslant (N_j : I_M) \leqslant (N_i : I_M)$ or $bQ \leqslant N_j < N_i$. Thus either $ab \leqslant \bigwedge_{i\in\mathbb{Z}_+} (N_i : I_M) = [(\bigwedge_{i\in\mathbb{Z}_+} N_i) : I_M]$ or $bQ \leqslant \bigwedge_{i\in\mathbb{Z}_+} N_i$ which proves that $\bigwedge_{i\in\mathbb{Z}_+} N_i$ is a 2-absorbing element of M.

(2) Since I_M is compact, $(\bigvee_{i\in\mathbb{Z}_+} N_i) \neq I_M$. Let $abQ \leqslant (\bigvee_{i\in\mathbb{Z}_+} N_i)$ and $aQ \nleqslant (\bigvee_{i\in\mathbb{Z}_+} N_i)$ for $a, b \in L$, $Q \in M$. Then as $\{N_i\}$ is a chain we have $abQ \leqslant N_j$ for some $j \in \mathbb{Z}_+$ but $aQ \nleqslant N_j$ which implies either $abI_M \leqslant N_j \leqslant (\bigvee_{i\in\mathbb{Z}_+} N_i)$ or $bQ \leqslant N_j \leqslant (\bigvee_{i\in\mathbb{Z}_+} N_i)$ as N_j is a 2-absorbing element and thus $\bigvee_{i\in\mathbb{Z}_+} N_i$ is a 2-absorbing element of M. □

The "weakly" version of above Theorem 2.9 is as follows.

Theorem 2.10. *Let $\{N_i \mid i \in \mathbb{Z}_+\}$ be a (ascending or descending) chain of weakly 2-absorbing elements of an L-module M. Then*

(1) $\bigwedge_{i\in\mathbb{Z}_+} N_i$ *is a weakly 2-absorbing element of M.*

(2) $\bigvee_{i\in\mathbb{Z}_+} N_i$ *is a weakly 2-absorbing element of M if I_M is compact.*

Proof. The proof is similar to the proof of Theorem 2.9 and hence omitted. □

Theorem 2.11. *If a proper element N of an L-module M is a 2-absorbing element then $(N : d)$ is a 2-absorbing element of M for every $d \in L$.*

Proof. Let $d, a, b \in L$ and $Q \in M$. Assume that $abQ \leqslant (N : d)$, $aQ \nleqslant (N : d)$ and $bQ \nleqslant (N : d)$. As $ab(dQ) \leqslant N$, $a(dQ) \nleqslant N$, $b(dQ) \nleqslant N$ and $N \in M$ is 2 absorbing we get $abI_M \leqslant N$ which implies $d(abI_M) \leqslant N$. It follows that $ab \leqslant ((N : d) : I_M)$ and hence $(N : d)$ is a 2-absorbing element of M. □

The following theorem shows that if an element in M (or L) is 2-absorbing then its corresponding element in L (or M) is also 2-absorbing.

Theorem 2.12. *Let M be a faithful multiplication PG-lattice L-module with I_M compact where L is also a PG-lattice. Then the following statements are equivalent:*

(1) *N is a 2-absorbing element of M.*

(2) *$(N : I_M)$ is a 2-absorbing element of L.*

(3) *$N = qI_M$ for some 2-absorbing element $q \in L$.*

Proof. (1) $\Rightarrow$ (2) Assume that N is a 2-absorbing element of M. Let $abc \leqslant (N : I_M)$ such that $ab \nleqslant (N : I_M)$ and $bc \nleqslant (N : I_M)$ for $a, b, c \in L$. Then as $ac(bI_M) \leqslant N$, $a(bI_M) \nleqslant N$, $c(bI_M) \nleqslant N$ and N is a 2-absorbing element we have $ac \leqslant (N : I_M)$ which implies $(N : I_M)$ is a 2-absorbing element of L.

(2) $\Rightarrow$ (1) Assume that $(N : I_M)$ is a 2-absorbing element of L. Let $abQ \leqslant N$ for $a, b \in L$, $Q \in M$. Since M is a multiplication lattice L-module, $Q = qI_M$ for some $q \in L$. Then as $abq \leqslant (N : I_M)$ and $(N : I_M)$ is a 2-absorbing element we have either $ab \leqslant (N : I_M)$ or $bq \leqslant (N : I_M)$ or $aq \leqslant (N : I_M)$ which implies either $ab \leqslant (N : I_M)$ or $bQ \leqslant N$ or $aQ \leqslant N$ and hence $N \in M$ is a 2-absorbing element.

(2) $\Rightarrow$ (3) Assume that $(N : I_M)$ is a 2-absorbing element of L. Then obviously (3) holds since in a multiplication lattice L-module M we have $N = (N : I_M)I_M$.

(3) $\Rightarrow$ (2) Assume that $N = qI_M$ for some 2-absorbing element $q \in L$. Also $N = (N : I_M)I_M$ since M is a multiplication lattice L-module. It follows that $qI_M = (N : I_M)I_M$. As I_M is compact, (2) holds by Theorem 5 of [7]. $\square$

In view of above Theorem 2.12 we give the following corollary without proof.

Corollary 2.13. *If a proper element N of an L-module M is 2-absorbing, then $(N : I_M)$ is a 2-absorbing element of L. The converse holds if M is a multiplication lattice L-module.*

The above Corollary 2.13 is true for "weakly" version provided M is faithful as shown below.

