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ON THE GENUS OF THE COMMUTING GRAPHS OF FINITE NON-ABELIAN GROUPS

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Abstract. The commuting graph of a non-abelian group is a simple graph in which the vertices are the non-central elements of the group, and two distinct vertices are adjacent if and only if they commute. In this paper, we determine (up to isomorphism) all finite non-abelian groups whose commuting graphs are acyclic, planar or toroidal. We also derive explicit formulas for the genus of the commuting graphs of some well-known class of finite non-abelian groups, and show that, every collection of isomorphism classes of finite non-abelian groups whose commuting graphs have the same genus is finite.

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1. Introduction

Let G be a non-abelian group and $Z(G)$ be its center. The *commuting graph* of G, denoted by $\Gamma_c(G)$, is a simple undirected graph in which the vertex set is $G \setminus Z(G)$, and two distinct vertices x and y are adjacent if and only if $xy = yx$. This graph is precisely the complement of the non-commuting graph of a group considered in [1] and [16]. The origin of this notion lies in a seminal paper by R. Brauer and K. A. Fowler [7] who were concerned primarily with the classification of the finite simple groups. However, the ever-increasing popularity of the topic is often attributed to a question, posed in 1975 by Paul Erdös and answered affirmatively by B. H. Neumann [18], asking whether or not a non-commuting graph having no infinite complete subgraph possesses a finite bound on the cardinality of its complete subgraphs. In recent years, the commuting graphs of groups have become a topic of research for many mathematicians (see, for example, [4], [13]). In [14], it was conjectured that the commuting graph of a finite group is either disconnected or has diameter bounded above by a constant independent of the group G. This conjecture was well-supported in [19] and [24]. However, in [11], it is shown that, for all positive integers d, there exists a finite special 2-group G such that the commuting graph of

G has diameter greater than d. But in [17], it is proved that for finite groups with trivial center the conjecture made in [14] holds good. The concept of commuting graphs of groups (taking, as the vertices, the non-trivial elements of the group in place of non-central elements) has also been recently used in [20] to show that finite quotients of the multiplicative group of a finite dimensional division algebra are solvable. There is also a ring theoretic version of commuting graphs (see, for example, [2], [3]).

Most of the works cited above on commuting graphs of groups deal with connectedness, diameter and some algebraic aspects of the graph. Also, some of the results for the non-commuting graphs of groups have their obvious analogues for the commuting counterparts, the commuting and non-commuting graphs being complements of each other. In the present paper, however, we deal with a topological aspect, namely, the genus of the commuting graphs of finite non-abelian groups, and on this count the commuting and the non-commuting graphs are independent of each other. Here we show that every collection of isomorphism classes of finite non-abelian groups whose commuting graphs have the same genus is finite. One of the sections in this paper is devoted entirely to the computation of the genus of the commuting graphs of some well-known families of finite non-abelian groups. The primary objective of this paper is, of course, to determine (up to isomorphism) all finite non-abelian groups whose commuting graphs are planar or toroidal, that is, can be drawn on the surface of a sphere or of a torus (without any crossing of edges). We, however, begin by classifying all non-abelian groups whose commuting graphs have no triangles, which, in fact, turns out to be equivalent to saying that the corresponding non-commuting graphs are planar. It may be mentioned here that the motivation for this paper comes from [9], [15], [25], [28] and [29], where similar problems for certain graphs associated to finite rings have been addressed.

2. Some prerequisites

In this section, we recall certain graph theoretic terminologies (see, for example, [26] and [27]) and some well-known results which have been used extensively in the forthcoming sections. Note that all graphs considered in this and the following sections are simple graphs, that is, graphs without loops or multiple edges.

Let Γ be a graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. Let $x, y \in V(\Gamma)$. Then x and y are said to be *adjacent* if $x \neq y$ and there is an edge $x - y$ in $E(\Gamma)$ joining x and y. A path between x and y is a sequence of adjacent vertices, often written as $x - x_1 - x_2 - \cdots - x_n - y$, where the vertices $x, x_1, x_2, \ldots, x_n, y$ are all distinct (except, possibly, x and y). Γ is said to be *connected* if there is a path between every pair of distinct vertices in Γ. A path between x and y is called a *cycle* if $x = y$. The number of edges in a path or a cycle, is called its *length*. A cycle of length n is called an n-cycle, and a 3-cycle is also called a triangle. The *girth* of Γ is the minimum of the lengths of all cycles in Γ, and is denoted by $\mathrm{girth}(\Gamma)$. If Γ is *acyclic*, that is, if Γ has no cycles, then we write $\mathrm{girth}(\Gamma) = \infty$.

A graph G is said to be *complete* if there is an edge between every pair of distinct vertices in G. We denote the complete graph with n vertices by K_n. A *bipartite graph* is the one whose vertex set can be partitioned into two disjoint parts in such a way that the two end vertices of every edge lie in different parts. Among the bipartite graphs, the *complete bipartite graph* is the one in which two distinct vertices are adjacent if and only if they lie in different parts. The complete bipartite graph, with parts of size m and n, is denoted by $K_{m,n}$.

A subset of the vertex set of a graph Γ is called a *clique* of Γ if it consists entirely of pairwise adjacent vertices. The least upper bound of the sizes of all the cliques of G is called the *clique number* of Γ, and is denoted by $\omega(\Gamma)$. The *chromatic number* of a graph Γ, written $\chi(\Gamma)$, is the minimum number of colors needed to label the vertices so that adjacent vertices receive different colors. Clearly, $\omega(\Gamma) \leq \chi(\Gamma)$.

Given a graph Γ, let U be a nonempty subset of $V(\Gamma)$. Then the *induced subgraph* of Γ on U is defined to be the graph $\Gamma[U]$ in which the vertex set is U and the edge set consists precisely of those edges in Γ whose endpoints lie in U. If $\{\Gamma_\alpha\}_{\alpha\in\Lambda}$ is a family of subgraphs of a graph Γ, then the union $\underset{\alpha\in\Lambda}{\cup}\Gamma_\alpha$ denotes the subgraph of Γ whose vertex set is $\underset{\alpha\in\Lambda}{\cup}V(\Gamma_\alpha)$ and the edge set is $\underset{\alpha\in\Lambda}{\cup}E(\Gamma_\alpha)$. Further, given a graph Γ, its complement is defined to be the graph in which the vertex set is the same as the one in Γ and two distinct vertices are adjacent if and only if they are not adjacent vertices in Γ.

