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CLUSTER ALGEBRAS ARISING FROM INFINITY-GON

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Abstract. We introduce a handy construction of cluster algebras of type $\mathbb{A}_\infty$, we give a complete classification of the cluster algebras arising from the infinity-gon, and finally we construct the category of the diagonals of the infinity-gon and show that it is triangle-equivalent to the infinite cluster category of type $\mathbb{A}_\infty$ described by Holm and Jørgensen.

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1. Introduction

Since their invention by Fomin and Zelevinsky in [12] cluster algebras have found applications in various areas of mathematics like Lie theory, quiver representations, Calabi-Yau algebras, Teichmller theory, tropical geometry, Poisson geometry, integrable systems, combinatorics and mathematical physics see [9], [10], [14], [21].

A cluster algebra is a commutative ring with a distinguished set of generators, called *cluster variables*. The set of all cluster variables is constructed recursively from an initial set using a procedure called *mutation*. All generators are organized into clusters and each cluster contains exactly n clusters variables. The study of cluster structures for 2-Calabi-Yau categories in [4] led however to the introduction of cluster algebras with countable clusters. The infinite cluster category $\mathcal{D}$ of Dynkin type $\mathbb{A}_\infty$ was constructed by P. Jørgensen in [18]. The cluster tilting subcategories of $\mathcal{D}$ were classified by using the triangulations of infinity-gon in [17]. The Caldero-Chapoton map has been introduced in [6] and [8] by Caldero, Chapoton and Keller to formalize the connection between the Fomin-Zelevinsky cluster algebras and the cluster category of Buan, Marsh, Reineke, Reiten and Todorov. The analogue of this map was realized between infinite cluster algebras of type $\mathbb{A}_\infty$ and the infinite cluster category $\mathcal{D}$ by Jørgensen and Palu, see [19].

This paper is devoted to the study of cluster algebras of type $\mathbb{A}_\infty$, cluster algebras arising from the triangulations and the categorification of the infinity-gon. In this paper we first give a handy construction of each cluster algebra $\mathcal{B}$ of type $\mathbb{A}_\infty$. More specifically we prove that the cluster algebra $\mathcal{B}$ is a particular subalgebra of the projective limit algebra $\mathcal{A}$ of a particular projective system $(\mathcal{A}^i, p_{i,j})_{i,j\geq 1}$,

where $\mathcal{A}^n$ is a cluster algebra of type $\mathbb{A}_n$, and $p_{i,j} : \mathcal{A}^j \longrightarrow \mathcal{A}^i$ is a surjective cluster morphism in the sense of Assem-Dupont-Schiffler.

Theorem 1.1. *The cluster algebra $\mathcal{B}$ is the proper $\mathbb{Z}$-subalgebra of the algebra $\mathcal{A} = \varprojlim \mathcal{A}^n$, consisting of the ultimately constant elements of $\mathcal{A}$.*

We next give the study of cluster algebras arising from the infinity-gon started by Grabowski and Gratz in [16]. In this study, we associate to each triangulation T of the infinity-gon, the cluster algebra $\mathcal{A}(T)$ as done by Fomin, Shapiro and Thurston in [11] for the case of the marked surfaces with a finite number of marked points. For the case of marked surfaces with a finite number of marked points, Fomin, Shapiro and Thurston have shown that the cluster algebra associated to a triangulated surface does not depend upon the choice of triangulation. We shall show that this result does not hold for cluster algebras arising from the triangulation of the infinity-gon. However, we give a complete classification of the clusters algebras arising from the infinity-gon using the notion of congruence between triangulations inside the set of all triangulations. In [13] Fomin and Zelevinsky have considered the notion of *strong isomorphisms*, by which they mean an isomorphism of the cluster algebras which maps every seed to an isomorphic seed. Two triangulations T and T' are said to be *congruent* if there exists a bijection $\theta : \overline{\mathbb{S}} \longrightarrow \overline{\mathbb{S}}$ which maps T to T' and preserves the flips of arcs; that is $\theta(T) = T'$ and $\theta(f_\gamma) = f_{\theta(\gamma)}$, where f_γ is the flip of γ.

Theorem 1.2. *Let T and T' be two triangulations of the infinity-gon, $\mathcal{A}(T)$ and $\mathcal{A}(T')$ the associated cluster algebras. Then T and T' are congruent if and only if the clusters algebras $\mathcal{A}(T)$ and $\mathcal{A}(T')$ are strongly isomorphic.*

We also defined the category of diagonals of the infinity-gon as the one of the $(n+3)$-gon constructed by Caldero, Chapoton and Schiffler in [7]. It is well-known, that the category of diagonals of the $(n+3)$-gon is equivalent to the cluster category of Buan-Marsh-Reineke-Reiten-Todorov for the quiver of type $\mathbb{A}_n$. Here we show that the category of diagonals of the infinity-gon $\mathcal{C}$ is equivalent to the infinite cluster category $\mathcal{D}$ of type $\mathbb{A}_\infty$ of Jørgensen [18].

Theorem 1.3. *The categories $\mathcal{C}$ and $\mathcal{D}$ are triangle-equivalent.*

As a consequence of this result, we give a description of the Auslander-Reiten triangles in geometric terms inspired by [3]. Our paper is organized as follows.

In Section 2, we introduce a special projective system of clusters algebras of type $\mathbb{A}_n$ and give the relation between the projective limit of this system and the corresponding cluster algebras of type $\mathbb{A}_\infty$; this gives rise to a handy construction of clusters algebras of type $\mathbb{A}_\infty$. This construction is handy because each algebra $\mathcal{B}$ is expressed as a classical sub-algebra of the product of $\mathbb{Z}$-algebras.

In Section 3, we give a complete classification of the cluster algebras arising from the infinity-gon.

Finally, in Section 4, we construct the category $\mathcal{C}$ of diagonals of the infinity-gon and show that it is triangle-equivalent to the category $\mathcal{D}$. Therefore, inspired by [3], we give a description of Auslander-Reiten triangles of $\mathcal{C}$ using the diagonals of the infinity-gon.

2. Cluster algebras of type $\mathbb{A}_\infty$

2.1. Basic construction. We recall that a *quiver* is a quadruple $Q = (Q_0, Q_1, s, t)$ consisting of two sets, Q_0 (whose elements are called *points*) and Q_1 (whose elements are called *arrows*) and two functions $s, t : Q_1 \longrightarrow Q_0$ associating to each arrow $\alpha \in Q_1$ its so-called *source* $s(\alpha)$ and *target* $t(\alpha)$. If $i = s(\alpha)$ and $j = t(\alpha)$, we denote this situation by $i \xrightarrow{\alpha} j$. Given a point i, we set $i^+ = \{\alpha \in Q_1 \mid s(\alpha) = i\}$ and $i^- = \{\alpha \in Q_1 \mid t(\alpha) = i\}$. We say that a quiver Q is *locally finite* if for each $i \in Q_0$, the sets i^+ and i^- are finite.

Let Q be a countably infinite, but locally finite quiver without cycles of length at most two, and let $X = \{x_n \mid n \geq 1\}$ be a countable set of undeterminates. where we agree that the point i of the quiver Q corresponds to the variable x_i. We define the mutation μ_k in $k \in Q_0$ exactly as in the case of a finite quiver, that is $\mu_k(Q, X) = (X^{'}, Q^{'})$, where $Q^{'}$ is the quiver obtained from Q by performing the following operations:
- for any path of $i \longrightarrow k \longrightarrow j$ of length two having k as midpoint, we insert a new arrow $i \longrightarrow j$,
- all arrows incident to the point k are reversed,
- all newly occurring cycles of length two are deleted.
Clearly, $Q^{'}$ is still locally finite.
On the other hand, $X^{'}$ is a countable set of variables defined as follows:
$X^{'} = (X \setminus \{x_k\}) \cup \{x_k^{'}\}$ where $x_k^{'}$ is obtained from X by the so-called *exchange relation*

$$x_k x_k^{'} = \prod_{\alpha \in i^+} x_{t(\alpha)} + \prod_{\alpha \in i^-} x_{s(\alpha)}.$$

These operations are performed inside the field $\mathbb{Q}(X)$ of rational functions over the undeterminates x_n, called the *ambient field*. One verifies exactly as in the case of a finite quiver that $\mu_k^2(Q, X) = (X, Q)$.
From now on, let Q be a quiver having as underlying graph the infinite half-path $\mathbb{A}_\infty$

$$1 \text{ —— } 2 \text{ —— } 3 \text{ —— } \cdots \text{ —— } n-1 \text{ —— } n \text{ —— } \cdots.$$

We call quiver of *type* $\mathbb{A}_\infty$ each quiver whose underline graph is a half-path defined above.

Let $X = \{x_n \mid n \in \mathbb{N}\}$ be a countable set of undeterminates. We denote by $\mathcal{F} = \mathbb{Q}(X)$ the ambient field, that is the elements of the form $\frac{P(x_{\sigma(1)},\ldots,x_{\sigma(n)})}{Q(x_{\varrho(1)},\ldots,x_{\varrho(m)})}$, where P and Q are polynomials (Q is not the zero polynomial) and σ, ϱ are equipotent maps from $\mathbb{N}$ to $\mathbb{N}$. Each pair $(X^{'}, Q^{'})$ obtained from (X, Q) by a finite sequence of mutations is called a *seed*, and the set $X^{'}$ is called a *cluster*. The elements of $X^{'}$ are called *cluster variables*. The pair (X, Q) is called the *initial seed*, and X is called the *initial cluster*.

Definition 2.1. The cluster algebra $\mathcal{B} = \mathcal{A}(Q, X)$ is the $\mathbb{Z}$-subalgebra of $\mathcal{F}$ generated by the set $\mathcal{X}$ which is the union of all possible sets of variables obtained from X by finite sequences of mutations.

Gekhtman, Shapiro and Vainshtein showed in [14] that for every seed $(\tilde{X}, \tilde{Q})$ of a given cluster algebra, the quiver $\tilde{Q}$ is uniquely defined by the cluster $\tilde{X}$. Because the mutation is a local operation inside a quiver, this result remains true for the countable seeds above. Our objective in this first section is to give a handy construction of the cluster algebra $\mathcal{B} = \mathcal{A}(X, Q)$.

2.2. A projective system of cluster algebras. In this subsection we denote by $\overrightarrow{\mathbb{A}}_n$ the linearly oriented quiver of type $\mathbb{A}_n$, $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n$ and by $X^n = \{x_1, x_2, \ldots, x_n\}$ an associated set of variables. Let $\mathcal{F}^n = \mathbb{Q}(x_1, x_2, \ldots, x_n)$ be the field of rational functions on the x_i (with rational coefficients) and $\mathcal{X}^n$ be the union of all possible sets of variables obtained from X^n by successive mutations. This data defines a cluster algebra $\mathcal{A}^n = \mathcal{A}(X^n, \overrightarrow{\mathbb{A}}_n)$ having $(X^n, \overrightarrow{\mathbb{A}}_n)$ as initial seed. We recall that the Laurent phenomenon asserts that each cluster variable in $\mathcal{A}^n$ can be expressed as a Laurent polynomial in the x_i, with $1 \leq i \leq n$, that is, such a variable is of the form

$$\frac{p(x_1, x_2, \ldots, x_n)}{\prod\limits_{l=1}^{n} x_l^{d_l}}$$

where $p \in \mathbb{Z}[x_1, x_2, \ldots, x_n]$ and $d_l \geq 0$ for all l, with $1 \leq l \leq n$. And the *positivity* [20, Theorem 1.1] asserts that all coefficients of the polynomial P are non-negative integers. The positivity theorem holds for the cluster algebra $\mathcal{A}^n$.