Theorem 2.14. *If a proper element N of a faithful L-module M is weakly 2-absorbing, then $(N : I_M)$ is a weakly 2-absorbing element of L. The converse holds if M is a multiplication lattice L-module.*

Proof. Assume that N is a weakly 2-absorbing element of M. Let $0 \neq abc \leqslant (N : I_M)$ such that $ab \nleqslant (N : I_M)$ and $bc \nleqslant (N : I_M)$ for $a, b, c \in L$. If $acbI_M = O_M$ then as M is faithful we have $abc \leqslant (O_M : I_M) = 0$; a contradiction. Now as N is a weakly 2-absorbing element with $O_M \neq ac(bI_M) \leqslant N$, $a(bI_M) \nleqslant N$ and $c(bI_M) \nleqslant N$ we have $ac \leqslant (N : I_M)$ which implies $(N : I_M)$ is a 2-absorbing element of L. Conversely assume that $(N : I_M)$ is a weakly 2-absorbing element of

L and M is a multiplication lattice L-module. Let $O_M \neq abQ \leqslant N$ for $a, b \in L$, $Q \in M$. Since M is a multiplication lattice L-module, $Q = qI_M$ for some $q \in L$. If $abq = 0$, then $abQ = O_M$, a contradiction. Now as $0 \neq abq \leqslant (N : I_M)$ and since $(N : I_M)$ is a weakly 2-absorbing element we have either $ab \leqslant (N : I_M)$ or $bq \leqslant (N : I_M)$ or $aq \leqslant (N : I_M)$ which implies either $ab \leqslant (N : I_M)$ or $bQ \leqslant N$ or $aQ \leqslant N$ and hence N is a 2-absorbing element of M. □

Result similar to Theorem 2.12 for a weakly 2-absorbing element of M is as follows.

Theorem 2.15. *Let M be a faithful multiplication PG-lattice L-module with I_M compact where L is also PG-lattice. Then the following statements are equivalent:*

(1) *N is a weakly 2-absorbing element of M.*
(2) *$(N : I_M)$ is a weakly 2-absorbing element of L.*
(3) *$N = qI_M$ for some weakly 2-absorbing element $q \in L$.*

Proof. The proof is similar to the proof of Theorem 2.12 and hence omitted. □

Thus a proper element N of a multiplication lattice L-module M is a 2-absorbing element if and only if $(N : I_M)$ is a 2-absorbing element of L and a proper element N of a faithful multiplication lattice L-module M is a weakly 2-absorbing element if and only if $(N : I_M)$ is a weakly 2-absorbing element of L.

Theorem 2.16. *If a proper element N of an L-module M is prime, then N is a $(2, 1)$-absorbing element. The converse holds if M is a multiplication lattice L-module.*

Proof. Assume that $N \in M$ is prime. Let $abQ \leqslant N$ for $a, b \in L$, $Q \in M$. Then as N is prime we have either $a \leqslant (N : I_M)$ or $b \leqslant (N : I_M)$ or $Q \leqslant N$ and we are done. Conversely assume that $N \in M$ is $(2, 1)$-absorbing. Let $aQ \leqslant N$ for $a \in L$, $Q \in M$. Since M is a multiplication lattice L-module, $Q = qI_M$ for some $q \in L$. Then as $a(qI_M) \leqslant N$ and N is $(2, 1)$-absorbing we have either $a \leqslant (N : I_M)$ or $q \leqslant (N : I_M)$ which implies either $a \leqslant (N : I_M)$ or $Q = qI_M \leqslant N$ and hence N is prime. □

Theorem 2.17. *If a proper element N of an L-module M is weakly prime, then N is a weakly $(2, 1)$-absorbing element. The converse holds if M is a multiplication lattice L-module.*

Proof. The proof is similar to the proof of Theorem 2.16 and hence omitted. □

Theorem 2.18. *If a proper element N of an L-module M is 2-absorbing, then N is a $(3,2)$-absorbing element. The converse holds if M is a multiplication lattice L-module.*

Proof. Assume that $N \in M$ is 2-absorbing. Let $abcQ \leqslant N$ for $a,b,c \in L$, $Q \in M$. Then by repeated use of the fact that N is 2-absorbing we get either $ab \leqslant (N : I_M)$ or $[a(cQ) \leqslant N]$ or $[b(cQ) \leqslant N]$ which implies either $ab \leqslant (N : I_M)$ or $[ac \leqslant (N : I_M)$ or $aQ \leqslant N$ or $cQ \leqslant N]$ or $[bc \leqslant (N : I_M)$ or $bQ \leqslant N]$. It follows that N is $(3,2)$-absorbing. Conversely assume that N is a $(3,2)$-absorbing element of a multiplication lattice L-module M. Let $abQ \leqslant N$ for $a,b \in L$, $Q \in M$. Since M is a multiplication lattice L-module, $Q = qI_M$ for some $q \in L$. Then as $ab(qI_M) \leqslant N$ and N is $(3,2)$-absorbing we have either $[abI_M \leqslant N$ or $bqI_M \leqslant N$ or $aqI_M \leqslant N]$ or $[aI_M \leqslant N$ or $bI_M \leqslant N$ or $qI_M \leqslant N]$ which implies either $[abI_M \leqslant N$ or $bqI_M \leqslant N$ or $aqI_M \leqslant N]$ or $[baI_M \leqslant N$ or $qbI_M \leqslant N$ or $aqI_M \leqslant N]$. It follows that either $ab \leqslant (N : I_M)$ or $bQ \leqslant N$ or $aQ \leqslant N$ and hence N is 2-absorbing. □

Theorem 2.19. *If a proper element N of an L-module M is weakly 2-absorbing, then N is a weakly $(3,2)$-absorbing element. The converse holds if M is a multiplication lattice L-module.*