The *genus* of a graph Γ, denoted by $\gamma(\Gamma)$, is the smallest non-negative integer n such that the graph can be embedded on the surface obtained by attaching n handles to a sphere. Clearly, if $\tilde{\Gamma}$ is a subgraph of Γ, then $\gamma(\tilde{\Gamma}) \leq \gamma(\Gamma)$. Graphs having genus zero are called *planar* graphs, while those having genus one are called *toroidal* graphs.

A *block* of a graph Γ is a connected subgraph B of Γ that is maximal with respect to the property that removal of a single vertex (and the incident edges) from B does not make it disconnected, that is, the graph $B \setminus \{v\}$ is connected for all $v \in V(B)$. Given a graph Γ, there is a unique finite collection $\mathfrak{B}$ of blocks of Γ, such that $\Gamma = \underset{B\in\mathfrak{B}}{\cup}B$. The collection $\mathfrak{B}$ is called the *block decomposition* of Γ.

In [5, Corollary 1], it has been proved that the genus of a graph is the sum of the genera of its blocks. Thus, it follows that the following.

Lemma 2.1. *If a graph Γ has two disjoint subgraphs Γ_1 and Γ_2 such that $\Gamma_1 \cong K_m$ and $\Gamma_2 \cong K_n$ for some positive integers m and n, then $\gamma(\Gamma) \geq \gamma(K_n) + \gamma(K_m)$.*

We conclude the section with the following two useful results.

Lemma 2.2. [27, Theorem 6-38] *If $n \geq 3$, then*

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil.$$

Lemma 2.3. [27, Theorem 6-37] *If $m, n \geq 2$, then*

$$\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil.$$

3. Some basic results

In this section we derive some results concerning the genus of the commuting graphs of finite groups which are not only of interest in their own right but also used extensively in the forthcoming sections. Our notations are quite standard; for example, given a group G and an element $x \in G$, we write $o(x)$, $C_G(x)$ and $Cl_G(x)$ to denote the order of x in G, the centralizer of x in G and the conjugacy class of x in G, respectively.

In the study of the genus of a graph, the cycles in the graph play a crucial role. Therefore, determining whether or not the graph is acyclic can be considered as the first step in this direction. Even otherwise, whether or not a graph associated to a group has a triangle is a topic of substantial interest (see, for example [10]). Keeping this is mind, we begin the section with the following result which, in view of [1, Proposition 2.3], also says that the commuting graph of a non-abelian group is acyclic if and only if its complement (that is, the non-commuting graph of the group) is planar.

Proposition 3.1. *Let G be a non-abelian group. Then, $\Gamma_c(G)$ has no 3-cycle if and only if G is isomorphic to the symmetric group S_3, the quaternion group Q_8, or the dihedral group D_8.*

Proof. If G is isomorphic to S_3, Q_8 or D_8, then it is easy to see that $\Gamma_c(G)$ has no 3-cycle; in fact, $\Gamma_c(G)$ is acyclic.

Conversely, suppose that $\Gamma_c(G)$ has no 3-cycle. Then, $|Z(G)| \leq 2$; otherwise, for all $x \in G \setminus Z(G)$, any three distinct elements of $xZ(G)$ would form a 3-cycle in $\Gamma_c(G)$.

Case 1. $|Z(G)| = 1$.

In this case, every element of G has order 2 or 3; otherwise $\{x, x^2, x^3\}$ would form a 3-cycle in $\Gamma_c(G)$ for all $x \in G$ with $o(x) > 3$. Therefore, G is a group of exponent dividing 6. Let H be a finitely generated subgroup of G. Then, H is a finite group (see [12, Sections 18.2, 18.4]), and so, we have $|H| = 2^m 3^n$ for some non-negative integers m and n. First assume that $m \geq 2$ and let H_2 be a Sylow 2-subgroup of H. Clearly, H_2 is elementary abelian of order at least 4. Hence, any three distinct elements of $H_2 \setminus \{1\}$ form a 3-cycle in $\Gamma_c(G)$. Therefore, we must have $m \leq 1$. Next assume that $n \geq 2$ and let H_3 be a Sylow 3-subgroup of H. Clearly, $|H_3| \geq 9$. If H_3 is abelian, then any three distinct elements of $H_3 \setminus \{1\}$ form a 3-cycle in $\Gamma_c(G)$. If H_3 is non-abelian, then, choosing $x \in H_3 \setminus Z(H_3)$ and noting that $|Z(H_3)| \geq 3$, we see that any three distinct elements of $xZ(H_3)$ form a 3-cycle in $\Gamma_c(G)$. Therefore, we must have $n \leq 1$. Thus, every finitely generated subgroup of G is of order at most 6. It follows that G itself is of order not exceeding 6. Since G non-abelian, we have $G \cong S_3$.

Case 2. $|Z(G)| = 2$.

In this case, $G/Z(G)$ is an elementary abelian 2-group; otherwise, for all $x \in G \setminus Z(G)$ with $x^2 \notin Z(G)$, any three distinct elements of $xZ(G) \sqcup x^2 Z(G)$ would form a 3-cycle in $\Gamma_c(G)$. It follows that every element of G is of order 2 or 4. Since G is non-abelian, there is an element $x \in G$ of order 4, and so, we have $Z(G) = \{1, x^2\}$. It is easy to see that $C_G(x) = \langle x \rangle$; otherwise $\{x, x^3, w\}$ would form a 3-cycle in $\Gamma_c(G)$ for all $w \in C_G(x) \setminus \langle x \rangle$. Thus, $|C_G(x)| = 4$. Let $z \in Cl_G(x) \setminus \{x\}$. Then, $1 \neq zx^{-1} \in G' \subseteq Z(G)$. Therefore, we have $zx^{-1} = x^2$, and so, $z = x^3$. Thus, $Cl_G(x) = \{x, x^3\}$, and so, $|G : C_G(x)| = |Cl_G(x)| = 2$. It follows that $|G| = 8$. Since G is non-abelian, we have $G \cong Q_8$ or D_8 . This completes the proof. □

It follows, in particular, from the above result that the girth of the commuting graph of a non-abelian group is 3 or ∞. Our next result is used not only in this section but also in the forthcoming sections.