Now let i, j be positive integers with $i \leq j$, we define the map $p_{i,j} : \mathcal{A}^j \longrightarrow \mathcal{A}^i$ on the generators of $\mathcal{A}^j$ as follows

$$p_{i,j}\left(\frac{p(x_1, x_2, \ldots, x_i, x_{i+1}, \ldots, x_j)}{\prod\limits_{l=1}^{j} x_l^{d_l}}\right) = \frac{p(x_1, x_2, \ldots, x_i, 1, \ldots, 1)}{\prod\limits_{l=1}^{i} x_l^{d_l}}.$$

It follows from Theorem 6.11 of [2] that the image of this map is non-trivial and is in the cluster algebra $\mathcal{A}^i$. Since $p_{i,j}$ is an evaluation, it is a morphism of $\mathbb{Z}$-algebras.

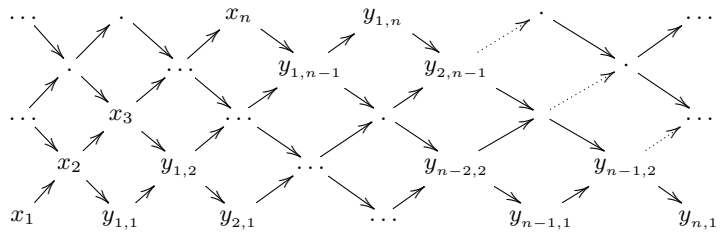
Clearly, we have $p_{i,i} = id_{\mathcal{A}^i}$ and if $i < j$, $p_{i,j} = p_{i,i+1}p_{i+1,i+2}\cdots p_{j-1,j}$. Thus, if $i \leq j \leq k$, then $p_{i,j}p_{j,k} = p_{i,k}$. Our first objective is to prove that, if $i \leq j$, then $p_{i,j}$ is actually a surjective morphism from $\mathcal{A}^j$ to $\mathcal{A}^i$.

Proposition 2.2. *With the above notation, $\mathcal{A}^i = \mathbf{Z}[p_{i,j}(\mathcal{X}^j)]$. In particular, $p_{i,j}$ is a surjective morphism of $\mathbb{Z}$-algebras from $\mathcal{A}^j$ to $\mathcal{A}^i$.*

Proof. Because of the above equalities, it suffices to show that, for each $n \geq 2$, we have $\mathcal{A}^{n-1} = \mathbb{Z}[p_{n-1,n}(\mathcal{X}^n)]$. This is done by induction on n.

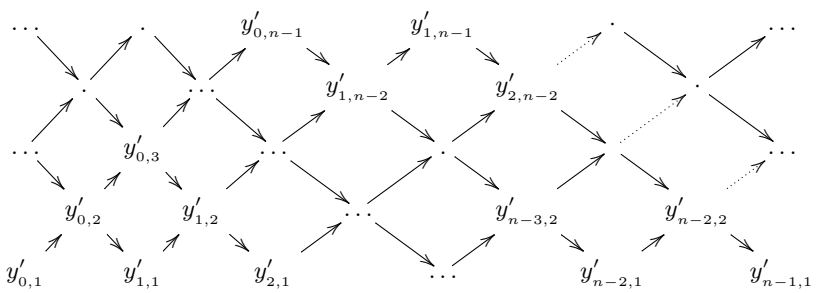
Assume first $n = 2$. In this case $\mathcal{A}^2 = \mathbb{Z}[x_1, x_2, \frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1x_2}, \frac{1+x_1}{x_2}]$ while $\mathcal{A}^1 = \mathbb{Z}[x_1, \frac{2}{x_1}]$. The morphism $p_{1,2} : \mathcal{A}^2 \longrightarrow \mathcal{A}^1$ is defined on the generators as follows:
$p_{1,2}(x_1) = x_1, p_{1,2}(x_2) = 1, p_{1,2}(\frac{1+x_2}{x_1}) = \frac{2}{x_1}, p_{1,2}(\frac{1+x_1+x_2}{x_1}) = 1+\frac{2}{x_1}$, and $p_{1,2}(\frac{1+x_1}{x_2}) = 1 + x_1$. Thus, clearly, $\mathcal{A}^1 = \mathbb{Z}[p_{1,2}(\mathcal{X}^2)]$ so that $p_{1,2} : \mathcal{A}^2 \longrightarrow \mathcal{A}^1$ is a surjective morphism of $\mathbb{Z}$-algebras.

We now assume that, for every $j < n$, we have $\mathcal{A}^{j-1} = \mathbb{Z}[p_{j-1,j}(\mathcal{X}^j)]$ and show that $\mathcal{A}^{n-1} = \mathbb{Z}[p_{n-1,n}(\mathcal{X}^n)]$. For this purpose, we use the categorification of the cluster algebras $\mathcal{A}^n$, and $\mathcal{A}^{n-1}$, as in [5]. We recall that the Auslander-Reiten quiver Γ^n of the cluster category attached to $\mathcal{A}^n$ is of the form



where we agree to identify each point of Γ^n with the corresponding cluster variable and $y_{0,i} = x_i$ for each i such that $1 \leq i \leq n$; and we denote by $y_{i,j}$ with $0 \leq i \leq n$, $1 \leq j \leq n$ and $i + j \leq n + 1$, the clusters variables of $\mathcal{A}^n$. By the Calaby-Yau duality of cluster category we have $y_{0,i} = y_{i+1,n-i+1}$, with $1 \leq i \leq n$.

We denote by Γ^{n-1} the Auslander-Reiten quiver of the cluster category attached to $\mathcal{A}^{n-1}$. It is of the form



where again each point is identified with the corresponding cluster variable. Therefore $y'_{0,i} = x_i$ for all i such that $1 \leq i \leq n-1$; and we denote by $y_{i,j}$ with $0 \leq i \leq n-1$, $1 \leq j \leq n-1$ and $i+j \leq n$, the clusters variables of $\mathcal{A}^{n-1}$. By the Calaby-Yau duality of cluster category we have $y'_{0,i} = y'_{i+1,n-i}$, with $1 \leq i \leq n-1$.

We say that a point $y_{i,j}$ of Γ^n is *stable* provided $p_{n-1,n}(y_{i,j}) = y'_{i,j}$. Clearly, all the points $y_{0,i}$ with $1 \leq i \leq n-1$ are stable. It then follows from the definition of mutation that, for every pair (i,j), such that $i+j \leq n-1$, the point $y_{i,j}$ is stable. Now it remains to consider the points $\{y_{i,j} | n \leq i+j \leq n+1\}$ of Γ^n. Because $p_{n-1,n}$ is an evaluation, we may write $p_{n-1,n}(y_{i,j}) = y_{i,j}(1)$ for brevity.

The proof is completed in the following three steps (a), (b), (c).

(a) We first claim that, if $i+j=n$ and $n \geq 1$, then $y_{i,j}(1) = y'_{i,j}$. This is done by induction on i.

If $i=1$, we have $y_{1,n-1} = \frac{1+y_{1,n-2}y_{0,n}}{y_{0,n-1}}$. The evaluation of $y_{1,n-1}$ at 1 is given by:

$$\begin{aligned} y_{1,n-1}(1) &= \frac{1+y_{1,n-2}y_{0,n}}{y_{0,n-1}}(1) \\ &= \frac{1+y_{1,n-2}}{y_{0,n-1}} \\ &= y'_{1,n-1}, \end{aligned}$$

where we used the stability of $y_{0,n-1}$ and $y_{1,n-2}$.

Assume now the result valid for all $i \leq n-1$. We have $y_{i+1,n-i-1} = \frac{1+y_{i+1,n-i-2}y_{i,n-i}}{y_{i,n-i-1}}$. The evaluation of $y_{i+1,n-i-1}$ at 1 is given by:

$$\begin{aligned} y_{i+1,n-i-1}(1) &= \frac{1+y_{i+1,n-i-2}y_{i,n-i}}{y_{i,n-i-1}}(1) \\ &= \frac{1+y_{i+1,n-i-2}(1)y_{i,n-i}(1)}{y_{i,n-i-1}(1)} \\ &= \frac{1+y'_{i+1,n-i-2}y'_{i,n-i}}{y'_{i,n-i-1}} \\ &= y'_{i+1,n-i-1}, \end{aligned}$$

where we used the induction hypothesis and the stability of $y_{i+1,n-i-2}$ and $y_{i+1,n-i-1}$. This establishes our claim for the step (a).

(b) Next we consider the particular case of the variable $y_{1,n}$. Because $y_{1,n} = \frac{1+y_{1,n-1}}{y_{0,n}}$, we have

$$\begin{aligned} y_{1,n}(1) &= \frac{1+y_{1,n}(1)}{y_{0,n}(1)} \\ &= 1+y'_{1,n-1} \end{aligned}$$

using point (a).

(c) Finally, we prove that, if $i+j=n+1$ and $i \geq 2$, then $p_{n-1,n}(y_{i,j}) = y'_{0,i} + y'_{i,n-i}$. Assume first $i=2$, then $y_{2,n-1}y_{1,n-1} = 1 + y_{1,n}y_{2,n-2}$ yields

$$\begin{aligned} y_{2,n-1}(1) &= \frac{1+y_{1,n}(1)y_{2,n-2}(1)}{y_{1,n-1}(1)} \\ &= \frac{1+(1+y'_{1,n-1})y'_{2,n-2}}{y'_{1,n-1}} \\ &= \frac{1+y'_{2,n-2}}{y'_{1,n-1}} + y'_{2,n-2} \\ &= y'_{0,1} + y'_{2,n-2} \end{aligned}$$

where we use points (a) and (b). Finally, if $i \geq 3$, then

$$\begin{aligned} y_{i+1,n-i}(1) &= \frac{1+y_{i+1,n-i-1}(1)y_{i,n-i+1}(1)}{y_{i,n-i}(1)} \\ &= \frac{1+y'_{i+1,n-i+1}(y'_{0,i}+y'_{i,n-i})}{y'_{i,n-i}} \\ &= \frac{1+y'_{i+1,n-i+1}y'_{i,n-i}}{y'_{i,n-i}} + y'_{i+1,n-i+1} \\ &= y'_{0,i+1} + y'_{i+1,n-i+1} \end{aligned}$$

This completes the proof of our claim.

Finally, it follows easily from (a), (b) and (c) that $\mathcal{A}^{n-1} = \mathbb{Z}[p_{n-1,n}(\mathcal{X}^n)]$, as asserted. □

We want to show that $p_{i,j}$ is a morphism of cluster algebras in the sense of Assem-Dupont-Schiffler in [2]. In order to define cluster morphisms, we recall the definitions of rooted cluster algebras and rooted cluster morphisms due to I. Assem, G. Dupont and R. Schiffler.

