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LIE CENTRAL TRIPLE RACKS

Guy Roger Biyogmam

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Abstract. This paper introduces Lie $\mathfrak{c}$-triple racks. These triple systems generalize both left and right racks to ternary algebras, and locally differentiates to gb-triple systems [3].

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1. Introduction

The generalization of Lie algebras to algebras such as Lie triple systems, Jordan triple systems [8] and 3-Lie algebras [7] suggests a natural generalization of Leibniz algebras (non commutative Lie algebras) [11] to ternary algebras. One generalization is provided by Leibniz 3-algebras [5] for which the characteristic identity expresses the adjoint action as a derivation of the algebra. A second generalization of Leibniz algebras to ternary algebras is provided by Leibniz triple systems [4]. They are defined in such a way that Lie triple systems are a particular case. Recently, the author introduced gb-triple systems [3], another generalization of Leibniz algebras in which the ternary operation $T : \mathfrak{g}^{\otimes 3} \to \mathfrak{g}$ expresses the map $T_{a,b}(x) = T(a, x, b)$ as a derivation of $\mathfrak{g}$ for all $a, b \in \mathfrak{g}$.

The local integration problem of these algebras generated from Lie's third theorem, which states that every finite-dimensional Lie algebra over the real numbers is associated with a Lie group. Partial solutions to this problem for Leibniz algebras (dubbed by Loday as the Coquecigrue problem) have been provided by several authors (see M. Kinyon [10], S. Covez [6]). The author extended Kinyon's results to Leibniz 3-algebras using Lie 3-racks [2]. In this paper we open the problem of integration of gb-triple systems. We follow Kinyon's approach [10] to open a path to a solution by defining an algebraic structure that locally differentiates to a gb-triple systems. We refer to these algebras as Lie $\mathfrak{c}$-triple racks. They appear to generalize both left and right Lie racks to ternary algebras; a particularity not supported by Lie 3-racks.

For the remainder of this paper, we assume that $\mathfrak{K}$ is a field of characteristic different to 2.

2. $\mathfrak{c}$-Triple racks

In this section we define $\mathfrak{c}$-triple racks and provide some examples. We also provide funtorial connections with the category of groups and the category of racks.

Recall that a *gb-triple system* [3] is a $\mathfrak{K}$ -vector space $\mathfrak{g}$ equipped with a trilinear operation $[-,-,-]_{\mathfrak{g}} : \mathfrak{g}^{\otimes 3} \longrightarrow \mathfrak{g}$ satisfying the identity

$$[A,B,[X,C,Y]_{\mathfrak{g}}]_{\mathfrak{g}} = [X,[A,B,C]_{\mathfrak{g}},Y]_{\mathfrak{g}} - [[X,A,Y]_{\mathfrak{g}},B,C]_{\mathfrak{g}} - [A,[X,B,Y]_{\mathfrak{g}},C]_{\mathfrak{g}} \quad (1)$$

Definition 2.1. A $\mathfrak{c}$-*triple rack* $(R,[-,-,-]_R)$ is a set R together with a ternary operation $[-,-,-]_R : R \times R \times R \longrightarrow R$ satisfying the following conditions:

(1) $[x,[a,b,c]_R,y]_R = [[x,a,y]_R,[x,b,y]_R,[x,c,y]_R]_R$ ($\mathfrak{c}$-distributive property),

(2) Given $a,c,d \in R$, there exists a unique $x \in R$ such that $[a,x,c,]_R = d$.

Definition 2.2. A $\mathfrak{c}$-triple rack $(R,[-,-,-]_R)$ is said to be *pointed* if there is a distinguished element $1 \in R$ satisfying

$$[1,y,1]_R = y \text{ and } [a,1,b]_R = 1 \quad \text{for all } a,b \in R.$$

A $\mathfrak{c}$-triple rack is said to be a *weak* $\mathfrak{c}$-*triple quandle* if it satisfies

$$[x,x,x]_R = x \text{ for all } x \in R.$$

A $\mathfrak{c}$-triple rack is a $\mathfrak{c}$-*triple quandle* if it satisfies

$$[a,y,b]_R = y \text{ if } a = y \text{ or } b = y.$$

Note that these generalize the notions of racks and quandles [9] to ternary operations. It is also clear that $\mathfrak{c}$-triple quandles are weak $\mathfrak{c}$-triple quandles but the converse is not true. See Example 2.4.

Definition 2.3. Let R and R' be two $\mathfrak{c}$-triple racks. A function $\alpha : R \longrightarrow R'$ is said to be a *homomorphism of* $\mathfrak{c}$-*triple racks* if

$$\alpha([x,y,z]_R) = [\alpha(x),\alpha(y),\alpha(z)]_{R'} \text{ for all } x,y,z \in R.$$

This provides a category ${}_{c}pRACK$ of pointed $\mathfrak{c}$-triple racks and pointed $\mathfrak{c}$-triple rack homomorphisms.