Lemma 3.2. *Let G be a finite non-abelian group whose commuting graph has genus g, where g is a non-negative integer. Then the following assertions hold:*

(a) *If $\emptyset \neq S \subseteq G \setminus Z(G)$ such that $xy = yx$ for all $x, y \in S$, then $|S| \leq \lfloor \frac{7+\sqrt{1+48g}}{2} \rfloor$.*
(b) $|Z(G)| \leq \frac{1}{t-1} \lfloor \frac{7+\sqrt{1+48g}}{2} \rfloor$, *where $t = \max\{o(xZ(G)) \mid xZ(G) \in G/Z(G)\}$.*
(c) *If A is an abelian subgroup of G, then $|A| \leq \lfloor \frac{7+\sqrt{1+48g}}{2} \rfloor + |A \cap Z(G)|$.*

Proof. Consider the induced subgraph $\Gamma_c(G)[S] \cong K_{|S|}$. If $g = 0$, then $\gamma(K_{|S|}) = \gamma(\Gamma_c(G)[S]) \leq \gamma(\Gamma_c(G)) = 0$, and so, it follows that $|S| \leq 4$. On the other hand, if $g > 0$, then, by Heawood's formula [26, Theorem 6.3.25], we have $|S| = \omega(\Gamma_c(G)[S]) \leq \omega(\Gamma_c(G)) \leq \chi(\Gamma_c(G)) \leq \lfloor \frac{7+\sqrt{1+48g}}{2} \rfloor$. This proves (a). The remaining two assertions follow from (a); in fact, for (b) we take $S = \bigsqcup_{i=1}^{t-1} y^i Z(G)$, where $y \in G \setminus Z(G)$ such that $o(yZ(G)) = t$, and for (c) we simply note that $A = (A \setminus Z(G)) \cup (A \cap Z(G))$. □

Our main result of this section says that every collection of isomorphism classes of finite non-abelian groups whose commuting graphs have the same genus is finite.

Theorem 3.3. *The order of a finite non-abelian group is bounded by a function of the genus of its commuting graph. Consequently, given a non-negative integer g, there are at the most finitely many (up to isomorphism) finite non-abelian groups whose commuting graphs have genus g.*

Proof. Let G be a finite non-abelian group whose commuting graph has genus g. Let us put $h = \lfloor \frac{7+\sqrt{1+48g}}{2} \rfloor$. Then, by Lemma 3.2(a), we have $|Z(G)| \leq h$. Let p be a prime divisor of $|G|$, and P be a Sylow p-subgroup of G with $|P| = p^n$, where n is a positive integer. If $P \subset Z(G)$, then $|P| \leq h$. So, let $P \nsubseteq Z(G)$. If P is abelian, then, by Lemma 3.2(a), we have $|P \setminus Z(G)| \leq h$, and hence, $|P| \leq 2h$. So, we assume that P is non-abelian. Then, $|Z(P)| = p^c$ for some positive integer $c < n$, and, by [8, Section I, Para 4], P has an abelian subgroup A of order p^v, where v is a positive integer such that $v \geq -\frac{1}{2} + \sqrt{2n + c^2 - c + \frac{1}{4}}$; in particular, $n < (2v+1)^2$. By Lemma 3.2(c), we have $p^v = |A| \leq 2h$; in particular, $v < 2h$ and $p < 2h$. Hence, it follows that $|P| = p^n < (2h)^{(4h+1)^2}$. Since the number of primes less than $2h$ is at most h, we have $|G| < (2h)^{h(4h+1)^2}$. This completes the proof. □

Recall that a group is said to be an *AC-group* if the centralizer of each of its non-central elements is abelian. The *AC*-groups have been extensively studied by many authors (see, for example, [1], [21], [23]). Our final result of this section deals with finite non-abelian *AC*-groups.

Proposition 3.4. *Let G be a finite non-abelian AC-group. Then*

$$\gamma(\Gamma_c(G)) = \sum_{X \in \mathcal{P}} \gamma(K_{|X|}),$$

where $\mathcal{P} = \{C_G(u) \setminus Z(G) \mid u \in G \setminus Z(G)\}$.

Proof. Let $X \in \mathcal{P}$. Then, $X = C_G(u) \setminus Z(G)$ for some $u \in G$. Clearly, for $x, y \in X$, we have $xy = yx$, since $C_G(u)$ is abelian. On the other hand, for $x \in X$ and $y \in G \setminus Z(G)$ with $xy = yx$, we have $y \in X$, since $y, u \in C_G(x)$ and $C_G(x)$ is abelian. It follows that the induced subgraph $\Gamma_c(G)[X] \cong K_{|X|}$ is a block of $\Gamma_c(G)$, and, since $G \setminus Z(G) = \underset{X \in \mathcal{P}}{\cup} X$, the collection $\{\Gamma_c(G)[X] \mid X \in \mathcal{P}\}$ is the block decomposition of $\Gamma_c(G)$. Therefore, by [5, Corollary 1], we have $\gamma(\Gamma_c(G)) = \sum\limits_{X \in \mathcal{P}} \gamma(K_{|X|})$. □

Remark 3.5. *If G is a finite non-abelian AC-group and A is a finite abelian group, then $A \times G$ is also a finite non-abelian AC-group with $C_{A\times G}(a, u) \setminus Z(A \times G) = A \times (C_G(u) \setminus Z(G))$ for all $(a, u) \in (A \times G) \setminus Z(A \times G)$. Therefore, it follows from Proposition 3.4 that*

$$\gamma(\Gamma_c(A \times G)) = \sum_{X \in \mathcal{P}} \gamma(K_{|A||X|}),$$

where $\mathcal{P} = \{C_G(u) \setminus Z(G) \mid u \in G \setminus Z(G)\}$.

We close this section with a couple of immediate corollaries to Proposition 3.4.

Corollary 3.6. *The genus of the commuting graph of a non-abelian group G of order pq, where p and q are primes with $p \mid q - 1$, is given by*

$$\gamma(\Gamma_c(G)) = \gamma(K_{q-1}) + q\gamma(K_{p-1}).$$

Proof. Note that G is an AC-group with $|Z(G)| = 1$, in which the centralizers of the non-central elements are precisely the Sylow subgroups of G, and so, the result follows from Proposition 3.4. □

Corollary 3.7. *The genus of the commuting graph of a non-abelian group G of order p^3, where p is a prime, is given by*

$$\gamma(\Gamma_c(G)) = (p + 1)\gamma(K_{p(p-1)}).$$

Proof. Note that G is an AC-group with $|Z(G)| = p$, in which the centralizers of the non-central elements are of order p^2. Since any two distinct centralizers of the non-central elements of G intersect at $Z(G)$, it follows that the number of such centralizers is $p + 1$. Hence, the result follows from Proposition 3.4. □

4. Genus of the commuting graphs of some well-known AC-groups

In this section, we determine the genus of the commuting graphs of some well-known finite non-abelian AC-groups. Some of the results obtained here play crucial role in the study of planarity and toroidality of the commuting graphs of finite non-abelian groups.