Definition 2.3. A seed is a triple $\Sigma = (X, ex, B)$ such that:

(1) X is a countable set of undeterminates over $\mathbb{Z}$, called the cluster of Σ;
(2) $ex \subset X$ is a subset of X whose elements are the exchangeable variables of Σ;
(3) $B = (b_{x,y})_{x,y \in X} \in M_X(\mathbb{Z})$ is a (locally finite) skew-symmetrisable matrix called the exchange matrix of Σ.

The elements of $X \backslash ex$ are called the *frozen variables*. Note that in the above definition, the matrix B can be replaced by a (locally finite) quiver without loops and 2-cycles.

Let $\Sigma = (X, ex, Q)$ be a seed. We say that $(x_1, x_2, \ldots, x_l)$ is Σ-admissible if x_1 is exchangeable in Σ and if, for every $i \geq 2$, the variable x_i is exchangeable in $\mu_{x_{i-1}} \circ \cdots \circ \mu_{x_1}(\Sigma)$. The mutations are made along finite admissible sequences of variables.

A rooted cluster algebra is defined similarly as the Fomin-Zelevinsky cluster algebras, but the definition of rooted cluster algebras authorises seeds whose clusters are empty. Such seeds are called empty seeds and by convention the rooted cluster algebra corresponding to an empty seed is $\mathbb{Z}$. The rooted cluster algebra is always viewed with its initial seed. To know more about the different points of view between Fomin-Zelevinsky cluster algebras and rooted cluster algebras, we refer to [2, Remark 1.7].

Let $\Sigma = (X, ex, Q)$ and $\Sigma' = (X', ex', Q')$ be two seeds and let $f : \mathcal{A}(\Sigma) \longrightarrow \mathcal{A}(\Sigma')$ be a map. A sequence $(x_1, x_2, \dots, x_l) \subset \mathcal{A}(\Sigma)$ is (f, Σ, Σ')-biadmissible if it is Σ-admissible and $(f(x_1), \dots, f(x_l))$ is Σ'-admissible. The following definition is due to Assem, Dupont and Schiffler.

Definition 2.4. A rooted cluster morphism from $\mathcal{A}(\Sigma)$ to $\mathcal{A}(\Sigma')$ is a ring homomorphism from $\mathcal{A}(\Sigma)$ to $\mathcal{A}(\Sigma')$ such that:

(CM1) $f(X) \subset X' \cup \mathbb{Z}$

(CM2) $f(ex) \subset ex' \cup \mathbb{Z}$

(CM3) For every (f, Σ, Σ')-biadmissible sequence $(x_1, x_2, \dots, x_l)$, we have $\mu_{x_l} \circ \cdots \circ \mu_{x_1,\Sigma}(y) = \mu_{f(x_l)} \circ \cdots \circ \mu_{f(x_1),\Sigma'}(f(y))$.

The rooted cluster algebras and the rooted cluster morphisms form a category denoted by *Clus*, see [2].

Let $\Sigma^n = (X^n, ex^n, Q^n)$, with $ex^n = X^n$, then Σ^n is a seed of the rooted cluster algebra $\mathcal{A}(\Sigma^n)$; the cluster algebra $\mathcal{A}(\Sigma^n)$ coincides with the cluster algebra $\mathcal{A}^n$. We also have $\Sigma^{n-1} = \Sigma^n \backslash \{x_n\}$, where the seed $\Sigma^n \backslash \{x_n\}$ is defined in [2, Section 6.2].

Corollary 2.5. *Each $\mathbb{Z}$-morphism $p_{i,j}$ is a surjective rooted cluster morphism.*

Proof. By Proposition 2.2, the map $p_{n-1,n}$ is a $\mathbb{Z}$-morphism induced by the specialization of x_n to 1. Because of [2, Proposition 6.10], the $\mathbb{Z}$-morphism $p_{n-1,n}$ is a surjective rooted cluster morphism. Since $p_{i,j} = p_{i,i+1} \circ p_{i+1,i+2} \circ \cdots \circ p_{j-1,j}$ and each $p_{n-1,n}$ is a surjective rooted cluster morphism, by [2, Proposition 2.5] $p_{i,j}$ is also a surjective rooted cluster morphism. $\square$

Corollary 2.6. *The family $(\mathcal{A}^i, p_{i,j})_{i,j\geq 1}$ forms a projective system of cluster algebras.*

We denote by $\mathcal{A} = \varprojlim \mathcal{A}^n$ the corresponding projective limit in the category of $\mathbb{Z}$-algebras, thus

$$\mathcal{A} = \{(a_n)_{n\geq 1} \in \prod_{n\geq 1} \mathcal{A}^n \mid p_{i,j}(a_j) = a_i\}.$$

We also denote by $p_i : \mathcal{A} \longrightarrow \mathcal{A}^i$ the canonical morphisms induced by the projective limit. Let a_l be a cluster variable of $\mathcal{A}^i$, then the element $(a_1, a_2, \dots, a_l, a_l, \dots)$ with $l \geq i$, is an element of $\mathcal{A}$ and $p_i(a_1, a_2, \dots, a_l, a_l, \dots) = a_i$; therefore p_i is an

epimorphism of $\mathbb{Z}$-algebras. The p_i are called the canonical projections morphisms. We thus have a commutative diagram

$$\begin{tikzcd}
\dots \arrow[r, two heads] & \mathcal{A}^i \arrow[r, two heads, "p_{i-1,i}"] & \dots \arrow[r, two heads] & \mathcal{A}^2 \arrow[r, two heads, "p_{1,2}"] & \mathcal{A}^1 \\
& \mathcal{A} \arrow[u, two heads, "p_i"] \arrow[urr, two heads, "p_2"] \arrow[urrr, two heads, "p_1"'] & & &
\end{tikzcd}$$

For more details about projective limits and their properties, we refer to [1, Chap.11].

It was shown that the category *Clus* admits countable coproducts [2, Lemma 5.1] and does not generally admit products [2, Proposition 5.4].

Definition 2.7. An element $(a_n)_{n\geq 1}$ of $\mathcal{A} = \varprojlim \mathcal{A}^n$ is ultimately constant if there exists $j \in \mathbb{N}$ such that $a_n = a_j$ for all $n \geq j$.

Let Q be a linear oriented quiver of type $\mathbb{A}_\infty$ with the unique source in 1. Here (X, Q) is a seed of the cluster algebra $\mathcal{B}$. We want to understand the relation between the cluster algebra $\mathcal{B}$ and the $\mathbb{Z}$-algebras $\mathcal{A}$. Because the algebra $\mathcal{A}$ is a projective limit of cluster algebras of finite type, it allows to express the cluster algebra $\mathcal{B}$ in term of $\mathcal{A}$. Our first result is the following.

Theorem 2.8. *The cluster algebra $\mathcal{B}$ is the proper $\mathbb{Z}$-subalgebra of the algebra $\mathcal{A}$, consisting of the ultimately constant elements of $\mathcal{A} = \varprojlim \mathcal{A}^n$.*

Proof. Assume that Q is the quiver having as underlying graph the infinite tree $\mathbb{A}_\infty$ with linear orientation and for unique source the vertex 1. Recall that $\mathcal{B} = \mathcal{A}(X, Q)$ is the cluster algebra associated to a seed (X, Q), where $X = \{x_n, n \geq 1\}$. Let $Y = \{y_n, n \geq 1\}$ be a new set of undeterminates whose elements are defined by $y_1 = (x_1, x_1, x_1, \dots)$, $y_2 = (1, x_2, x_2, x_2, \dots)$, $y_3 = (1, 1, x_3, x_3, x_3, \dots)$, $\dots$, $y_i = (1, 1, 1, \dots, 1, x_i, x_i, x_i, \dots), \dots$ By definition, all y_i are elements of $\mathcal{A}$ and the family $(y_n)_{n\geq 1}$ is algebraically independent. Let $\tilde{\mathcal{F}} = \mathbb{Q}(Y)$ be the field of rational functions over y_i (with rational coefficients), we call $\tilde{\mathcal{F}}$ the *ambient field*.

The cluster algebra $\tilde{\mathcal{A}} = \mathcal{A}(Y, Q)$ is the $\mathbb{Z}$-subalgebra of $\tilde{\mathcal{F}}$ generated by the set $\mathcal{Y}$ which is the union of all possible sets of variables obtained from Y by successive mutations. We define the map $\varphi : \mathcal{B} \longrightarrow \tilde{\mathcal{A}}$ by setting $\varphi(x_i) = y_i$ and we extend it to all cluster variables of $\mathcal{B}$ by respecting mutations, that is if $z_i = \mu_{x_{i_k}} \dots \mu_{x_{i_2}} \mu_{x_{i_1}}(x_i)$ then $\varphi(z_i) = \mu_{\varphi(x_{i_k})} \dots \mu_{\varphi(x_{i_2})} \mu_{\varphi(x_{i_1})}(\varphi(x_i))$. We extend again φ to an injective morphism of $\mathbb{Z}$-algebras. Thus φ is a monomorphism of $\mathbb{Z}$-algebras.

Let $x = p(x_1, x_2, \dots, x_k)$ be an element of $\mathcal{X}$ then $y = p(y_1, y_2, \dots, y_k)$ is an element of $\mathcal{Y}$. By definition we have $\varphi(x) = y$. This shows that the morphism φ is an isomorphism between $\mathcal{B}$ and $\varphi(\mathcal{B}) = \tilde{\mathcal{A}}$.

Each ultimately constant element of $\mathcal{A}$ belongs to $\tilde{\mathcal{A}}$. But the element $a = (x_1, x_1x_2, x_1x_2x_3, \dots, x_1x_2x_3 \dots x_k, \dots)$ is an element of $\mathcal{A}$ which does not belong to $\tilde{\mathcal{A}}$. Therefore $\tilde{\mathcal{A}}$ is a proper $\mathbb{Z}$-subalgebra of the algebra $\mathcal{A}$.

Now let $\underline{x} = \{q_n \mid n \geq 1\}$ be a cluster of $\mathcal{B}$, by the definition of φ we have $\varphi(\mu_{q_k}(\underline{x})) = \mu_{\varphi(q_k)}(\varphi(\underline{x}))$ and $\varphi(\underline{x})$ is a cluster of $\tilde{\mathcal{A}}$. It follows that φ is a cluster isomorphism. For more details about cluster isomorphisms, we refer to [2]. This completes the proof. □

Remark 2.9. *Let Q be a quiver of type $\mathbb{A}_\infty$ and Q^n the full sub-quiver of Q whose set of vertices is $Q_0^n = \{1, 2, \dots, n\}$. We denote by $\mathcal{B}'$ the cluster algebra of seed (X, Q) and $\mathcal{A}'^n$ the cluster algebra of seed (X^n, Q^n). We reproduce the above construction with $\mathcal{B}'$ playing the role of $\mathcal{B}$ and $\mathcal{A}'^n$ playing the role of $\mathcal{A}^n$. Then the Theorem 2.8 remains true for any cluster algebra of type $\mathbb{A}_\infty$.*

Corollary 2.10. *The Laurent phenomenon and the positivity theorem hold for the cluster algebra $\mathcal{B}$.*

Proof. Let a be a cluster variable of the cluster algebra $\mathcal{B}$. By Theorem 2.8, there exists a non-negative integer k such that $\varphi(a) = (a_1, a_2, \dots, a_k, a_k, \dots)$ is a cluster variable of $\tilde{\mathcal{A}}$ with $1 \leq i \leq k$ and $p_{i,j}(a_j) = a_i$. Since the element a_k is a cluster variable of $\mathcal{A}^k$, then $a_k = p(x_1, x_2, \dots, x_k)$ is the Laurent polynomial with nonnegative coefficients by the positivity [20, Theorem 1.1]. By the isomorphism introduced in the proof of Theorem 2.1, we have $P(y_1, y_2, \dots, y_k, y_k) = \varphi(P(x_1, x_2, \dots, x_k, x_k))$; Hence $a = P(x_1, x_2, \dots, x_k, x_k)$. Then the Laurent phenomenon and the positivity theorem hold for the cluster algebra $\mathcal{B}$. □

The above corollary was proved by using other technics in [19, Theorem 6.8] and [15, Proposition 3.2].