Proof. The proof is similar to the proof of Theorem 2.18 and hence omitted. □

Theorem 2.20. *Let N be a proper element of an L-module M and $n,k \in \mathbb{Z}_+$ such that $n > k$.*

1. *If N is (n,k)-absorbing, then N is $(k+1,k)$-absorbing.*
2. *If N is (n,k)-absorbing, then N is (n,k')-absorbing for every positive integer $k' > k$.*

Proof. (1) Assume that $N \in M$ is (n,k)-absorbing. Let $a_1a_2\cdots a_nQ \leqslant N$ where $a_1,a_2,\cdots,a_n \in L$, $Q \in M$. Since N is (n,k)-absorbing it follows that either the product of any k of the $a_i's$ is less than or equal to $(N : I_M)$ or there are $(k-1)$ of the $a_i's$ whose product with Q is less than or equal to N and hence N is $(k+1,k)$-absorbing.

(2) Assume that $N \in M$ is (n,k)-absorbing. Let $k' \in \mathbb{Z}_+$ such that $k' > k$. Let $a_1a_2\cdots a_nQ \leqslant N$ where $a_1,a_2,\cdots,a_n \in L$, $Q \in M$. Since N is (n,k)-absorbing, we have either $b_1b_2\cdots b_k \leqslant (N : I_M)$ or $c_1c_2\cdots c_{k-1}Q \leqslant N$ where these $b_i's$ and $c_i's$ are some of the $a_i's$ obtained on renaming. It follows that either $bb_1b_2\cdots b_k \leqslant (N : I_M)$ for any element b among $a_i's$ but other than $b_i's$ or $cc_1c_2\cdots c_{k-1}Q \leqslant N$ for any element c among $a_i's$ but other than $c_i's$ and hence continuing the same argument we get N is (n,k')-absorbing. □

Theorem 2.21. *Let N be a proper element of an L-module M and $n, k \in \mathbb{Z}_+$ such that $n > k$.*

(1) *If N is weakly (n, k)-absorbing, then N is weakly $(k+1, k)$-absorbing.*
(2) *If N is weakly (n, k)-absorbing, then N is weakly (n, k')-absorbing for every positive integer $k' > k$.*

Proof. The proof is similar to the proof of Theorem 2.20 and hence omitted. $\square$

Corollary 2.13 for an n-absorbing element of an L-module M is as follows.

Theorem 2.22. *Let $n \in \mathbb{Z}_+$. If a proper element N of an L-module M is n-absorbing, then $(N : I_M)$ is an n-absorbing element of L. The converse holds if M is a multiplication lattice L-module.*

Proof. Let N be an n-absorbing element of M and let $\widehat{a}_i = a_1 \cdots a_{i-1}a_{i+1} \cdots a_n$ where i $(1 \leqslant i \leqslant n)$ and $a_1, \cdots, a_n \in L$. Assume that $a_1 \cdots a_n a_{n+1} \leqslant (N : I_M)$ and $\widehat{a}_i a_{n+1} \nleqslant (N : I_M)$ for every i $(1 \leqslant i \leqslant n)$. Then as N is n-absorbing, $a_1 \cdots a_n(a_{n+1}I_M) \leqslant N$ and $\widehat{a}_i a_{n+1} I_M \nleqslant N$ we have $a_1 \cdots a_n \leqslant (N : I_M)$ which implies $(N : I_M)$ is an n-absorbing element of L. Conversely assume that $(N : I_M)$ is an n-absorbing element of L and M is a multiplication lattice L-module. Let $a_1 \cdots a_n Q \leqslant N$ for $a_1, \cdots, a_n \in L$, $Q \in M$. Since M is a multiplication lattice L-module, $Q = qI_M$ for some $q \in L$. Then as $a_1 \cdots a_n q \leqslant (N : I_M)$ and since $(N : I_M)$ is an n-absorbing element we have either $a_1 \cdots a_n \leqslant (N : I_M)$ or there exist $(n-1)$ of $a_i's$ whose product with q is less than or equal to $(N : I_M)$ which implies either $a_1 \cdots a_n \leqslant (N : I_M)$ or there exist $(n-1)$ of $a_i's$ whose product with $qI_M = Q$ is less than or equal to N and hence N is an n-absorbing element of M. $\square$

Lemma 2.23. *Let $m, n \in \mathbb{Z}_+$. If a proper element N of an L-module M is n-absorbing then N is an m-absorbing element of M for all $m > n$.*

Proof. Let $m, n \in \mathbb{Z}_+$ be such that $m > n$. Let $x_1 \cdots x_m Q = x_1 \cdots x_n(x_{n+1} \cdots x_m Q) \leqslant N$ for $x_1, \cdots, x_m \in L$, $Q \in M$. Then as N is n-absorbing, we have either $x_1 \cdots x_n \leqslant (N : I_M)$ or $x_1 \cdots x_{i-1}x_{i+1} \cdots x_n(x_{n+1} \cdots x_m Q) \leqslant N$ for some i $(1 \leqslant i \leqslant n)$ which implies either $x_1 \cdots x_n \cdots x_m \leqslant (N : I_M)$ or $(x_1 \cdots x_{i-1}x_{i+1} \cdots x_n x_{n+1} \cdots x_m)Q \leqslant N$ for some i $(1 \leqslant i \leqslant m)$ and thus N is an m-absorbing element of M. $\square$

In view of above Lemma 2.23, we have the following definition.