Proposition 4.1. *The genus of the commuting graph of the dihedral group* $D_{2n} = \langle x, y \mid y^n = x^2 = 1, xyx^{-1} = y^{-1}\rangle$, *where* $n \geq 3$, *is given by*

$$\gamma(\Gamma_c(D_{2n})) = \begin{cases} \gamma(K_{n-2}) & \text{if } n \text{ is even,} \\ \gamma(K_{n-1}) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Note that D_{2n} is a non-abelian AC-group. If n is even, then $Z(D_{2n}) = \{1, y^{\frac{n}{2}}\}$, $C_{D_{2n}}(y^i) = \langle y\rangle$ for $1 \leq i \leq n-1$ ($i \neq \frac{n}{2}$), and $C_{D_{2n}}(xy^j) = \{1, xy^j, y^{\frac{n}{2}}, xy^{j+\frac{n}{2}}\}$ for $0 \leq j \leq n-1$. If n is odd, then $Z(D_{2n}) = \{1\}$, $C_{D_{2n}}(y^i) = \langle y\rangle$ for $1 \leq i \leq n-1$, and $C_{D_{2n}}(xy^j) = \{1, xy^j\}$ for $0 \leq j \leq n-1$. Thus, if n is even, the distinct centralizers of the non-central elements in D_{2n} are $\langle y\rangle$ and $\{1, xy^j, y^{\frac{n}{2}}, xy^{j+\frac{n}{2}}\}$, where $0 \leq j \leq \frac{n}{2} - 1$, and so, by Proposition 3.4, we have $\gamma(\Gamma_c(D_{2n})) = \gamma(K_{n-2}) + \frac{n}{2}\gamma(K_2) = \gamma(K_{n-2})$. On the other hand, if n is odd, the distinct centralizers in D_{2n} are $\langle y\rangle$ and $\{1, xy^j\}$, where $0 \leq j \leq n-1$, and so, by Proposition 3.4, we have $\gamma(\Gamma_c(D_{2n})) = \gamma(K_{n-1}) + n\gamma(K_1) = \gamma(K_{n-1})$. □

Proposition 4.2. *The genus of the commuting graph of the dicyclic group or the generalized quaternion group* $Q_{4n} = \langle x, y \mid y^{2n} = 1, x^2 = y^n, xyx^{-1} = y^{-1}\rangle$, *where* $n \geq 2$, *is given by*

$$\gamma(\Gamma_c(Q_{4n})) = \gamma(K_{2(n-1)}).$$

Proof. It is well-known that Q_{4n} is a non-abelian AC-group with $Z(Q_{4n}) = \{1, y^n\}$, $C_{Q_{4n}}(y^i) = \langle y\rangle$ for $1 \leq i \leq 2n-1$ ($i \neq n$), and $C_{Q_{4n}}(xy^j) = \{1, xy^j, y^n, xy^{j+n}\}$ for $0 \leq j \leq 2n-1$. Therefore, the distinct centralizers of the non-central elements in Q_{4n} are $\langle y\rangle$ and $\{1, xy^j, y^n, xy^{j+n}\}$, where $0 \leq j \leq n-1$, and so, by Proposition 3.4, we have $\gamma(\Gamma_c(Q_{4n})) = \gamma(K_{2(n-1)}) + n\gamma(K_2) = \gamma(K_{2(n-1)})$. □

Proposition 4.3. *The genus of the commuting graph of the semidihedral group* $SD_{2^n} = \langle r, s \mid r^{2^{n-1}} = s^2 = 1, srs = r^{2^{n-2}-1}\rangle$, *where* $n \geq 4$, *is given by*

$$\gamma(\Gamma_c(SD_{2^n})) = \gamma(K_{2^{n-1}-2}).$$

Proof. Note that SD_{2^n} is a non-abelian AC-group with $Z(SD_{2^n}) = \{1, r^{2^{n-2}}\}$, $C_{SD_{2^n}}(r^i) = \langle r\rangle$, for $1 \leq i \leq 2^{n-1} - 1$ ($i \neq 2^{n-2}$), and $C_{SD_{2^n}}(sr^j) = \{1, sr^j, r^{2^{n-2}}, sr^{j+2^{n-2}}\}$ for $0 \leq j \leq 2^{n-1} - 1$. Therefore, the distinct centralizers of the non-central elements in SD_{2^n} are $\langle r\rangle$ and $\{1, sr^j, r^{2^{n-2}}, sr^{j+2^{n-2}}\}$, where $0 \leq j \leq 2^{n-2} - 1$, and so, by Proposition 3.4, we have $\gamma(\Gamma_c(SD_{2^n})) = \gamma(K_{2^{n-1}-2}) + 2^{n-2}\gamma(K_2) = \gamma(K_{2^{n-1}-2})$. □

Proposition 4.4. *The genus of the commuting graph of the projective special linear group $PSL(2,2^k)$, where $k \geq 2$, is given by*

$$\gamma(\Gamma_c(PSL(2,2^k))) = (2^k+1)\gamma(K_{2^k-1})+2^{k-1}(2^k+1)\gamma(K_{2^k-2})+2^{k-1}(2^k-1)\gamma(K_{2^k}).$$

Proof. It is well-known that $PSL(2,2^k)$ is a non-abelian group of order $2^k(2^{2k}-1)$ with $Z(PSL(2,2^k)) = \{1\}$. Moreover, in view of [1, Proposition 3.21], the following assertions hold for $PSL(2,2^k)$:

(a) $PSL(2,2^k)$ has an elementary abelian 2-subgroup P of order 2^k such that the number of conjugates of P in $PSL(2,2^k)$ is 2^k+1.

(b) $PSL(2,2^k)$ has a cyclic subgroup A of order 2^k-1 such that the number of conjugates of A in $PSL(2,2^k)$ is $2^{k-1}(2^k+1)$.

(c) $PSL(2,2^k)$ has a cyclic subgroup B of order 2^k+1 such that the number of conjugates of B in $PSL(2,2^k)$ is $2^{k-1}(2^k-1)$.

(d) The centralizers of the non-trivial elements of $PSL(2,2^k)$ constitute precisely the family $\{xPx^{-1}, xAx^{-1}, xBx^{-1} \mid x \in G\}$; in particular, $PSL(2,2^k)$ is an AC-group.