3. Cluster algebras arising from infinity-gon

Fomin, Shapiro and Thurston initiated a study of the cluster algebras arising from triangulations of a surface with boundary and finitely many marked points in [11]. In this approach, it was shown that the cluster algebra associated to a triangulation of a marked surface (S, M) depends only on the surface (S, M) and not on the choice of triangulation. This result is not true as we shall see for the case of the infinity-gon. Our objective in this section is to classify the cluster algebras arising from the infinity-gon.

3.1. Triangulations of the infinity-gon. In this subsection, we classify the triangulations of the infinity-gon $\mathbb{S}$ by using the notions of connected component and frozen arc which will be defined later.

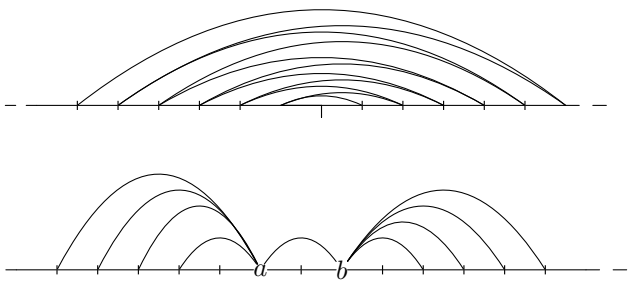
We adopt the same philosophy as that of [17], that is, we view the integers as the vertices of the infinity-gon and the pairs of integers as the arcs.

Let (m, n) be an arc of the infinity-gon, with $m < n$. If $n - m = 1$, we say that the arc (m, n) is a *boundary* arc, and if $n - m \geq 2$, we say that (m, n) is a *diagonal* of the infinity-gon. The diagonals of $\mathbb{S}$ will be simply called the *arcs*.

Two arcs (m, n) and (p, q) are said to *cross* if we have either $m < p < n < q$ or $p < m < q < n$. A *triangulation* of $\mathbb{S}$ is a maximal set of non-crossing arcs.

Definition 3.1. A triangulation T of $\mathbb{S}$ is called a *zigzag triangulation* if it is of the form $T = \{(-n + n_0, n + n_0), (-n + n_0 - 1, n + n_0)/n \geq 1\}$ or $T = \{(-n + n_0, n + n_0), (-n + n_0, n + n_0 + 1)/n \geq 1\}$, where n_0 is a given integer.

Example 3.2. *Let T and T' be the sets of arcs defined by $T = \{(-n, n), (-n, n + 1)/n \geq 1\}$ and $T' = \{(-n, -1), (-1, 1), (1, n)/n \geq 3\}$. One can show that T and T' are two triangulations of $\mathbb{S}$ whose illustrations are respectively the following:*



The triangulation T is called a zigzag triangulation.

The following definition is due to Holm and Jørgensen in [17].

Definition 3.3. Let T be a triangulation of $\mathbb{S}$.

(a) If for each integer n there are only finitely many arcs in T which are incident to n, then T is called *locally finite*.
(b) If n is an integer such that T contains infinitely many arcs of the form (m, n), then n is called a *left-fountain of T*.
(c) If n is an integer such that T contains infinitely many arcs of the form (n, p), then n is called a *right-fountain of T*.
(d) If n is both a left-fountain and a right-fountain of T, then it is called a *fountain*.

It is shown in [17] that if a triangulation of $\mathbb{S}$ has a right-fountain, then it also has a left-fountain and vice versa. The following result in [17, Lemma 3.3] characterizes the triangulations of infinity-gon.

Lemma 3.4. *Let T be a triangulation of $\mathbb{S}$. Then T has at most one right-fountain and at most one left-fountain.*

Now we introduce the notion of connected components of the triangulations of $\mathbb{S}$ before giving a classification of the triangulations of $\mathbb{S}$. Let T be a triangulation of $\mathbb{S}$ and γ an arc of T. We say that the arc γ' is the *flip* of the arc γ if the set $(T \backslash \{\gamma\}) \cup \{\gamma'\}$ is a triangulation of $\mathbb{S}$.

Following [19], we say that the arc $\omega = (s,t)$ *spans* the arc $\delta = (u,v)$ if $s \leq u < v < t$ or $s < u < v \leq t$. We denote by $\mathcal{B}(\omega)$ the set of all arcs which are spanned by the given arc ω.

Definition 3.5. Let T be a triangulation of $\mathbb{S}$, and τ an arc of T. We say that an arc γ is reachable by τ if for all sequence of arcs $\gamma_1, \gamma_2, \ldots, \gamma_k$, such that $\gamma_{j+1} \in f_{\gamma_j} \ldots f_{\gamma_1} f_\tau(T)$, with $0 \leq j \leq k-1$ and $\gamma \in f_{\gamma_k} \ldots f_{\gamma_1} f_\tau(T)$, then there exists an arc ω of T which spans the set $\{\tau, \gamma_1, \ldots, \gamma_k\}$.

If so we say that the sequence of flips $f_{\gamma_k}, \ldots, f_{\gamma_1}$ transforms τ to γ and we denote $\gamma = f_{\gamma_k} \ldots f_{\gamma_1}(\tau)$.

The *connected component* of an arc τ of T denoted C_τ is the set of all arcs which are reachable by τ. The connected components of a given arc of T is called simply the *connected component of* T. So two distinct connected components of T are disjoint.

An arc of T which can not be flipped to any other arc is called a *frozen* arc.

Lemma 3.6. *Any triangulation of* $\mathbb{S}$ *has at most one frozen arc.*

Proof. Let T be a triangulation of $\mathbb{S}$. Assume that T is locally finite. Let γ be an arc of $\mathbb{S}$, then there exists an arc ζ such that γ is an arc of the polygon $\mathbb{P}_\zeta$ bounded by ζ. The restriction T_ζ of the triangulation T to $\mathbb{P}_\zeta$ is a triangulation. Because γ is an arc of $\mathbb{P}_\zeta$, it is joined by a finite sequence of flips of arcs of T_ζ. Since $T_\zeta \subset T$, then γ is joined by a finite sequence of flips of arcs of T. Thus, each arc of $\mathbb{S}$ is reachable; hence T has no frozen arc. If T has a left-fountain m_0 and a right-fountain n_0 such that $n_0 - m_0 = 1$, then T has two connected components and no frozen arc. If T has a left-fountain m_0 and a right-fountain n_0 such that $n_0 - m_0 \geq 2$, then T has three connected components and the arc (m_0, n_0) is a frozen arc. Assume that T possesses another frozen arc (m_1, n_1), then (m_1, n_1) crosses an infinity of arcs of T incident to m_0 or an infinity of arcs of T incident to n_0.

Assume now that T is a triangulation with two frozen arcs ω_1 and ω_2. The arc ω_1 does not spans ω_2 and vice versa, because if not, then one of the two frozen arcs can be flipped to another arc. Each frozen arc bounds a finite connected component, and then it is finite. Therefore, the triangulation T has more than one left-fountain or more than one right-fountain. This is a contradiction to Lemma 3.4. $\square$

Let T be a triangulation of $\mathbb{S}$ with a frozen arc τ, we say that T is of Type $(III)_k$ with $k = |\mathcal{B}(\tau)|$, where is the number of arcs spanned by the frozen arc $\omega = (s,t)$, more precisely $k = |t - s - 1|$.

Definition 3.7. A triangulation T of $\mathbb{S}$ is called

a) of Type (I) if it has only one connected component.
b) of Type (II) if it has two connected components and no frozen arc.

c) of Type $(III)_k$ if it has two connected components and one frozen arc or if it has three connected components and one frozen arc, where k is nonnegative integer.

The following result gives a classification of the triangulations of $\mathbb{S}$ which uses the notions of connected components and the frozen arc.

Proposition 3.8. *Any triangulation of* $\mathbb{S}$ *is one of the three types* (I), (II), $(III)_k$.

Proof. Let T be a triangulation of $\mathbb{S}$. If T is locally finite, then every arc of $\mathbb{S}$ can be reached by a sequence of flips of arcs of T. Then T is of type (I). If T has a fountain or a left-fountain m_0 and a right-fountain n_0 with $n_0 - m_0 = 1$, then T has two connected components, and any arc of each component is reachable. In this case T has no frozen arc, hence T is of type(II).

If T has a left-fountain m_0 and a right-fountain n_0 with $n_0 - m_0 \geq 2$, then T has three connected components. The arc (m_0, n_0) is an arc of T. Assume that (m_0, n_0) is not in T; then one of the arcs $(m_0 + 1, n_0), (m_0, n_0 - 1)$ belongs to T and one of the arcs $(m_0 - 1, n_0), (m_0, n_0 + 1)$ belongs to T. If the arc $(m_0 + 1, n_0)$ belongs to T, then $(m_0 - 1, n_0) \in T$ and $(m_0 - 1, n_0)$ crosses an infinite number of arcs of T incident to a left fountain m_0. This is a contradiction because T is a triangulation. Similarly, if $(m_0, n_0 - 1)$ belongs to T, we get a contradiction. Hence the arc (m_0, n_0) is a frozen arc, and it is unique by Lemma 3.6. Thus T is of type $(III)_k$.

Assume now that T has l connected components, where $l \geq 4$. Then only one of the l components is finite, because if not, T would have more than one frozen arc, and this is a contradiction to Lemma 3.6. Therefore, the triangulation T has at least three infinite connected components. Each of the infinite connected components of T contains either a right-fountain or a left-fountain, thus T has more than two fountains, this contradicts Lemma 3.3 of [17]. □

Fomin, Shapiro and Thurston in [11] associated to a triangulation of a marked surface (S, M) a finite quiver without cycles of length at most two. Similarly, we associate to each triangulation of $\mathbb{S}$ an infinite quiver without cycles of length at most two.

Let T be a triangulation of the infinity-gon $\mathbb{S}$, we associate to T a quiver Q_T. The classification of quivers Q_T is given in [16, Theorem 3.11]. We associate to the triangulation T the cluster algebra $\mathcal{A}(T)$ of seed (X_T, Q_T) in the same way as the one for a marked surface (S, M).