Definition 2.24. If a proper element N is an n-absorbing element of M for some $n \in \mathbb{Z}_+$, then we define $\omega(N) = \min\{n \in \mathbb{Z}_+ \mid N \text{ is an } n\text{-absorbing element of } M\}$ otherwise we write $\omega(N) = \infty$. Moreover we define $\omega(I_M) = 0$.

Thus for any element $N \in M$ we have $\omega(N) \in \mathbb{Z}_+ \cup \{0, \infty\}$ with $\omega(N) = 1$ if and only if N is a prime element of M and $\omega(N) = 0$ if and only if $N = I_M$. So $\omega(N)$ measures in some sense how far $'N'$ is from being a prime element of M.

Theorem 2.25. *If a proper element N of an L-module M is p-primary such that $p^n I_M \leqslant N$ where $n \in \mathbb{Z}_+$, then N is an n-absorbing element of M. Moreover, $\omega(N) \leqslant n$.*

Proof. Let $a_1 \cdots a_n Q \leqslant N$ with $\widehat{a}_i Q \nleqslant N$ for every i $(1 \leqslant i \leqslant n)$ where $\widehat{a}_i$ is the element $a_1 \cdots a_{i-1} a_{i+1} \cdots a_n$ and $a_1, \cdots, a_n \in L$, $Q \in M$. As N is p-primary, $a_i(\widehat{a}_i Q) \leqslant N$ and $\widehat{a}_i Q \nleqslant N$, we have $a_i \leqslant \sqrt{N : I_M} = p$ for every i $(1 \leqslant i \leqslant n)$ which implies $a_1 \cdots a_n \leqslant p^n$. It follows that $a_1 \cdots a_n \leqslant (N : I_M)$ and thus N is an n-absorbing element of M. The "moreover" statement is clear. □

Corollary 2.26. *Let a proper element N of an L-module M be p-primary. Then N is 2-absorbing if and only if $p^2 I_M \leqslant N$.*

Proof. Let a p-primary element $N \in M$ be 2-absorbing. Then by Corollary 2.13 $(N : I_M)$ is a 2-absorbing element of L which implies $(\sqrt{N : I_M})^2 \leqslant (N : I_M)$ by Lemma 2(iii) of [11] and thus $p^2 I_M \leqslant N$. The converse part is clear by Theorem 2.25. □

We define a classical prime element of an L-module M as follows.

Definition 2.27. A proper element $N \in M$ is said to be *classical prime* if for each element $K \in M$ and elements $a, b \in L$; $abK \leqslant N$ implies either $aK \leqslant N$ or $bK \leqslant N$.

Theorem 2.28. *Let N be a proper element of an L-module M. Then N is prime implies N is classical prime implies N is 2-absorbing implies N is weakly 2-absorbing.*

Proof. Assume that $N \in M$ is prime. Let $abK \leqslant N$ for $a, b \in L$, $K \in M$. Then as N is prime we have either $a \leqslant (N : I_M) \leqslant (N : K)$ or $bK \leqslant N$ which implies either $aK \leqslant N$ or $bK \leqslant N$ and thus N is classical prime. Now let N be classical prime and let $abK \leqslant N$ for $a, b \in L$, $K \in M$. Then as N is classical prime we have either $aK \leqslant N$ or $bK \leqslant N$ and thus N is 2-absorbing. Last implication is obvious since every 2-absorbing element is weakly 2-absorbing. □

From the above Theorem 2.28, it is clear that every 2-absorbing element is weakly 2-absorbing. But the converse is not true as shown in the following example.

Example 2.29. *Let $R = \mathbb{Z}$ and $M = \mathbb{Z}/(30\mathbb{Z})$. Then M is a module over $\mathbb{Z}$. Suppose that $L(R)$ is the set of all ideals of R and $L(M)$ is the set of all submodules of M. Then $L(M)$ is a lattice module over $L(R)$. Obviously $N = \{30\mathbb{Z}\}$ being the zero element of $L(M)$ is weakly 2-absorbing. However N is not a 2-absorbing element of $L(M)$ since $(2)(3)(5+30\mathbb{Z}) \subseteq N$ and $(2)(3) \nsubseteq (N : M)$, $(2)(5+30\mathbb{Z}) \nsubseteq N$, $(3)(5+30\mathbb{Z}) \nsubseteq N$.*

The following theorem shows that under particular condition a weakly 2-absorbing element of an L-module M is 2-absorbing.

Theorem 2.30. *If a weakly 2-absorbing element N of an L-module M is such that $(N : I_M)^2 N \neq O_M$, then N is a 2-absorbing element.*