Hence, the result follows from Proposition 3.4. □

Proposition 4.5. *The genus of the commuting graph of the general linear group $GL(2,q)$, where $q = p^n > 2$ (p is a prime), is given by*

$$\gamma(\Gamma_c(GL(2,q))) = \frac{q(q+1)}{2}\gamma(K_{(q-1)(q-2)})+\frac{q(q-1)}{2}\gamma(K_{q(q-1)})+(q+1)\gamma(K_{(q-1)^2}).$$

Proof. Note that $GL(2,q)$ is a non-abelian AC-group (see [1, Lemma 3.5]) with $|GL(2,q)| = (q^2-1)(q^2-q)$ and $|Z(GL(2,q))| = q-1$. Also, in view of [1, Proposition 3.26], the centralizers of the non-central elements of $GL(2,q)$ are precisely the members of the family $\{xDx^{-1}, xIx^{-1}, xPZ(GL(2,q))x^{-1} \mid x \in G\}$, where

(a) D is the subgroup of $GL(2,q)$ consisting of all diagonal matrices, $|D| = (q-1)^2$, and the number of conjugates of D in $GL(2,q)$ is $\frac{q(q+1)}{2}$,

(b) I is a cyclic subgroup of $GL(2,q)$, $|I| = q^2-1$, and the number of conjugates of I in $GL(2,q)$ is $\frac{q(q-1)}{2}$,

(c) P is the Sylow p-subgroup of $GL(2,q)$ consisting of all upper triangular matrices with 1 in the diagonal, $|PZ(GL(2,q))| = q(q-1)$, and the number of conjugates of $PZ(GL(2,q))$ in $GL(2,q)$ is $q+1$.

Hence, the result follows from Proposition 3.4. □

In view of Remark 3.5 and the results obtained in this section, one can easily compute the genus of the commuting graph of the group $A \times G$, where A is a finite abelian group and G is any one of the groups considered in the Propositions 4.1 to 4.5.

5. Finite non-abelian groups whose commuting graphs are planar

In this section, we characterize all finite non-abelian groups whose commuting graphs are planar. However, we begin the section with a lemma containing a couple of elementary properties of finite 2-groups.

Lemma 5.1. *Let G be a finite 2-group. Then, the following assertions hold:*

(a) *If $|G| \geq 16$, then G contains an abelian subgroup of order 8.*
(b) *If $|G| \geq 32$ and $|Z(G)| \geq 4$, then G contains an abelian subgroup of order 16.*

Proof. If $|G| = 32$ and $|Z(G)| = 4$, then, using GAP [30] or otherwise (see, for example [6, Theorem 35.4]), it is not difficult to see that G contains an abelian subgroup of order 16. The rest of the lemma follows immediately from [8, Section I, Para 4]. □

If G is a finite non-abelian group whose commuting graph is planar, then, by Lemma 3.2(b), we have $1 \leq |Z(G)| \leq 4$. Our next lemma of this section provides some useful information regarding the size of G and its abelian subgroups.

Lemma 5.2. *Let G be a finite non-abelian group whose commuting graph is planar. Then the following assertions hold:*

(a) *If p is a prime divisor of $|G|$, then $p \leq 5$.*
(b) *Neither 9 nor 25 divides $|G|$, and hence, $|G|$ is even with $|G| \geq 6$.*

Proof. If $p \geq 7$ is a prime divisor of $|G|$, then $G/Z(G)$ has an element of order p, and so, by Lemma 3.2(b), we have $|Z(G)| \leq \frac{4}{p-1} < 1$, which is impossible. This proves (a). For (b), note that if 9 or 25 divides $|G|$, then, a Sylow 3-subgroup or a Sylow 5-subgroup of G contains a subgroup of order 9 or 25. Since such a subgroup is abelian, we have, in view of Lemma 3.2(c), a contradiction in either situation. That $|G|$ is even with $|G| \geq 6$, follows from the fact that G is non-abelian. □

Given a finite non-abelian group G, whose commuting graph is planar, it follows from Lemma 5.2 that $|G| = 2^r 3^s 5^t$, where $r \geq 1$ and $s, t \in \{0, 1\}$. However, depending on the values of $|Z(G)|$, the range of possible values of $|G|$ gets reduced further.

Lemma 5.3. *Let G be a finite non-abelian group whose commuting graph is planar. Then the possible values of $|G|$ are given as follows:*

(a) *If* $|Z(G)| = 1$*, then* $|G| = 2^r 3^s 5^t$*, where* $1 \le r \le 3$ *and* $s, t \in \{0, 1\}$.
(b) *If* $|Z(G)| = 2$*, then* $|G| \in \{8, 12, 24\}$.
(c) *If* $|Z(G)| = 4$*, then* $|G| = 16$.
(d) $|Z(G)| \neq 3$.

Proof. We have $|G| = 2^r 3^s 5^t$, where $r \ge 1$ and $s, t \in \{0, 1\}$. Let H be a Sylow 2-subgroup of G. If $|Z(G)| \le 3$ and $r \ge 4$, then, by Lemma 5.1(a), H has an abelian subgroup of order 8. However, by Lemma 3.2(c), the size of an abelian subgroup of G does not exceed 7 if $|Z(G)| \le 3$. Thus, $r \le 3$ if $|Z(G)| \le 3$. On the other hand, if $|Z(G)| = 4$ and $r \ge 5$, then, using Lemma 5.1(b) and noting that $Z(G) \subseteq Z(H)$, there is an abelian subgroup of H of order 16. But, by Lemma 3.2(c), this is impossible. Thus, $r \le 4$ if $|Z(G)| = 4$. If 5 divides $|G|$, then $G/Z(G)$ has an element of order 5, and so, by Lemma 3.2(b), we have $|Z(G)| = 1$. Also, if $|Z(G)| = 4$, then 3 does not divide $|G|$; otherwise $G/Z(G)$ would have an element of order 3, which, by Lemma 3.2(b), is impossible. Now, it is a routine matter to see that the assertions (a), (b) and (c) hold. Finally, note that if $|Z(G)| = 3$, then, by the above argument, we have $|G| = 12$ or 24. Therefore, G has a subgroup A of order 4, and hence, an abelian subgroup $AZ(G)$ of order 12, which, by Lemma 3.2(c), is impossible. Thus, (d) holds as well. □

Note that some of the possibilities mentioned in Lemma 5.3 are not maintainable; for example, in (a), it is obviously not possible to have $s = t = 0$. In fact, the following small result helps us in avoiding few more finite groups as far as the planarity of their commuting graphs is concerned.