Remark 3.9. *Proposition 3.8 is equivalent to the Theorem 3.11 of [16].*

An isomorphism f between two $\mathbb{Z}$-algebras is called a *strong isomorphism* of clusters algebras if f maps each cluster to a cluster and preserves mutations. For more details we refer to [13].

Example 3.10. *We consider* $T = \{(-n, n), (-n-1, n) \mid n \geq 1\}$ *and*

$$T' = \{(-n, 0), (0, n) \mid n \geq 2\},$$

two triangulations of $\mathbb{S}$. *The quiver associated to the triangulation* T *is the quiver* Q_T *given by:*

$$1 \longrightarrow 2 \longleftarrow 3 \longrightarrow \ldots \longleftarrow n-1 \longrightarrow n \longleftarrow \ldots$$

and the quiver associated to the triangulation T' *is the non connected quiver* $Q_{T'}$ *given by:*

$$2 \longrightarrow 4 \longrightarrow 6 \longrightarrow \ldots \longrightarrow 2n \longrightarrow 2n+2 \longrightarrow \ldots$$
$$1 \longleftarrow 3 \longleftarrow 5 \longleftarrow \ldots \longleftarrow 2n-1 \longleftarrow 2n+3 \longleftarrow \ldots .$$

There is no strong isomorphism between the cluster algebras $\mathcal{A}(T)$ *and* $\mathcal{A}(T')$. *This shows that the theorem of Fomin, Shapiro and Thurston mentioned above does not hold for the infinity-gon.*

3.2. The specificity of cluster algebras arising from the infinity-gon. In this section we find a criterion that allows us to decide whether two triangulations give rise to isomorphic cluster algebras. We denote $\overline{\mathbb{S}}$ the set of all arcs of $\mathbb{S}$.

Definition 3.11. Two triangulations T and T' are said to be *congruent* if there exists a bijection $\theta : \overline{\mathbb{S}} \longrightarrow \overline{\mathbb{S}}$ which maps T to T' and preserves the flips of arcs; that is $\theta(T) = T'$ and $\theta(f_\gamma) = f_{\theta(\gamma)}$, where f_γ is the flip of γ.

The bijection θ is called an *admissible* map. If T and T' are *congruent*, we denote by $T \simeq T'$. Congruence is an equivalence relation.

Example 3.12. *Let* Γ *be the triangulation defined by* $\Gamma = \{(-n, 0), (0, n) \mid n \geq 2\}$, *and* Γ' *the triangulation given by* $\Gamma' = \{(-n, 0), (-2, 0), (1, n) \mid n \geq 3\}$.

We observe that the sets $S_\Gamma = \{(m, n) \mid m \leq -1, n \geq 1\}$ *and*

$$S_{\Gamma'} = \{(-1, 1), (m, n), (0, n) \mid m \leq -1, n \geq 0\}$$

are respectively the complements of the triangulations Γ *and* Γ'. *Since* S_Γ *and* $S_{\Gamma'}$ *are countable sets, they are equipotent; that is there exists a bijection* ξ *which maps each element* (m, n) *of* S_Γ *to a unique element* $\xi(m, n)$ *of* $S_{\Gamma'}$.

We define the map $\theta : \overline{\mathbb{S}} \longrightarrow \overline{\mathbb{S}}$ *by:*

$$\begin{cases} \theta(m, n) = (m, n) \quad if \quad m \leq -2, n \leq 0 \\ \theta(m, n) = (m+1, n+1) \quad if \quad m \geq 0, n \geq 2 \\ \theta(m, n) = \xi(m, n) \qquad otherwise \end{cases}$$

The map θ *is a bijection such that* $\theta(\Gamma) = \Gamma'$ *and preserves the flips of arcs. The two triangulations* Γ *and* Γ' *are congruent.*

Let $(\gamma_n)_{n \geq 1}$ be a sequence of arcs in $\overline{\mathbb{S}}$, we say that the sequence $(\gamma_n)_{n \geq 1}$ spans $\overline{\mathbb{S}}$ if for any arc γ there exists an integer k such that γ is spanned by γ_k.

Lemma 3.13. *Let T be a triangulation of type (I), then there exist a zigzag triangulation Z and a sequence of common arcs $(\gamma_n)_{n\geq 1}$ in T and Z that spans $\overline{\mathbb{S}}$.*

Proof. Let T be a triangulation of type (I), we shall show that there exists a sequence of distinct arcs $(\gamma_{k_n})_{n\geq 1}$ of T such that $\gamma_{k_n} = (s_{k_n}, t_{k_n})$, where $s_{k_{n+1}} < s_{k_n} < 0$ and $t_{k_{n+1}} > t_{k_n} > 0$.

We assume first that any arc $\gamma = (s,t)$, with $s < 0$ and $t > 0$ does not belong to T. Because (s,t) does not belong to T and T is a triangulation, there is an arc $\gamma_1 = (s_1, t_1)$ of de T which crosses (s,t) and γ_1 is closer to a vertex t and $t_1 > t$. Analogously, there exists an arc $\gamma_2 = (s_2, t_2)$ of de T which crosses (s,t) and γ_2 is closer to a vertex s and $s_2 < s$. The connected components C_{γ_1} and C_{γ_2} respectively of γ_1 and γ_2 are distinct, this is a contradiction because T is of type (I).

We show that T has an infinite sequence of arcs $(\gamma_{k_n})_{n\geq 1}$ where $s_{k_{n+1}} < s_{k_n} < 0$ and $t_{k_{n+1}} > t_{k_n} > 0$. Assume that any sequence of arcs $(\gamma_{k_n})_{n\geq 1}$ with the above property is finite. Let $\gamma_{k_l} = (s_{k_l}, t_{k_l})$, with $s_{k_l} < 0$ and $t_{k_l} > 0$ be an arc of T such for all arc of $(\gamma_{k_n})_{n\geq 1}$, we have $s_{k_l} < s_{k_n}$ and $t_{k_l} > t_{k_n}$. The same argument used for γ to γ_{k_l} gives rise to a contradiction. Thus there exists a sequence of arcs $(\gamma_{k_n})_{n\geq 1}$ of T which can be chosen such that $s_{k_{n+1}} < s_{k_n<0}$ and $t_{k_{n+1}} > t_{k_n>0}$. For now we construct a zigzag triangulation having infinitely many common arcs with T. Because $(\gamma_{k_n})_{n\geq 1}$ is sequence of infinite non-crossing arcs, we can extend it in each polygon delimited by γ_{k_n} and $\gamma_{k_{n+1}}$. This extension gives rise to a new set of non-crossing arcs Z. In fact, Z is a triangulation of S by the construction, and each γ_{k_l} is a common arc of T and Z.

Finally, we show that $(\gamma_{k_n})_{n\geq 1}$ spans $\overline{\mathbb{S}}$. Let $\delta = (u,v)$ be an arc of $\mathbb{S}$, since $(\gamma_{k_n})_{n\geq 1}$ is infinite, there is an integer l such that the arc δ is spanned by γ_{k_l}. This completes the proof. □

Lemma 3.14. *Let T and T' be two triangulations of type (II) or T and T' be two triangulations of type $(III)_k$, and let C_T and $C_{T'}$ their connected components respectively. Then there is a sequence of common arcs $(\gamma_n)_{n\geq 1}$ in T and T' that spans C_T and $C_{T'}$.*

Proof. (i) Let T and T' be two triangulations of type (II), suppose that T has a left-fountain m_0 and a right-fountain n_0, $n_0 - m_0 = 1$. There is a triangulation Γ with one fountain such that its associated quiver Q_Γ is isomorphic to the associated quiver Q_T of T.

Because of the above argument, it is sufficient to give a proof just for the case where each triangulation has a left-fountain and a right-fountain. Let T and T' be two triangulations of type (II) such that T has a left-fountain m_0 and a right-fountain n_0 and T' has a left-fountain m_0' and a right-fountain n_0'. Because n_0 and n_0' are two integers, we can assume $n_0 \leq n_0'$; and there is a non negative integer l

such that $l = n_0' - n_0$. We define the map $\sigma : \overline{\mathbb{S}} \longrightarrow \overline{\mathbb{S}}$ by $\sigma(m,n) = (m+l, n+l)$. The map σ is a bijection and the image $\sigma(T)$ of T is a triangulation. Moreover, σ preserves the flips of arcs. Thus σ is an admissible map. Therefore, we can suppose without loss of generality that T and T' have the same left-fountain m_0 and the same right-fountain n_0. Because the triangulations T and T' have the same left-fountain m_0 and the same right-fountain n_0, there exists infinitely many common arcs of the form (n_0, n) and so, there exists infinitely many common arcs of the form (m, m_0). Thus there is a sequence of distinct common arcs $(\gamma_{k_n})_{n\geq 1}$ which spans C_T and $C_{T'}$.

(ii) Now we suppose that T and T' are two triangulations of type $(III)_k$. By using the argument above, we can assume without loss of generality that T and T' have the same left-fountain m_0 and the same right-fountain n_0.

If k is equal to zero, the proof is similar as in (i). If $k \geq 1$, the frozen arc ω bounds a polygon $\mathbb{P}_\omega$; and by the definition of k, each arc of the restriction of T to $\mathbb{P}_\omega$ is spanned by ω and each arc of the restriction of T' to $\mathbb{P}_\omega$ is spanned by ω. By combining this argument and the one used in (i) to define the common arcs, there exists a sequence of distinct arcs $(\gamma_{k_n})_{n\geq 1}$ which spans C_T and $C_{T'}$. □

Proposition 3.15. *Let T and T' be two triangulations of $\mathbb{S}$, then T and T' are congruent if and only if T and T' are of the same type.*

Proof. Let T and T' be two triangulations of $\mathbb{S}$, and assume that $T \simeq T'$. Since $T \simeq T'$, there exists an admissible map θ which maps T to T'. Let C_γ be a connected component of T, then $\theta(C_\gamma)$ is a connected component of T'. Because $\theta(\gamma) \in \theta(C_\gamma)$, we have $\theta(C_\gamma) = C_{\theta(\gamma)}$. If C_γ and C_δ are two distinct connected components of T, then $\theta(C_\gamma)$ and $\theta(C_\delta)$ are distinct connected components of T'. The number of connected components is invariant by θ. Moreover the connect components C_γ and $C_{\theta(\gamma)}$ are both either finite, or infinite. Moreover, if T has a finite component C_γ, then $|C_\gamma| = |C_{\theta(\gamma)}| = k$. Hence T and T' are of the same type. Conversely, let T and T' be two triangulations of $\mathbb{S}$, we have three cases.

(i) If T and T' are of the type (I), then by Lemma 3.13 there exists a zigzag triangulation Z and a sequence of common distinct arcs $(\gamma_{k_n})_{n\geq 1}$ in T and Z that spans $\overline{\mathbb{S}}$.