Proof. Assume that $(N : I_M)^2 N \neq O_M$. Let $abQ \leqslant N$ for $a, b \in L$, $Q \in M$. If $abQ \neq O_M$, then as N is weakly 2-absorbing we get either $ab \leqslant (N : I_M)$ or $aQ \leqslant N$ or $bQ \leqslant N$ and we are done. So let $abQ = O_M$. First assume that $abN \neq O_M$. Then $abN_0 \neq O_M$ for some $N_0 \leqslant N$ in M. As $O_M \neq ab(Q \vee N_0) \leqslant N$ and N is weakly 2-absorbing we have either $ab \leqslant (N : I_M)$ or $a(Q \vee N_0) \leqslant N$ or $b(Q \vee N_0) \leqslant N$ which implies either $ab \leqslant (N : I_M)$ or $aQ \leqslant N$ or $bQ \leqslant N$ and we are done. Hence we may assume that $abN = O_M$. If $a(N : I_M)Q \neq O_M$, then $ar_0Q \neq O_M$ for some $r_0 \leqslant (N : I_M)$ in L. Since $O_M \neq ar_0Q \leqslant a(b \vee r_0)Q \leqslant N$ and N is weakly 2-absorbing we have either $a(b \vee r_0) \leqslant (N : I_M)$ or $aQ \leqslant N$ or $(b \vee r_0)Q \leqslant N$ which implies either $ab \leqslant (N : I_M)$ or $aQ \leqslant N$ or $bQ \leqslant N$ and we are done. So we can assume that $a(N : I_M)Q = O_M$. Likewise we can assume that $b(N : I_M)Q = O_M$. As $(N : I_M)^2 N \neq O_M$, there exist $a_0, b_0 \leqslant (N : I_M)$ and $X_0 \leqslant N$ with $a_0b_0X_0 \neq O_M$. If $ab_0X_0 \neq O_M$ then $O_M \neq ab_0X_0 \leqslant a(b \vee b_0)(Q \vee X_0) \leqslant N$. As N is weakly 2-absorbing we get either $a(b \vee b_0) \leqslant (N : I_M)$ or $a(Q \vee X_0) \leqslant N$ or $(b \vee b_0)(Q \vee X_0) \leqslant N$ which implies either $ab \leqslant (N : I_M)$ or $aQ \leqslant N$ or $bQ \leqslant N$ and we are done. So we can assume that $ab_0X_0 = O_M$. Likewise we can assume that $a_0b_0Q = O_M$ and $a_0bX_0 = O_M$. Then as $O_M \neq a_0b_0X_0 \leqslant (a \vee a_0)(b \vee b_0)(Q \vee X_0) \leqslant N$ and N is weakly 2-absorbing we get either $(a \vee a_0)(b \vee b_0) \leqslant (N : I_M)$ or $(a \vee a_0)(Q \vee X_0) \leqslant N$ or $(b \vee b_0)(Q \vee X_0) \leqslant N$ which implies either $ab \leqslant (N : I_M)$ or $aQ \leqslant N$ or $bQ \leqslant N$ and thus N is a 2-absorbing element. □

We define a nilpotent element of an L-module M in the following manner.

Definition 2.31. A proper element N of an L-module M is said to be *nilpotent* if $(N : I_M)^k N = O_M$ for some $k \in \mathbb{Z}_+$.

The consequences of Theorem 2.30 are presented in the form of following corollaries.

Corollary 2.32. *If a proper element N of an L-module M is weakly 2-absorbing but not 2-absorbing, then N is a nilpotent element of M.*

Proof. The proof is obvious. □

Corollary 2.33. *If a proper element N of an L-module M is weakly 2-absorbing but not 2-absorbing, then $(N : I_M)^3 N = O_M$.*

Proof. As $(N : I_M)^3 \leqslant (N : I_M)^2$, we have $(N : I_M)^3 N \leqslant (N : I_M)^2 N = O_M$ by Theorem 2.30 and hence $(N : I_M)^3 N = O_M$. □

Corollary 2.34. *If a proper element N of an L-module M is weakly 2-absorbing but not 2-absorbing, then $(N : I_M)^n N = O_M$ for every $n \geqslant 3$.*

Proof. The proof is obvious. □

Corollary 2.35. *If a proper element N of a multiplication lattice L-module M is weakly 2-absorbing but not 2-absorbing, then $(N : I_M)^3 I_M = O_M$.*

Proof. Since M is a multiplication lattice L-module, we have $N = (N : I_M)I_M$. By Theorem 2.30, we have $(N : I_M)^2 N = O_M$ which implies $(N : I_M)^3 I_M = O_M$. □

Corollary 2.36. *If a proper element N of a faithful multiplication lattice L-module M is weakly 2-absorbing but not 2-absorbing, then $(N : I_M) \leqslant \sqrt{0}$ and hence $\sqrt{N : I_M} = \sqrt{0}$. Moreover, if L is a reduced lattice then $(N : I_M) = 0$.*

Proof. The proof is obvious. □

Corollary 2.37. *Let L be a reduced lattice. If $O_M < N < I_M$ is a weakly 2-absorbing element of a faithful multiplication lattice L-module M, then N is a 2-absorbing element of M.*

Proof. The proof is obvious. □

3. $rad(N)$ as a 2-absorbing element of M

In this section, we prove $rad(N)$ is a 2-absorbing element of an L-module M if $N \in M$ is a 2-absorbing element. We begin with defining the radical of an element of a lattice module. In view of the definition of the M-radical of a submodule of an R-module M in [12], the definition of the radical of an element of an L-module M is as follows.

Definition 3.1. Let N be a proper element of an L-module M. Then the radical of N is denoted as $rad(N)$ and is defined as the element $\wedge\{P \in M \mid P \textit{ is a prime}$ element and $N \leqslant P\}$. If $N \nleqslant P$ for any prime $P \in M$, then we write $rad(N) = I_M$.

Before proving $rad(N)$ is a 2-absorbing element of M, we prove the results required to show that $rad(aI_M) = \sqrt{a}I_M$ as proved in an R-module M in [10].

Lemma 3.2. *Let L be a PG-lattice and M be a faithful multiplication PG-lattice L-module. Then* $\underset{\alpha\in\triangle}{\wedge}(a_\alpha I_M) = (\underset{\alpha\in\triangle}{\wedge} a_\alpha)I_M$ *where* $\{a_\alpha \in L \mid \alpha \in \triangle\}$.