Lemma 5.4. *Let* G *be a finite non-abelian group. If* $|G| = 30$*, or if* G *is a solvable group with* $|G| = 60$ *or* 120*, then* G *has an subgroup of order* 15 *(which is obviously abelian). Also, if* $|G| = 40$*, then* G *has an abelian subgroup of order* 10.

Proof. If $|G| = 30$, or if G is a solvable group with $|G| = 60$ or 120, then, by a theorem of Hall (see [22, Theorem 5.28]), G has a subgroup of order 15. On the other hand, if $|G| = 40$, then G has a unique Sylow 5-subgroup, and so, considering the centralizer and the number of conjugates of an element of order 5, one can show that G has an element (hence, an abelian subgroup) of order 10. □

In view of Lemma 3.2(c) and Lemma 5.3, it follows from Lemma 5.4 that if G is a finite non-abelian group whose commuting graph is planar, then $|G| \notin \{30, 40\}$; in addition, if G is solvable, then $|G| \notin \{60, 120\}$.

We also have the following useful result concerning the groups of order 16.

Lemma 5.5. *Let G be a finite non-abelian group with $|Z(G)| = 4$. Then, the commuting graph of G is planar if and only if $|G| = 16$.*

Proof. Let G be a finite group with $|G| = 16$ and $|Z(G)| = 4$. Note that, for each $x \in G \setminus Z(G)$, we have $|C_G(x)| = 8$ and $C_G(x) = \langle x \rangle Z(G)$, which is abelian. Thus, G is an AC-group with $|C_G(x) \setminus Z(G)| = 4$. Hence, it follows from Proposition 3.4 that $\gamma(\Gamma_c(G)) = 0$, that is, the commuting graph of G is planar. This, in view of Lemma 5.3(c), completes the proof. □

Remark 5.6. *Up to isomorphism, there are exactly six non-abelian groups of order 16 with centers of order 4, namely, the two direct products $\mathbb{Z}_2 \times D_8$ and $\mathbb{Z}_2 \times Q_8$, the Small Group $SG(16, 3) = \langle a, b \mid a^4 = b^4 = 1, ab = b^{-1}a^{-1}, ab^{-1} = ba^{-1} \rangle$, the semi-direct product $\mathbb{Z}_4 \rtimes \mathbb{Z}_4 = \langle a, b \mid a^4 = b^4 = 1, bab^{-1} = a^{-1} \rangle$, the central product $D_8 * \mathbb{Z}_4 = \langle a, b, c \mid a^4 = b^2 = c^2 = 1, ab = ba, ac = ca, bc = a^2cb \rangle$ and the modular group $M_{16} = \langle a, b \mid a^8 = b^2 = 1, bab = a^5 \rangle$.*

We now state and prove the main result of this section, where two new groups make their appearance, namely, the Suzuki group $Sz(2) = \langle a, b \mid a^5 = b^4 = 1, bab^{-1} = a^2 \rangle$, and the special linear group $SL(2, 3) = \langle a, b, c \mid a^3 = b^3 = c^2 = abc \rangle$.

Theorem 5.7. *Let G be a finite non-abelian group. Then, the commuting graph of G is planar if and only if G is isomorphic to either S_3, D_{10}, A_4, $Sz(2)$, S_4, A_5, D_8, Q_8, D_{12}, Q_{12}, $SL(2, 3)$, $\mathbb{Z}_2 \times D_8$, $\mathbb{Z}_2 \times Q_8$, $SG(16, 3)$, $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$, $D_8 * \mathbb{Z}_4$ or M_{16}.*

Proof. In view of Lemma 5.3, Lemma 5.5, Remark 5.6 and the paragraph following Lemma 5.4, it is enough to study the planarity of the commuting graph of a finite group G that belongs to one of the following categories:

I. $|Z(G)| = 1$ and $|G| \in \{6, 10, 12, 20, 24\}$.
II. $|Z(G)| = 1$, $|G| \in \{60, 120\}$ and G is not solvable.
III. $|Z(G)| = 2$ and $|G| \in \{8, 12, 24\}$.

We use GAP [30] to examine the groups that belong to the above categories and look into some of their properties which eventually help in concluding whether their commuting graphs are planar or not.

There are exactly five groups that belong to category I, namely, S_3, D_{10}, A_4, $Sz(2)$ and S_4. Among these groups, S_3, D_{10}, A_4 and $Sz(2)$ are AC-groups such that, in each case, the size of the centralizer of every non-central element is at most 5, and so, by Proposition 3.4, the commuting graph of each of these groups is planar; on the other hand, the commuting graph $\Gamma_c(S_4)$ has a block decomposition given by

$$\Gamma_c(S_4)[H] \cup \bigcup_{\sigma\in\mathcal{F}} \Gamma_c(S_4)[H_\sigma],$$

with $\mathcal{F} = \{(1\,2), (1\,3), (1\,4), (1\,2\,3\,4), (1\,2\,4\,3), (1\,3\,2\,4), (1\,2\,3), (1\,2\,4), (1\,3\,4), (2\,3\,4)\}$, $H = \{(1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$ and $H_\sigma = C_\sigma(S_4) \setminus \{(1)\}$ for all $\sigma \in \mathcal{F}$, and so, by [5, Corollary 1], it follows that $\gamma(\Gamma_c(S_4)) = 7\gamma(K_3) + 4\gamma(K_2) = 0$.

There are exactly two groups that belong to category II, namely, A_5 and S_5. Of the two groups, A_5 is an AC-group in which the centralizer of every non-central element is at most 5, and so, by Proposition 3.4, its commuting graph is planar; on the other hand, S_5 has an abelian subgroup of order 6, namely, $C_{S_5}(1\,2) = \langle(1\,2), (3\,4\,5)\rangle$, and so, by Lemma 3.2(c), its commuting graph is not planar.

Finally, there are exactly nine groups that belong to category III. However, except D_8, Q_8, D_{12}, Q_{12} and $SL(2, 3)$, each of the remaining four groups has an abelian centralizer of order at least 8, and so, by Lemma 3.2(c), has commuting graph of positive genus. The groups D_8, Q_8, D_{12}, Q_{12} and $SL(2, 3)$, on the other hand, are all AC-groups such that, in each case, the size of the centralizer of every non-central element is at most 6, and so, by Proposition 3.4, the commuting graph of each of these groups is planar. This completes the proof. □

6. Finite non-abelian groups whose commuting graphs are toroidal

In this section, we characterize all finite non-abelian groups whose commuting graphs are toroidal.

The following result is analogous to Lemma 5.2.