Let $\mathbb{P}_{k_n}$ be the polygon bounded by the arc γ_{k_n}. The restriction T_{k_n} and Z_{k_n} of the triangulations T and Z are triangulations of $\mathbb{P}_{k_n}$. We want to show that $T = \bigcup\limits_{n\geq 1} T_{k_n}$ and $Z = \bigcup\limits_{n\geq 1} Z_{k_n}$. Because $T_{k_n} \subset T$, then $\bigcup\limits_{n\geq 1} T_{k_n} \subset T$. For now let us show that $\bigcup\limits_{n\geq 1} T_{k_n}$ is a triangulation. Assume that $\bigcup\limits_{n\geq 1} T_{k_n}$ is not a triangulation, since $\bigcup\limits_{n\geq 1} T_{k_n} \subset T$ and the maximality of T implies the existence of arc $\varsigma \in T$ and ς does not belong to $\bigcup\limits_{n\geq 1} T_{k_n}$. Because the sequence $(\gamma_{k_n})_{n\geq 1}$ spans $\overline{\mathbb{S}}$, there exist

an integer l such that the arc ς is spanned by γ_{k_l}. Hence ς is an arc of T_{k_l}, this is a contradiction. Thus $T = \bigcup_{n\geq 1} T_{k_n}$. Analogously, one can show that $Z = \bigcup_{n\geq 1} Z_{k_n}$.

We know that T_{k_n} and Z_{k_n} are related by a sequence of flips. This sequence of flip induces an admissible map θ_n which maps T_{k_n} to Z_{k_n}. Let $\overline{\mathbb{P}}_{k_n}$ be the set of all arcs of $\mathbb{P}_{k_n}$, we have $\overline{\mathbb{P}}_{k_n} \subset \overline{\mathbb{P}}_{k_{n+1}}$. We have also $\bigcup_{n\geq 1} \overline{\mathbb{P}}_{k_n} = \overline{\mathbb{S}}$.

We define the map $\theta : \overline{\mathbb{S}} \longrightarrow \overline{\mathbb{S}}$ by the following: let γ be an element of $\overline{\mathbb{S}}$, then there is a minimal integer n such that $\gamma \in \overline{\mathbb{P}}_{k_n}$, we set $\theta(\gamma) = \theta_n(\gamma)$. By construction, θ is a bijection which maps T to Z and preserves the flips of arcs. Hence $T \simeq Z$.

We reproduce the same reasoning above with T' playing the role of T and Z' playing the role of Z.

If (m_0, n_0) is the arc of Z and (m'_0, n'_0) is the arc of Z' such that $n_0 - m_0 = 2 = n'_0 - m'_0$. We set $l = n'_0 - n_0$ and define the map $\sigma : \overline{\mathbb{S}} \longrightarrow \overline{\mathbb{S}}$ by $\sigma(m, n) = (m+l, n+l)$. Then σ is an admissible map which maps Z to Z'. Then $Z \simeq Z'$ and thus $T \simeq T'$.

(ii) If T and T' are of the type (II), because $C_T = C_{T'}$ we use Lemma 3.14 and we have a bijection θ in C_T which maps T to T' and preserves the flips of arcs. We define θ on each unreachable arc γ by $\theta(\gamma) = \gamma$. Then we have extended the bijection θ to $\overline{\mathbb{S}}$. Hence θ is an admissible map, thus $T \simeq T'$.

(iii) If T and T' are of the type (III_k), the restrictions of the triangulations T and T' to the polygon bounded by the ice arc are congruency. Because $C_T = C_{T'}$, by using the same principle as in (b), we construct an admissible map θ which maps T to T'. □

Corollary 3.16. *Let T and T' be two triangulations of $\mathbb{S}$, if T and T' are mutation equivalent, then T and T' are congruent.*

Proof. Let T and T' be two triangulations of $\mathbb{S}$. Assume that T and T' are flip equivalent. Because the triangulations T and T' are flip equivalence, they are of the same type. By Proposition 3.15, we have $T \simeq T'$. □

Remark 3.17. *The congruence relation generalize the notion of triangulations flip equivalent. Thus, the notion of congruency can be used for the triangulations of polygons.*

Now we are in position to proof our second main theorem.

Theorem 3.18. *Let T and T' be two triangulations of $\mathbb{S}$, $\mathcal{A}(T)$ and $\mathcal{A}(T')$ the associated cluster algebras. Then T and T' are congruent if and only if the clusters algebras $\mathcal{A}(T)$ and $\mathcal{A}(T')$ are strongly isomorphic.*

Proof. Let T and T' be two triangulations of $\mathbb{S}$, assume that $T \simeq T'$. Let $\mathcal{A}(T)$ be a cluster algebra of seed $(X_T, Q_T) := \sum_T$, where Q_T is a quiver of T and

$X_T = \{x_\gamma \mid \gamma \in T\}$ the set of undeterminates. Let $\mathcal{A}(T')$ be a cluster algebra of seed $(X_T, Q_{T'}) := \sum_T$ where $Q_{T'}$ is a quiver of T' and $X_{T'} = \{u_\lambda \mid \lambda \in T\}$ the set of undeterminates.

Since $T \simeq T'$, there exists an admissible bijection θ, such that $\theta(T) = T'$. We denote by f_γ the flip of the arc γ. We define $\varphi_\theta : \mathcal{A}(T) \longrightarrow \mathcal{A}(T')$ on the cluster variable by $\varphi_\theta(x_\gamma) = u_{\theta(\gamma)}$. We have in one hand

$$\begin{aligned} \varphi_\theta(x_T) &= \{\varphi_\theta(x_\gamma) \mid \gamma \in T\} \\ &= \{u_{\theta(\gamma)} \mid \gamma \in T\} \\ &= \{u_{\theta(\gamma)} \mid \theta(\gamma) \in \theta(T)\} \\ &= \{u_\lambda \mid \lambda \in T'\}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \varphi_\theta(\mu_{x_\gamma}) &= \varphi_\theta(x_{f(\gamma)}) \\ &= u_{\theta(f(\gamma)} \\ &= u_{f(\theta(\gamma))} \\ &= \mu_{u_{\theta(\gamma)}}(u_{\theta(\gamma)}). \end{aligned}$$

We extend φ_θ to an isomorphism of $\mathbb{Z}$-algebras from $\mathcal{A}(T)$ to $\mathcal{A}(T')$. Therefore $\mathcal{A}(T)$ and $\mathcal{A}(T')$ are strongly isomorphic.

Conversely assume that $\mathcal{A}(T)$ and $\mathcal{A}(T')$ are strongly isomorphic and that T and T' are not congruency. Then there exists a strong isomorphism $\psi : \mathcal{A}(T) \longrightarrow \mathcal{A}(T')$. According to Proposition 3.16, T and T' are not of the same type. We have four cases to enumerate.

(a) T is of type (I) and T' is of type (II). Because T' is of type (II), it has two disjoint connected components. Let u_{λ_1} and u_{λ_2} be the cluster variables such that λ_1 and λ_2 do not belong to the same connected component. Since ψ is a strong isomorphism, there are two arcs γ_1 and γ_2 such that $\psi(x_{\gamma_1}) = u_{\lambda_1}$ and $\psi(x_{\gamma_2}) = u_{\lambda_2}$. The two arcs γ_1 and γ_2 are related to a sequence of flips and then the two variables x_{γ_1} and x_{γ_2} are related to a sequence of mutations, because T is of type (I). In fact, ψ is a strong isomorphism, then u_{λ_1} and u_{λ_2} are related to a sequence of mutations. Thus λ_1 and λ_2 are related by a sequence of flip. This is a contradiction, because λ_1 and λ_2 do not belong to the same connected component.

(b) T is of type (I) and T' is of type $(III)_k$. The proof is analogous of the one in the case (a).

(c) T is of type (II) and T' is of type $(III)_k$. The triangulation T' has a frozen arc ω, then the cluster algebra $\mathcal{A}(T')$ has a frozen cluster variable u_ω in the sense of [2]. Because ψ is strong isomorphism, there is a frozen variable x_γ in $\mathcal{A}(T)$ such that $\psi(x_\gamma) = u_\omega$ and $\mathcal{A}(T)$ is without frozen variable. This is a contradiction.

(d) T is of type $(III)_k$ and T' is of type $(III)_{k'}$ with $k \neq k'$. We assume without loss of generality that $k > k'$. Let ω and ω' be respectively the frozen arcs of T and T'. Because ψ is a strong isomorphism, we have $\psi(x_\omega) = u_{\omega'}$. Since $k > k'$, there

is an arc γ spanned by ω such that $\psi(x_\gamma) = u_\lambda$, and λ is not spanned by ω'. The arc γ is related by a sequence of flips of arcs spanned by ω and the arc λ is not related by a sequence of flips of arcs spanned by ω. This is also a contradiction, because ψ is a strong isomorphism. □

Now we want to show that each cluster algebras of type $\mathbb{A}_\infty$ can be embedded in a cluster algebra arising from $\mathbb{S}$.

Lemma 3.19. *Let Q be a connected quiver mutation equivalent to a quiver of type $\mathbb{A}_\infty$. Then Q is not a quiver associated to a triangulation of $\mathbb{S}$ if and only if Q has a subquiver of type $\mathbb{A}_\infty$ with linear orientation.*

Proof. Assume that Q connected quiver and is mutation equivalent to a quiver of type $\mathbb{A}_\infty$ with non linear orientation. Because Q is a connected quiver of type $\mathbb{A}_\infty$, its underlying graph is an infinite half-path $\mathbb{A}_\infty$. Since Q is a quiver with no linear orientation, then Q is locally finite. Thus, Q can be associated to a locally finite triangulation of $\mathbb{S}$, that is there exists a triangulation T of $\mathbb{S}$ such that $Q_T \cong Q$.

Conversely, assume that Q has a subquiver of type $\mathbb{A}_\infty$ with linear orientation. It is sufficient to show that the quiver R: $1 \longrightarrow 2 \longrightarrow \dots$ is not the quiver associated to any triangulation. Suppose that there exists a triangulation T such that $Q_T = R$. We denote by τ_i the arc of T corresponding to the vertex i. All τ_i, where i is a non-negative integer have the same origin and are the arc of the same half-line. $T = \{\tau_i \mid i \geq 1\}$ is a triangulation of $\mathbb{S}$ with left-fountain, but not right-fountain. This is the contradiction see [17]. □

Corollary 3.20. *Let Q be a quiver mutation equivalent to a quiver of type $\mathbb{A}_\infty$. Let $\underline{u} = \{u_i \mid i \geq 1\}$ the set of undeterminates attached to a vertices of Q. Then there exists a seed $\Sigma_T = (\underline{x}_T, Q_T)$ associated to a triangulation T of $\mathbb{S}$ and an embedding $\eta : \mathcal{A}(\underline{u}, Q) \hookrightarrow \mathcal{A}(\Sigma_T)$.*

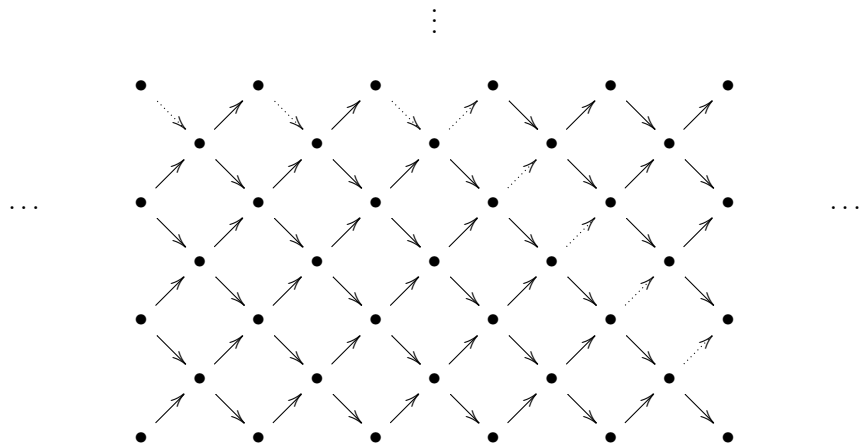
Proof. Assume that Q is a mutation equivalent to a quiver of type $\mathbb{A}_\infty$. If Q has a subquiver with orientation not necessarily linear, then Q is quiver associated to a triangulation of $\mathbb{S}$; hence the result.