Proof. Clearly $(\underset{\alpha\in\triangle}{\wedge} a_\alpha)I_M \leqslant \underset{\alpha\in\triangle}{\wedge}(a_\alpha I_M)$. Let $X \leqslant \underset{\alpha\in\triangle}{\wedge}(a_\alpha I_M)$ where $X \in M$. We may suppose that X is a principal element. Assume that $((\underset{\alpha\in\triangle}{\wedge} a_\alpha)I_M : X) \neq 1$. Then there exists a maximal element $q \in L$ such that $((\underset{\alpha\in\triangle}{\wedge} a_\alpha)I_M : X) \leqslant q$. As M is a multiplication lattice L-module and $q \in L$ is maximal, by Theorem 4 of [7], two cases arise:

Case 1. For principal element $X \in M$, there exists a principal element $r \in L$ with $r \nleqslant q$ such that $rX = O_M$. Then $r \leqslant (O_M : X) \leqslant ((\underset{\alpha\in\triangle}{\wedge} a_\alpha)I_M : X) \leqslant q$ which is a contradiction.

Case 2. There exists a principal element $Y \in M$ and a principal element $b \in L$ with $b \nleqslant q$ such that $bI_M \leqslant Y$. Then $bX \leqslant Y$, $bX \leqslant b[\underset{\alpha\in\triangle}{\wedge}(a_\alpha I_M)] \leqslant \underset{\alpha\in\triangle}{\wedge}(a_\alpha bI_M) \leqslant \underset{\alpha\in\triangle}{\wedge}(a_\alpha Y)$ and $(O_M : Y)bI_M \leqslant (O_M : Y)Y = O_M$ since Y is meet principal. As M is faithful it follows that $b(O_M : Y) = 0$. Since Y is meet principal, $(bX : Y)Y = bX \wedge Y = bX$. Let $s = (bX : Y)$ then $sY = bX \leqslant \underset{\alpha\in\triangle}{\wedge}(a_\alpha Y)$. So $s = (bX : Y) = (sY : Y) \leqslant [\underset{\alpha\in\triangle}{\wedge}(a_\alpha Y) : Y] = \underset{\alpha\in\triangle}{\wedge}(a_\alpha Y : Y) = \underset{\alpha\in\triangle}{\wedge}[a_\alpha \vee (O_M : Y)]$ since Y is join principal. Therefore $bs \leqslant b[\underset{\alpha\in\triangle}{\wedge}[a_\alpha \vee (O_M : Y)]] \leqslant \underset{\alpha\in\triangle}{\wedge}[b[a_\alpha \vee (O_M : Y)]] = \underset{\alpha\in\triangle}{\wedge}[(ba_\alpha) \vee b(O_M : Y)] = \underset{\alpha\in\triangle}{\wedge}(ba_\alpha) \leqslant b \wedge (\underset{\alpha\in\triangle}{\wedge} a_\alpha) \leqslant (\underset{\alpha\in\triangle}{\wedge} a_\alpha)$ and so $b^2X = b(bX) = bsY \leqslant (\underset{\alpha\in\triangle}{\wedge} a_\alpha)Y \leqslant (\underset{\alpha\in\triangle}{\wedge} a_\alpha)I_M$. Hence $b^2 \leqslant ((\underset{\alpha\in\triangle}{\wedge} a_\alpha)I_M : X) \leqslant q$ which implies $b \leqslant \sqrt{q} = q$; a contradiction.

Thus the assumption that $((\underset{\alpha\in\triangle}{\wedge} a_\alpha)I_M : X) \neq 1$ is absurd and so we must have $((\underset{\alpha\in\triangle}{\wedge} a_\alpha)I_M : X) = 1$ which implies $X \leqslant (\underset{\alpha\in\triangle}{\wedge} a_\alpha)I_M$. It follows that $\underset{\alpha\in\triangle}{\wedge}(a_\alpha I_M) \leqslant (\underset{\alpha\in\triangle}{\wedge} a_\alpha)I_M$ and hence $\underset{\alpha\in\triangle}{\wedge}(a_\alpha I_M) = (\underset{\alpha\in\triangle}{\wedge} a_\alpha)I_M$. □

Lemma 3.3. *Let L be a PG-lattice and M be a faithful multiplication PG-lattice L-module with I_M compact. If a proper element $q \in L$ is a prime element, then qI_M is a prime element of M.*

Proof. As I_M is compact and $q \in L$ is proper by Theorem 5 of [7] we have $qI_M \neq I_M$. Let $aX \leqslant qI_M$ and $a \nleqslant (qI_M : I_M)$ for $a \in L, X \in M$. Then $a \nleqslant q$. We may suppose that X is a principal element. Assume that $((qI_M) : X) \neq 1$. Then there exists a maximal element $m \in L$ such that $((qI_M) : X) \leqslant m$. As M is a multiplication lattice L-module and $m \in L$ is maximal, by Theorem 4 of [7], two cases arise:

Case 1. For principal element $X \in M$, there exists a principal element $r \in L$ with $r \nleqslant m$ such that $rX = O_M$. Then $r \leqslant (O_M : X) \leqslant ((qI_M) : X) \leqslant m$ which is a contradiction.