Lemma 6.1. *Let G be a finite non-abelain group whose commuting graph is toroidal. Then, the following assertions hold:*

(a) $|Z(G)| \le 3$.

(b) *If p is a prime divisor of $|G|$, then $p \le 7$.*

(c) *None of* 25, 27 *and* 49 *is a divisor of* $|G|$.

Proof. Suppose that $|Z(G)| = 4$. If p is an odd prime divisor of $|G|$, then $G/Z(G)$ has an element of order at least 3, and so, by Lemma 3.2(b), we have a contradiction. Therefore, in view of Lemma 5.5, $|G| = 2^r$ for some $r \ge 5$. But, by Lemma 5.1(b) and Lemma 3.2(c), we again have a contradiction. So, let $|Z(G)| \ge 5$. Choose $x, y \in G\setminus Z(G)$ such that $xy \ne yx$. Then, $xZ(G)$ and $yZ(G)$ are two disjoint subsets of $G\setminus Z(G)$, and the induced subgraph $\Gamma_c(G)[xZ(G)] \cong K_m \cong \Gamma_c(G)[yZ(G)]$, where $m = |Z(G)|$. Hence, by Lemma 2.1 and Lemma 2.2, it follows that $\gamma(\Gamma_c(G)) \ge 2$, which is impossible. Thus, (a) holds.

If $p \geq 11$ is a prime divisor of $|G|$, then, by (a), there is an element of order p in $G/Z(G)$. Therefore, by Lemma 3.2(b), we have $|Z(G)| \leq \frac{7}{p-1} < 1$, which is impossible. This proves (b).

For (c), note that if 25 or 49 divides $|G|$, then G has an abelian subgroup of order 25 or 49. Since such a subgroup is obviously abelian, we have a contradiction according to (a) and Lemma 3.2(c). On the other hand, if 27 divides $|G|$, then G has a subgroup of order 27. Therefore, since the commuting graph of a subgroup of G is a subgraph of the commuting graph of G, we have, by Corollary 3.7, a contradiction. This completes the proof. □

Analogous to Lemma 5.5, we have the following result concerning the groups of order 16.

Lemma 6.2. *Let G be a finite non-abelian 2-group with $|Z(G)| = 2$. Then, the commuting graph of G is toroidal if and only if $|G| = 16$, that is, if and only if G is isomorphic to either D_{16}, Q_{16} or SD_{16}.*

Proof. Let $|G| \geq 32$. Then, by the class equation [22, page 74], there exists $x \in G \setminus Z(G)$ such that $|G : C_G(x)| = 2$, and so, $|C_G(x)| \geq 16$. Clearly $|Z(C_G(x))| \geq 4$. First, let us assume that $|Z(C_G(x))| = 4$. Let $v \in C_G(x) \setminus Z(C_G(x))$. Then, there exists $w \in C_G(x) \setminus Z(C_G(x))$ such that $vw \neq wv$. Let z denote the non-trivial element of $Z(G)$. Consider the two disjoint subsets $H_1 = \{x, v, vz, xv, xvz\}$ and $H_2 = \{xz, w, wz, xw, xwz\}$ of $G \setminus Z(G)$. Clearly, the induced subgraph $\Gamma_c(G)[H_1] \cong K_5 \cong \Gamma_c(G)[H_2]$. Hence, by Lemma 2.1 and Lemma 2.2, it follows that $\gamma(\Gamma_c(G)) \geq 2$. Next, let us assume that $|Z(C_G(x))| \geq 8$. Consider a subset V of $Z(C_G(x)) \setminus Z(G)$ such that $|V| = 3$ and put $W = C_G(x) \setminus (V \cup Z(G))$. Clearly, the induced subgraph $\Gamma_c(G)[V \cup W]$ has a subgraph isomorphic to the complete bipartite graph $K_{3,n}$, where $n = |C_G(x)| - 5 \geq 11$. This, by Lemma 2.3, implies that the genus of the commuting graph of G is at least 3. Thus, in view of Theorem 5.7, it follows that if the commuting graph of G is toroidal, then $|G| = 16$. On the other hand, it is well-known (using GAP [30], for example) that if $|G| = 16$ and $|Z(G)| = 2$, then G is isomorphic to either D_{16}, Q_{16} or SD_{16}, and, by Proposition 4.1, Proposition 4.2 and Proposition 4.3, the commuting graph of each of these groups is toroidal. This completes the proof. □

We also have the following result concerning the finite groups that are not 2-groups.

Lemma 6.3. *Let G be a finite non-abelian group with $|G| = 2^r m$, where $r \geq 0$, $m > 1$ and m is odd. If the commuting graph of G is toroidal, then $r \leq 3$.*

Proof. Suppose that the commuting graph of G is toroidal and that $r \geq 4$. Let H be a Sylow 2-subgroup of G. In view of Lemma 3.2(c), H is non-abelian. Moreover, the commuting graph of H, being a subgraph of the commuting graph of G, is either planar or toroidal.

Case 1. $\Gamma_c(H)$ is planar.

In this case, by Lemma 5.3, we have $|Z(H)| = 4$. Therefore, by Lemma 6.1(a), we have $|Z(H) \setminus Z(G)| \geq 2$. Let $v_1, v_2 \in Z(H) \setminus Z(G)$ such that $v_1 \neq v_2$. Also, let $x, y \in H \setminus Z(H)$ such that $xy \neq yx$. Then, it is easy to see that $\{v_1\} \cup xZ(H)$ and $\{v_2\} \cup yZ(H)$ are two disjoint subsets of $G \setminus Z(G)$, and the induced subgraph $\Gamma_c(G)[\{v_1\} \cup xZ(H)] \cong K_5 \cong \Gamma_c(G)[\{v_2\} \cup yZ(H)]$. This implies that $\gamma(\Gamma_c(G)) \geq 2$, which is a contradiction.

Case 2. $\Gamma_c(H)$ is toroidal.