Assume now that Q has a subquiver of type $\mathcal{A}_\infty$ with linear orientation. By Lemma 3.20, Q is not the quiver of any triangulation of $\mathbb{S}$. We defined the quiver R of type $\mathbb{A}_\infty$ with linear orientation distinct to the one of Q. The Quiver $Q \cup R$ is a quiver associate to a triangulation T of $\mathbb{S}$. The inclusion of the quiver $Q \subset Q \cup R$ induces an embedding of cluster algebras $\eta : \mathcal{A}(\underline{u}, Q) \hookrightarrow \mathcal{A}(\Sigma_T)$. □

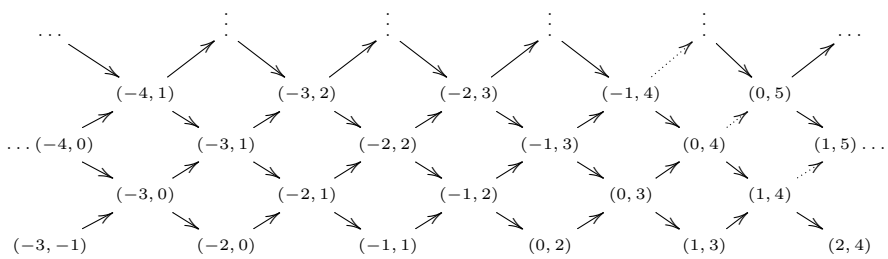
4. The cluster category of associated to $\mathbb{S}$

4.1. The infinite cluster category of type A_∞. We recall the description of the infinite cluster category given in [17, 18]. Let K be a field and $R = K[T]$ be the polynomial algebra. We view R as a differential graded algebra with zero differential

and T placed in homological degree 1. Then we set $D^f(R)$ be the derived category of differential graded R-modules with finite dimensional homology over K, then $\mathcal{D} = D^f(R)$ is the infinite cluster category of type A_∞. The suspension and the Serre functor of $\mathcal{D}$ are denoted by Σ and S respectively. The category $\mathcal{D}$ is a K-linear, Hom-finite, Krull-Schmidt, triangulated and 2-Calabi-Yau category whose Auslander-Reitein quiver is of the form $\mathbb{Z}A_\infty$, we refer to [18]. The Auslander-Reiten translation of $\mathcal{D}$ is $\tau = S\Sigma^{-1} = \Sigma$. For a given integer $r \geq 0$, we have a differential graded R-module $X_r = R/(T^{r+1})$ which is concentrated in homological degrees from 0 to r. The indecomposable objects of $\mathcal{D}$ are $\Sigma^j X_r$ for j, r integers, $r \geq 0$ and Σ the shift of $\mathcal{D}$. The Auslander-Reiten quiver $\Gamma(\mathcal{D})$ of $\mathcal{D}$ is of the form



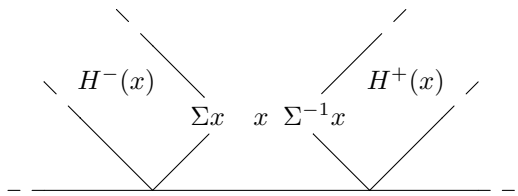
By using the identification $(m, n) := \Sigma^{-n} X_{n-m-2}$, we have the following representation of the quiver $\Gamma(\mathcal{D})$



The identification of the indecomposable objects $\Sigma^{-n} X_{n-m-2}$ of $\mathcal{D}$ given by $(m, n) := \Sigma^{-n} X_{n-m-2}$ is called the *standard coordinates* system on $\Gamma(\mathcal{D})$. The morphisms between indecomposable objects are described as follows: Let $x = \Sigma^{-j} X_{j-i-2}$ be a vertex of the Auslander-Reiten quiver of $\mathcal{D}$, we define the sets $H^-(x)$ and $H^+(x)$ of vertices of the Auslander-Reiten quiver as

$$H^-(x) = \{\Sigma^{-n} X_{n-m-2} / m \leq i-1, i+1 \geq n \leq j-1\}$$

and $H^+(x) = \{\Sigma^{-n}X_{n-m-2}/i+1 \geq m \leq j-1, j+1 \leq n\}$. We write $H(x) = H^-(x) \cup H^+(x)$. This situation can be sketched as follows.



Moreover we have $H^-(x) = \{\Sigma^{-n}X_{n-m-2}/m \leq i-1, i+1 \geq n \leq j-1\}$ and $H^+(x) = \{\Sigma^{-n}X_{n-m-2}/i+1 \geq m \leq j-1, j+1 \leq n\}$. We write $H(x) = H^-(x) \cup H^+(x)$.

The following proposition in [17] characterizes the morphisms of $\mathcal{D}$.

Proposition 4.1. *Let x and y be two indecomposable objects of $\mathcal{D}$. Then*

$$Hom_{\mathcal{D}}(x,y) = \begin{cases} K & \textit{if} \quad y \in H(\Sigma x) \\ 0 & \textit{if not} \end{cases}$$

The following remark is due to Holm and Jørgensen [17, Remark 2.4].

Remark 4.2. *There are two distinct types of non-zero morphisms going from x to indecomposable objects of $\mathcal{D}$: those going to objects in $H^+(x)$ are called forward morphisms, and those going to objects $H^-(x)$ are called backward morphisms.*

The forward morphisms have an easy model: up to multiplication by a nonzero scalar, they are induced by certain canonical morphisms of differential graded modules.

The *backward morphisms* cannot be seen in the Auslander-Reiten quiver; they are in the infinite radical of $\mathcal{D}$.

4.2. The category of diagonals of ∞-gon. In this section we provide a geometric realization of the category $\mathcal{D}$.

We adopt the same philosophy as that of [17], that is, the integers can be viewed as the vertices of the ∞-gon and the pairs of integers can be viewed as the arcs of the infinity-gon. Let (m,n) be an arc of ∞-gon, with $m < n$. If $n - m = 1$, we say that the arc (m,n) is a boundary arc, and if $m \leq n-2$, we say that (m,n) is a diagonal of the infinity-gon. Our construction is similar to that of [7] in the case of the $(n+3)$-gon.

One can define a combinatorial K-linear category $\mathcal{C}$ as follows:
The indecomposable objects are the arcs (m,n) of $\mathbb{S}$, with $m,n \in \mathbb{Z}$ and $m \leq n-2$; the objects of $\mathcal{C}$ are direct sums of the arcs and each arc is stable by the product of the scalars of K. The boundary arcs are identified to zero. The space of morphisms between two arcs (m,n) and (p,q) is given by:

$$Hom_{\mathcal{C}}((m,n),(p,q)) \quad = \quad \begin{cases} K & \texttt{if} \quad (p,q) \in F_R^{(m,n)} \cup F_L^{(m,n)} \\ 0 & \texttt{if not} \end{cases}$$

where $F_R^{(m,n)} = \{(l,k) \mid m \leq l \leq n-2, k \geq n\}$ and $F_L^{(m,n)} = \{(k,s) \mid m+2 \leq s \leq n, k \leq m\}$. The morphisms between two objects are direct sums of morphisms between arcs. The composition of morphisms between arcs is given by the product of scalars in K. The construction of $\mathcal{C}$ is inspired by the standard coordinates used in [17]. The category $\mathcal{C}$ is a category generated by all the diagonals of $\mathbb{S}$. Therefore by construction $\mathcal{C}$ is K-linear, Hom-finite and Krull-Schmidt. Our main result is the following.

Theorem 4.3. *The categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent.*

Proof. Let $F_0 : ind\mathcal{C} \longrightarrow ind\mathcal{D}$ be such that, for $(m,n) \in ind\mathcal{C}$ we have $F_0(m,n) = \Sigma^{-n} X_{n-m-2}$. According to [17], F_0 is a bijection. One can define the additive functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ as follows:

$F(m,n) = F_0(m,n)$, and we extend F by additivity and K-linearity to all objects of $\mathcal{C}$. Let $u_\alpha : (m,n) \longrightarrow (p,q)$ be a morphism of $\mathcal{C}$ which is identified with the scalar α of K.

We recall that, via the standard coordinates defined above, if $F(m,n) = x$ and $F(p,q) = y$, then $(p,q) \in F_R^{(m,n)}$ if and only if $y \in H^+(\Sigma x)$. We have also $(p,q) \in F_L^{(m,n)}$ if and only if $y \in H^-(\Sigma x)$.

On the one hand, if $(p,q) \in F_R^{(m,n)}$, then $y \in H^+(\Sigma x)$; let $f : x \longrightarrow y$ be a forward morphism of $\mathcal{D}$ that is f is induced by a canonical morphism of DG-modules. Then each morphism from x to y is of the form λf where $\lambda \in K$ and we set $F(u_\alpha) = \alpha f = f_\alpha$. On the other hand, if $(p,q) \in F_L^{(m,n)}$, then $y \in H^-(\Sigma x)$; let $\bar{g} : x \longrightarrow y$ be a backward morphism, because the category $\mathcal{D}$ is 2-Calabi-Yau that is $Hom_{\mathcal{D}}(x,y) = DHom_{\mathcal{D}}(y, S(x))$, where D=Hom$(-,K)$ is the usual duality. The morphism $\bar{g}$ is the isomorphic image of a forward morphism $g : y \longrightarrow \Sigma^2 x$. We set $F(u_\alpha) = \alpha\bar{g} = \bar{g}_\alpha$ and $F(1_{(m,n)}) = 1_x$. Let show now that F is a functor. Let $u_\alpha : (m,n) \longrightarrow (p,q)$ and $u_\beta : (p,q) \longrightarrow (r,s)$ where $F(m,n) = x$, $F(p,q) = y$ and $F(r,s) = z$. The proof is completed in three steps $(a), (b), (c)$.

(a) If $(p,q),(r,s) \in F_R^{(m,n)}$ and $(r,s) \in F_R^{(p,q)}$, then $y, z \in H^+(\Sigma x)$ and $z \in H^+(\Sigma y)$. We have $F(u_\alpha) = \alpha f$ and $F(u_\beta) = \beta g$, where $f : x \longrightarrow y$ and $g : y \longrightarrow z$ are forward morphisms. The morphism $u_\beta u_\alpha = u_{\beta\alpha}$ is a morphism from (m,n) to (p,q). Then $F(u_{\beta\alpha}) = \beta\alpha h$, where $h : x \longrightarrow z$ is a forward morphism of $\mathcal{D}$. According to [17, Lemma 2.5], the morphism g is a nonzero morphism and we have the following commutative triangle

$$\begin{tikzcd} x \arrow[rr, "h"] \arrow[dr, "f'"'] & & z \\ & y \arrow[ur, "g"'] & \end{tikzcd}$$

where f' is the morphism induced by a canonical morphism of differential graded modules. By uniqueness of the canonical morphism between two indecomposables objects, we have $f' = f$ and thus $F(u_{\beta\alpha}) = F(u_\beta)F(\alpha)$.