Case 2. There exists a principal element $Y \in M$ and a principal element $b \in L$ with $b \nleqslant m$ such that $bI_M \leqslant Y$. Then $bX \leqslant Y$, $baX \leqslant bqI_M = q(bI_M) \leqslant qY$ and $(O_M : Y)bI_M \leqslant (O_M : Y)Y = O_M$ since Y is meet principal. As M is faithful it follows that $b(O_M : Y) = 0$. Since Y is meet principal, $(bX : Y)Y = bX$. Let $s = (bX : Y)$ then $sY = bX$ and so $asY = abX \leqslant qY$. Since Y is meet principal, $abX = (abX : Y)Y = cY$ where $c = (abX : Y)$. Since $cY = abX \leqslant qY$ and Y is join principal we have $c \vee (O_M : Y) = (cY : Y) \leqslant (qY : Y) = q \vee (O_M : Y)$. So $bc \leqslant bq \leqslant q$. On the other hand since Y is join principal, $c = (abX : Y) = (asY : Y) = as \vee (O_M : Y)$ and so $abs \leqslant abs \vee b(O_M : Y) = b(as \vee (O_M : Y)) = bc \leqslant q$. If $b \leqslant q$, then $b \leqslant q \leqslant ((qI_M) : X) \leqslant m$ which contradicts $b \nleqslant m$ and so $b \nleqslant q$. Now as $abs \leqslant q, a \nleqslant q, b \nleqslant q$ and q is prime, we have $s \leqslant q$. Hence $bX = sY \leqslant qY \leqslant (qI_M)$ which implies $b \leqslant ((qI_M) : X) \leqslant m$; a contradiction.

Thus the assumption that $((qI_M) : X) \neq 1$ is absurd and so we must have $((qI_M) : X) = 1$ which implies $X \leqslant (qI_M)$. Therefore qI_M is a prime element of M. □

Lemma 3.4. *In an L-module M, if a proper element $Q \in M$ is prime such that $X \leqslant Q$, then $(Q : I_M) \in L$ is prime such that $\sqrt{X : I_M} \leqslant (Q : I_M)$ where $X \in M$ is a proper element.*

Proof. Obviously, $(Q : I_M) \in L$ is prime by Proposition 3.6 of [1]. Further, if $a \leqslant \sqrt{X : I_M}$, then $a^n \leqslant (X : I_M) \leqslant (Q : I_M)$ for some $n \in \mathbb{Z}_+$ which implies $a \leqslant (Q : I_M)$ and so $\sqrt{X : I_M} \leqslant (Q : I_M)$. □

Lemma 3.5. *For every proper element N of an L-module M, $(\sqrt{N : I_M})I_M \leqslant rad(N)$.*

Proof. Let $P \in M$ be prime such that $N \leqslant P$. Then by Lemma 3.4, $(P : I_M) \in L$ is prime such that $\sqrt{N : I_M} \leqslant (P : I_M)$ which implies $(\sqrt{N : I_M})I_M \leqslant P$. Thus whenever $P \in M$ is prime such that $N \leqslant P$ we have $(\sqrt{N : I_M})I_M \leqslant P$. It follows that $(\sqrt{N : I_M})I_M \leqslant rad(N)$. □

Theorem 3.6. *Let L be a PG-lattice and M be a faithful multiplication PG-lattice L-module with I_M compact. Then $rad(N) = \sqrt{a}I_M$ for every proper element $N = aI_M$ of M where $a = (N : I_M) \in L$.*

Proof. Let $b = \wedge\{p \in L \mid p \textit{ is a prime element and } a \leqslant p\} = \sqrt{a}$. Then by Lemma 3.2, $bI_M = (\underset{p \textit{ is prime};a \leqslant p}{\wedge} p)I_M = \underset{p \textit{ is prime};a \leqslant p}{\wedge} (pI_M)$. Let $p \in L$ be prime such that $a \leqslant p$. Also as $p \in L$ is a prime element by Lemma 3.3 we have $pI_M \in M$ is a prime element. Then $N = aI_M \leqslant pI_M$ and so $rad(N) \leqslant pI_M$. It follows that $rad(N) \leqslant \underset{p \text{ is prime}; a \leqslant p}{\wedge} (pI_M) = bI_M$ and hence $rad(N) \leqslant \sqrt{a}I_M$. But by Lemma 3.5 we have $\sqrt{a}I_M \leqslant rad(N)$. Therefore $rad(N) = \sqrt{a}I_M$. □

Following corollary is an outcome of Corollary 2.35 and Theorem 3.6.

Corollary 3.7. *Let L be a PG-lattice and M be a faithful multiplication PG-lattice L-module with I_M compact. If a proper element N of M is weakly 2-absorbing but not 2-absorbing, then $N \leqslant rad(O_M)$.*

Proof. As $O_M = (O_M : I_M)I_M = 0I_M$, we have $rad(O_M) = \sqrt{0}I_M$ by Theorem 3.6. By Corollary 2.35, we have $(N : I_M)^3 I_M = O_M$ which implies $(N : I_M)^3 \leqslant (O_M : I_M) = 0$ and hence $(N : I_M) \leqslant \sqrt{0}$. It follows that $N = (N : I_M)I_M \leqslant \sqrt{0}I_M = rad(O_M)$. □

Lemma 3.8. *In a multiplication lattice L-module M, the meet of each pair of distinct prime elements of M is a 2-absorbing element.*

Proof. Let N and K be any two distinct prime elements of M. Let $abQ \leqslant (N \wedge K)$ with $aQ \nleqslant (N \wedge K)$ and $bQ \nleqslant (N \wedge K)$ for $a, b \in L$, $Q \in M$. Since M is a multiplication lattice L-module, $Q = qI_M$ for some $q \in L$. Clearly $aQ \nleqslant N$ and $bQ \nleqslant N$ lead us to a contradiction because N is prime and $a(bQ) \leqslant (N \wedge K) \leqslant N$ gives $aI_M \leqslant N$ which implies $qaI_M = aQ \leqslant N$. Similarly $aQ \nleqslant K$ and $bQ \nleqslant K$ lead us to a contradiction. So assume that $aQ \nleqslant N$ and $bQ \nleqslant K$. Now $a(bQ) \leqslant (N \wedge K) \leqslant K$, $bQ \nleqslant K$, K is prime gives $a \leqslant (K : I_M)$ and $b(aQ) \leqslant (N \wedge K) \leqslant N$, $aQ \nleqslant N$, N is prime gives $b \leqslant (N : I_M)$. Hence $ab \leqslant (a \wedge b) \leqslant [(K : I_M) \wedge (N : I_M)] = [(N \wedge K) : I_M]$ which implies $(N \wedge K)$ is a 2-absorbing element of M. □

Now we are in a position to prove $rad(N)$ is a 2-absorbing element of M which is the main aim of this section.