In this case, by Lemma 6.1(a), we have $|Z(H)| = 2$. Therefore, by Lemma 6.2, we have $|H| = 16$ and there exists an element $x \in H$ with $o(x) = 8$. Note that, for each $y \in G$ with $o(y) = 8$, we have $\langle x \rangle = \langle y \rangle$; otherwise, choosing $M = \{x, x^3, x^5, x^7, x^2\}$ and $N = \{y, y^3, y^5, y^7, w\}$ with $w \in \{y^2, y^6\} \setminus \{x^2\}$, we would have the induced subgraph $\Gamma_c(G)[M] \cong K_5 \cong \Gamma_c(G)[N]$, which, by Lemma 2.1 and Lemma 2.2, implies that $\gamma(\Gamma_c(G)) \geq 2$, a contradiction. Also, in view of Lemma 3.2, we have $|C_G(x)| = 8$; otherwise either $Z(H)$ would have an element of order 8 or G would have an abelian subgroup of order at least 24. Hence, it follows that the number of conjugates of x in G is $2m \geq 6$, that is, there are at least six elements of order 8 in G. This contradiction completes the proof. □

If G is a finite non-abelian group whose commuting graph is toroidal, then it follows from Lemma 6.1 that $|G| = 2^r 3^s 5^t 7^u$, where $r \geq 0$, $0 \leq s \leq 2$ and $t, u \in \{0, 1\}$. However, as in Lemma 5.3, the range of possible values of $|G|$ gets reduced further depending on the values of $|Z(G)|$.

Lemma 6.4. *Let G be a finite non-abelian group whose commuting graph is toroidal. Then the possible values of $|G|$ are given as follows:*

(a) *If $|Z(G)| = 1$, then $|G| = 2^r 3^s 5^t 7^u$ where $0 \leq r \leq 3$ and $s, t, u \in \{0, 1\}$.*
(b) *If $|Z(G)| = 2$, then $|G| \in \{16, 24\}$.*
(c) *If $|Z(G)| = 3$, then $|G| = 18$.*

Proof. If 5 or 7 divides $|G|$, then $G/Z(G)$ has an element of order 5 or 7, and so, by Lemma 3.2(b), we have $|Z(G)| = 1$. If $|Z(G)| \leq 2$, then 9 does not divide $|G|$; otherwise G would have an abelian subgroup T of order 9, which, by Lemma 3.2(c), is impossible noting that $|T \cap Z(G)| = 1$. If $|Z(G)| = 3$, then 4 does not divide $|G|$; otherwise G would have a subgroup A of order 4, and hence, an abelian subgroup $AZ(G)$ of order 12, which, by Lemma 3.2(c), is impossible. In view of Theorem 5.7, Lemma 6.2 and Lemma 6.3, it is now not difficult to see that all the three assertions hold. □

Needless to mention that some of the possibilities mentioned in Lemma 6.4 are clearly not maintainable; for example, in (a), it is impossible to have $s = t = u = 0$, $r = u = 0$ or $r = s = 0$. Moreover, in view of Lemma 3.2(c) and Lemma 6.4, it follows from Lemma 5.4 that if G is a finite non-abelian group whose commuting graph is toroidal, then $|G| \notin \{30, 40\}$; in addition, if G is solvable, then $|G| \notin \{60, 120\}$.

The following result, along with Lemma 5.4, helps us in rejecting some more possibilities.

Lemma 6.5. *Let G be a finite non-abelian group whose commuting graph is toroidal. If $|G| = 7m$, where $m \geq 2$ and $7 \nmid m$, then $m = 2$ or 3.*

Proof. By Lemma 6.4, we have $|Z(G)| = 1$. Let H be a Sylow 7-subgroups of G. If S is a Sylow 7-subgroups of G such that $S \neq H$, then it is easy to see that the induced subgraph $\Gamma_c(G)[S \setminus Z(G)] \cong K_6 \cong \Gamma_c(G)[H \setminus Z(G)]$, and so, we have a contradiction to the toroidality of $\Gamma_c(G)$. Therefore, H is the unique (hence, normal) Sylow 7-subgroup of G. Note that $C_G(H) = H$; otherwise $C_G(H)$ (hence, G) would have an element (hence, an abelian subgroup) of order at least 14, which, by Lemma 3.2(c), is impossible. Therefore, by [22, Theorem 7.1(i)], G/H is isomorphic to a subgroup of the cyclic group $\mathbb{Z}_6 \cong \mathrm{Aut}(H)$. Since $|G/H| = m$, it follows that $m|6$ and G has an element x of order m. If $m = 6$, then the induced subgraph $\Gamma_c(G)[\langle x\rangle \setminus Z(G)] \cong K_5$, and so, we have a contradiction to the toroidality of $\Gamma_c(G)$ since $\Gamma_c(G)[H \setminus Z(G)] \cong K_6$. Hence, we have $m = 2$ or 3. □

We now state and prove the main result of this section.

Theorem 6.6. *Let G be a finite non-abelian group. Then, the commuting graph of G is toroidal if and only if G is isomorphic to either D_{14}, $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$, $\mathbb{Z}_2 \times A_4$, $\mathbb{Z}_3 \times S_3$, D_{16}, Q_{16} or SD_{16}.*

Proof. In view of Lemma 6.2, Lemma 6.4 (and the paragraph following it), Lemma 6.5 and the proof of Theorem 5.7, it is enough to study the toroidality of the commuting graph of a finite group G that belongs to one of the following categories:

I. $|Z(G)| = 1$ and $|G| \in \{14, 21\}$.

II. $|Z(G)| = 1$, $|G| = 120$ and G is not solvable.

III. $|Z(G)| = 2$, $|G| = 24$ and $G \ncong SL(2,3)$.

IV. $|Z(G)| = 3$ and $|G| = 18$.

As in the proof of Theorem 5.7, we use GAP [30] to determine the groups belonging to the above categories whose commuting graphs are toroidal.

D_{14} and $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ are the only groups that belong to category I and, by Proposition 4.1 and Corollary 3.6, the commuting graphs of these groups are toroidal.

S_5 is the only group that belongs to the category II. However, S_5 has two abelian subgroups $S = \langle(1\ 2)(3\ 4\ 5)\rangle$ and $T = \langle(4\ 5)(1\ 2\ 3)\rangle$ such that $|S| = |T| = 6$ and $S \cap T$ is trivial. It follows that the commuting graph of S_5 is not toroidal.

There are exactly four groups that belong to category III and all of them are AC-groups. However, except $\mathbb{Z}_2 \times A_4$, each of the remaining three groups have an abelian centralizer of order 12, whereas $\mathbb{Z}_2 \times A_4$ has only one abelian centralizer of order 8 and the rest of order 6. Therefore, by Proposition 3.4, it follows that $\mathbb{Z}_2 \times A_4$ is the only group in category III whose commuting graph is toroidal.

$\mathbb{Z}_3 \times S_3$ is the only group that belongs to the category IV and it is an AC-group with only one abelian centralizer of order 9 and the rest of order 6. Therefore, by Proposition 3.4, it follows that the commuting graph of $\mathbb{Z}_3 \times S_3$ is toroidal. This completes the proof. □

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