(b) If $(p,q),(r,s) \in F_g^{(m,n)}$ and $(r,s) \in F_d^{(p,q)}$, then $y,z \in H^-(\Sigma x)$ and $z \in H_+(\Sigma y)$. We have $F(u_\alpha) = \alpha\bar{f}$ and $F(u_\beta) = \beta g$ where $g : y \longrightarrow z$ is a morphism induced by a canonical morphism of differential graded modules. So, $\bar{f} : x \longrightarrow y$ is the isomorphic image of a morphism $f : y \longrightarrow \Sigma^2 x$ induced by a canonical morphism of differential graded modules. Since g is a nonzero morphism, in accordance with [17, Lemma 2.7], we have the following commutative triangle

$$\begin{tikzcd} x \arrow[rr, "\bar{h}"] \arrow[dr, "\bar{f}'"'] & & z \\ & y \arrow[ur, "g"'] & \end{tikzcd}$$

where $\bar{h}$ is the isomorphic image of a forward morphism h and $\bar{f}'$ is the image of the morphism $f' : y \longrightarrow \Sigma^2 x$ which is induced by the canonical morphism of DG-modules from y to $\Sigma^2 x$. By uniqueness of the canonical morphism between two indecomposables objects, we have $\bar{f}' = \bar{f}$ and hence $F(u_{\beta\alpha}) = F(u_\beta)F(\alpha)$. For all other cases not mentioned above, the composition of morphisms are equal to zero see [17, Corollary 2.3]. This show that F is a functor.

(c) F is essentially surjective because by the definition, each indecomposable module of $\mathcal{D}$ is the image of an arc of $\mathcal{C}$ under F. The map $F : \mathrm{Hom}_{\mathcal{C}}((m,n),(p,q)) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(x,y)$ which associates to u_α, the function $F(u_\alpha)$ is a bijection because of the step (a) and (b). Therefore F is full and faithful.

Finally, it follows from $(a),(b),(c)$ that F is an equivalence. □

We can give now the description of the category $\mathcal{C}$, via the equivalence established above; clearly, the category $\mathcal{C}$ is triangulated, 2-Calabi-Yau and has Auslander-Reiten triangles. In addition, the suspension is given by $(m,n)[1] = (m-1,n-1)$ and the Serre functor is given by $S(m,n) = (m-2,n-2)$; this situation was predictable from Holm and Jørgensen in [17].

We have also the following operations between the arcs of $\mathbb{S}$ defined by: ${}_s(m,n) = (m+1,n)$ and $(m,n)_e = (m,n+1)$. These operations are defined for the $n+3$-gon in [7] and for marked surfaces without punctures in [3]. The operations ${}_s(m,n)$ and $(m,n)_e$ can be extended as functors in the category $\mathcal{C}$.

Proposition 4.4. *The following statements are equivalent.*

(a) $Hom((i,j),(p,q)[1]) \neq 0$
(b) (i,j) *and* (p,q) *cross*
(c) $(p,q) = {}_{s^z}(i,j)_{e^n}$ *or* $(p,q) = {}_{s^{-n-2}}(i,j)_{e^{-r}}$ *where* $n \geq 0,\ 0 \leq z \leq l-2,\ 0 \leq r \leq l$ *and* $j - i = l$.

Proof. Let (i,j) and (p,q) be two arcs of $\mathcal{C}$. Then we have $\mathrm{Ext}^1((i,j),(p,q)) = \mathrm{Hom}((i,j),(p,q)[1])$. By Theorem 4.3 and [17, Lemma 3.5], $\mathrm{Hom}((i,j),(p,q)[1]) \neq 0$ if and only if the arcs (i,j) and (p,q) cross; that means, (a) and (b) are equivalent.

Now assume that $\mathrm{Hom}((i,j),(p,q)[1]) \neq 0$. By the definition of the morphisms spaces of $\mathcal{C}$, we have $m \leq p-1 \leq n-2$ and $n \leq q-1$, or $m+2 \leq q-1 \leq n$ and $q-1 \leq m$. Because i and j are integers, $l = j-l$ is a positive integer. If we consider the integers n, z, r such that $n \geq 0$, $0 \leq z \leq l-2$ and $0 \leq r \leq l$, then we have $(p,q) = (i+z, j+n)$ or $(p,q) = (i-n-2, j-r)$. By definition, we have ${}_s(i,j) = (i+1,j)$ and $(i,j)_e = (i,j+1)$, thus $(p,q) = {}_{s^z}(i,j)_{e^n}$ or $(p,q) = {}_{s^{-n-2}}(i,j)_{e^{-r}}$. It follows that (a) and (c) are equivalent. $\square$

Corollary 4.5. *Let (m,n) be a diagonal of the infinity-gon, then there is an Auslander-Reiten triangle in $\mathcal{C}$ as follows*

$$(m,n) \longrightarrow {}_s(m,n) \oplus (m,n)_e \longrightarrow {}_s(m,n)_e \longrightarrow (m,n)[1].$$

Moreover, all Auslander-Reiten triangles of $\mathcal{C}$ are of this form.

Proof. It is shown in [18] that the following triangle

$$\Sigma^{-n}X_u \longrightarrow \Sigma^{-n}X_v \oplus \Sigma^{-n}X_{n-m-1} \longrightarrow \Sigma^{-n-1}X_u \longrightarrow \Sigma^{-n+1}X_u$$

is an Auslander-Reiten in $\mathcal{D}$, where $u = n-m-2, v = n-m-3$. By using the equivalence F of Theorem 4.3, we have

$$(m,n) \longrightarrow (m+1,n) \oplus (m,n+1) \longrightarrow (m,n+1) \longrightarrow (m-1,n-1).$$

That is

$$(m,n) \longrightarrow {}_s(m,n) \oplus (m,n)_e \longrightarrow {}_s(m,n)_e \longrightarrow (m,n)[1].$$

Assuming now that

$$(m,n) \longrightarrow \bigoplus_{i=1}^{n}(m_i,n_i) \longrightarrow (p,q) \longrightarrow (m,n)[1]$$

is an Auslander-Reiten triangle of $\mathcal{C}$. Since F is an equivalence of categories,

$$F(m,n) \longrightarrow \bigoplus_{i=1}^{n} F(m_i,n_i) \longrightarrow (p,q) \longrightarrow F((m,n)[1])$$

is an Auslander-Reiten triangle of $\mathcal{D}$; that is

$$\Sigma^{-n}X_{n-m-2} \longrightarrow \bigoplus_{i=1}^{l} \Sigma^{-n_i}X_{n_i-m_i-2} \longrightarrow \Sigma^{-q}X_{q-p-2} \longrightarrow \Sigma^{-n+1}X_{n-m-2}$$

is an Auslander-Reiten triangle of $\mathcal{D}$. The form of the Auslander-Reiten triangle of $\mathcal{D}$ is well known; by identification, we have $\Sigma^{-q}X_{q-p-2} = \Sigma^{-n-1}X_{n-m-2}$. There exist r,s with $1 \leq r,s \leq l$ such that $F(m_r,n_r) = \Sigma^{-n}X_{n-m-3}, F(m_s,n_s) = \Sigma^{-n}X_{n-m-1}$ and $F(m_i,n_i) = 0$, for all i different from r and s. This completes the proof of our assertion. $\square$

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References

[1] J. Adámek, H. Herrlich and G. E. Strecker, Abstract and Concrete Categories, Pure and Applied Mathematics, a Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1990.

[2] I. Assem, G. Dupont and R. Schiffler, *On a category of cluster algebras*, J. Pure Appl. Algebra, 218(3) (2014), 553-582.

[3] T. Brüstle and J. Zhang, *On the cluster category of a marked surface without punctures*, Algebra Number Theory, 5(4) (2011), 529-566.

[4] A. B. Buan, O. Iyama, I. Reiten and J. Scott, *Cluster structures for 2-Calabi-Yau categories and unipotent groups*, Compos. Math., 145(4) (2009), 1035-1079.

[5] A. B. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov, *Tilting theory and cluster combinatorics*, Adv. Math., 204(2) (2006), 572-618.

[6] P. Caldero and F. Chapoton, *Cluster algebras as Hall algebras of quiver representations*, Comment. Math. Helv., 81(3) (2006), 595-616.

[7] P. Caldero, F. Chapoton and R. Schiffler, *Quivers with relations arising from clusters (A_n case)*, Trans. Amer. Math. Soc., 358(3) (2006), 1347-1364.

[8] P. Caldero and B. Keller, *From triangulated categories to cluster algebras*, Invent. Math., 172(1) (2008), 169-211.

[9] G. Dupont, *Transverse quiver Grassmannians and bases in affine cluster algebras*, Algebra Number Theory, 4(5) (2010), 599-624.

[10] V. Fock and A. Goncharov, *Moduli spaces of local systems and higher Teichmller theory*, Publ. Math. Inst. Hautes Études Sci., 103 (2006), 1-211.

[11] S. Fomin, M. Shapiro and D. Thurston, *Cluster algebras and triangulated surfaces. I. Cluster complexes*, Acta. Math., 201(1) (2008), 83-146.

[12] S. Fomin and A. Zelevinsky, *Cluster algebras I: Foundations*, J. Amer. Math. Soc., 15(2) (2002), 497-529.

[13] S. Fomin and A. Zelevinsky, *Cluster algebras II: Finite type classification*, Invent. Math., 154(1) (2003), 63-121.

[14] M. Gekhtman, M. Shapiro and A. Vainshtein, *Cluster algebras and Poisson geometry*, Mosc. Math. J., 3(3) (2003), 899-934.

[15] M. Gorsky, *On Young diagrams, flips and cluster algebras of type A*, arXiv:1106.2458.

[16] J. E. Grabowski and S. Gratz, *Cluster algebras of infinite rank*, J. Lond. Math. Soc., 89(2) (2014), 337-363.

[17] T. Holm and P. Jørgensen, *On a cluster category of infinite Dynkin type, and the relation to triangulations of the infinity-gon*, Math. Z., 270(1-2) (2012), 277-295.

[18] P. Jørgensen, *Auslander-Reiten theory over topological spaces*, Comment. Math. Helv., 79(1) (2004), 160-182.

[19] P. Jørgensen and Y. Palu, *A Caldero-Chapoton map for infinite clusters*, Trans. Amer. Math. Soc., 365(3) (2013), 1125-1147.

[20] K. Lee and R. Schiffler, *Positivity for cluster algebras*, Ann. of Math., 182 (2015), 73-125.

[21] Y. Palu, *Cluster characters for 2-Calabi-Yau triangulated categories*, Ann. Inst. Fourier (Grenoble), 58(6) (2008), 2221-2248.

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