Theorem 3.9. *Let L be a PG-lattice and M be a faithful multiplication PG-lattice L-module with I_M compact. If a proper element $N \in M$ is a 2-absorbing element, then $rad(N)$ is a 2-absorbing element of M.*

Proof. By Corollary 2.13, $(N : I_M)$ is a 2-absorbing element of L. By Theorem 3 of [11], two cases arise:

Case 1. $\sqrt{N : I_M} = p$ is a prime element of L. Then by Lemma 3.3 and Theorem 3.6, we have $rad(N) = (\sqrt{N : I_M})I_M = pI_M$ is prime and hence $rad(N)$ is a 2-absorbing element of M.

Case 2. $\sqrt{N : I_M} = p_1 \wedge p_2$ where p_1, p_2 are the only distinct prime elements of L that are minimal over $(N : I_M)$. Then by Lemma 3.3, $p_1 I_M$ and $p_2 I_M$ are distinct prime elements of M and are minimal over N. So by Theorem 3.6 and Lemma 3.2, we have $rad(N) = (\sqrt{N : I_M})I_M = (p_1 \wedge p_2)I_M = p_1 I_M \wedge p_2 I_M$. Hence by Lemma 3.8, $rad(N)$ is a 2-absorbing element of M. □

Theorem 3.10. *Let L be a PG-lattice and M be a faithful multiplication PG-lattice L-module with I_M compact. If a proper element $N \in M$ is 2-absorbing, then one of the following statement holds true:*

(1) *$rad(N) = pI_M$ is a prime element of M such that $p^2 I_M \leqslant N$.*
(2) *$rad(N) = p_1 I_M \wedge p_2 I_M$ and $(p_1 p_2)I_M \leqslant N$ where $p_1 I_M$ and $p_2 I_M$ are the only distinct prime elements of M that are minimal over N.*

Proof. By Corollary 2.13, $(N : I_M)$ is a 2-absorbing element of L. Then by Theorem 3 of [11], we have either $\sqrt{N : I_M} = p$ is a prime element of L such that $p^2 \leqslant (N : I_M)$ or $\sqrt{N : I_M} = p_1 \wedge p_2$ and $p_1 p_2 \leqslant (N : I_M)$ where p_1 and p_2 are the only distinct prime elements of L that are minimal over $(N : I_M)$. By Theorem 3.6, Lemma 3.3 and Lemma 3.2, it follows that either $rad(N) = pI_M$ is a prime element of M such that $p^2 I_M \leqslant N$ or $rad(N) = (p_1 \wedge p_2)I_M = p_1 I_M \wedge p_2 I_M$ and $(p_1 p_2)I_M \leqslant N$ where $p_1 I_M$ and $p_2 I_M$ are the only distinct prime elements of M that are minimal over N. □

Note that if N is a 2-absorbing element of a faithful multiplication PG-lattice L-module M with I_M compact, then $(\sqrt{N : I_M})rad(N) \leqslant N \leqslant rad(N)$.

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References

[1] E. A. Al-Khouja, *Maximal elements and prime elements in lattice modules*, Damascus University Journal for BASIC SCIENCES, 19(2) (2003), 9-21.

[2] F. Alarcón, D. D. Anderson and C. Jayaram, *Some results on abstract commutative ideal theory*, Period. Math. Hungar., 30(1) (1995), 1-26.

[3] D. F. Anderson and A. Badawi, *On n-absorbing ideals of commutative rings*, Comm. Algebra, 39(5) (2011), 1646-1672.

[4] A. Badawi, *On 2-absorbing ideals of commutative rings*, Bull. Austral. Math. Soc., 75(3) (2007), 417-429.

[5] A. Badawi and A. Y. Darani, *On weakly 2-absorbing ideals of commutative rings*, Houston J. Math., 39(2) (2013), 441-452.

[6] S. Ballal and V. Kharat, *On generalization of prime, weakly prime and almost prime elements in multiplicative lattices*, Int. J. Algebra, 8(9) (2014), 439-449.

[7] F. Callialp and U. Tekir, *Multiplication lattice modules*, Iran. J. Sci. Technol. Trans. A Sci., 35(4) (2011), 309-313.

[8] A. Y. Darani and F. Soheilnia, 2-*absorbing and weakly* 2-*absorbing submodules*, Thai J. Math., 9(3) (2011), 577-584.

[9] A. Y. Darani and F. Soheilnia, *On* n-*absorbing submodules*, Math. Commun., 17(2) (2012), 547-557.

[10] Z. A. El-Bast and P. F. Smith, *Multiplication modules*, Comm. Algebra, 16(4) (1988), 755-779.

[11] C. Jayaram, U. Tekir and E. Yetkin, 2-*absorbing and weakly* 2-*absorbing elements in multiplicative lattices*, Comm. Algebra, 42(6) (2014), 2338-2353.

[12] R. L. McCasland and M. E. Moore, *On radicals of submodules of finitely generated modules*, Canad. Math. Bull., 29(1) (1986), 37-39.

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