Example 2.4. *Let* $\Gamma := \mathbf{Z}[t^{\pm 1},s]/(2s^2+ts-s)$. *Any* Γ-*module* M *together with the ternary operation* $[-,-,-]_M$ *defined by*

$$[a,b,c]_M = sa + tb + sc$$

is a $\mathfrak{c}$*-triple rack. Indeed,*

$$\begin{aligned}[][x,a,y]_R,[x,b,y]_R,[x,c,y]_R]_R &= s(sx+ta+sy)+t(sx+tb+sy)+s(sx+tc+sy)\\ &= (2s^2+st)(x+y)+ts(a+c)+t^2b\\ &= [x,[a,b,c]_R,y]_R \text{ since } 2s^2+st=s.\end{aligned}$$

Therefore the $\mathfrak{c}$*-distributive property is satisfied. For the second axiom, given* $a,c,d\in R$ *one checks that* $x:=t^{-1}(d-s(a+c))$ *uniquely satisfies* $[a,x,c]_M=d$*. Note that* M *is a weak* $\mathfrak{c}$*-triple quandle that is not a* $\mathfrak{c}$*-triple quandle.*

Example 2.5. *Let* G *be a group with identity 1, and define on* G *the operation* $[-,-,-]_G$ *by*

$$[a,b,c]_G = acbc^{-1}a^{-1}.$$

Then $(G,[-,-,-],1)$ *is a pointed weak* $\mathfrak{c}$*-triple quandle. Indeed, we have on one hand*

$$[x,[a,b,c]_G,y]_G = xy[a,b,c]_G y^{-1}x^{-1} = xyacb^{-1}c^{-1}a^{-1}y^{-1}x^{-1}.$$

On the other hand,

$$\begin{aligned}[][x,a,y]_G,[x,b,y]_G,[x,c,y]_G]_G &= [xyay^{-1}x^{-1},xyby^{-1}x^{-1},xycy^{-1}x^{-1}]_G\\ &= xyay^{-1}x^{-1}xycy^{-1}x^{-1}xyb^{-1}y^{-1}x^{-1}xyc^{-1}y^{-1}x^{-1}xya^{-1}y^{-1}x^{-1}\\ &= xyacb^{-1}c^{-1}a^{-1}y^{-1}x^{-1} \text{ by cancellation.}\end{aligned}$$

Therefore the $\mathfrak{c}$*-distributive property is satisfied. For the second axiom, given* $a,c,d\in G$ *one checks that* $x:=c^{-1}a^{-1}dac$ *uniquely satisfies* $[a,x,c]_M=d$*. Finally, it is clear that* $[x,x,x]_R=x$ *for all* $x\in R$*.*

As a consequence, we have the following:

Proposition 2.6. *There is a faithful functor* $\mathfrak{F}$ *from the category of groups to the category of pointed* $\mathfrak{c}$*-triple racks.*

Proof. Define $\mathfrak{F}$ by $\mathfrak{F}(G)=(G,[-,-,-],1)$ as in Example 2.5. Its left adjoint $\mathfrak{F}'$ is defined as follows: Given a pointed $\mathfrak{c}$-triple rack R, consider the quotient group

$$G_R = < R > /I$$

where $< R >$ is the free group on R and I is the normal subgroup of $< R >$ generated by the set $\{(a^{-1}c^{-1}b^{-1}ca)([a,b,c]_R):\ \text{with } a,b,c\in R\}$. Indeed, given a morphism of $\mathfrak{c}$-triple racks $\alpha: R\longrightarrow \mathfrak{F}(G)$, there is a unique morphism of groups $\beta :< R >\longrightarrow G$ such that $\alpha=\beta|_R$ by the universal property of free groups. So

$$\beta((a^{-1}c^{-1}b^{-1}ca)([a,b,c]_R)) = \alpha((a^{-1}c^{-1}b^{-1}ca)([a,b,c]_R)) = 1 \text{ for all } a,b,c\in R.$$

Now by the universal property of quotient groups, there is a unique morphism of groups $\alpha_* : \mathfrak{F}'(R) \longrightarrow G$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathfrak{F}'(R) & \xrightarrow{\alpha_*} & G \\ \uparrow & & \uparrow{\scriptstyle id} \\ R & \xrightarrow{\alpha} & \mathfrak{F}(G) \end{array}$$

□

Example 2.7. *Let $(R, \circ, 1)$ be a pointed rack. Then define on R the ternary operation by*

$$[a, b, c]_R = a \circ (c \circ b).$$

It is easy to show that $(R, [-, -, -], 1)$ is a pointed $\mathfrak{c}$-triple rack.

As a consequence we have the following:

Proposition 2.8. *There is a faithful functor $\mathfrak{H}$ from the category of pointed racks to the category of $\mathfrak{c}$-triple racks.*

Proof. Define $\mathfrak{H}$ by $\mathfrak{H}\big((R, \circ, 1)\big) = (R, [-, -, -], 1)$ as in Example 2.7. Now, given a pointed $\mathfrak{c}$-triple rack $(R, [-, -, -], 1)$, one easily checks that the set $R^{\times(3)}$ together with the binary operation

$$(a, b, c) \circ (x, y, z) = ([a, x, c]_R, [a, y, c]_R, [a, z, c]_R)$$

is a rack pointed at $(1, 1, 1)$. We then define the left adjoint $\mathfrak{H}'$ of $\mathfrak{H}$ by

$$\mathfrak{H}'\big((R, [-, -, -], 1)\big) = (R^{\times(3)}, \circ, (1, 1, 1)).$$

□

Let us observe that in the proof of Proposition 2.8 the set $R^{\times(3)}$ is a quandle if R is a $\mathfrak{c}$-triple quandle.

3. From Lie $\mathfrak{c}$-triple racks to gb-triple systems

In this section we define the notion of Lie $\mathfrak{c}$-triple racks. We show that the tangent functor T_1 locally (at a specific point) maps Lie $\mathfrak{c}$-triple racks to gb-triple system.

Definition 3.1. A pointed $\mathfrak{c}$-triple rack $(R, [-, -, -]_R, 1)$ is called a *Lie $\mathfrak{c}$-triple rack* if the underlying set R is a differentiable manifold and the ternary operation $[-, -, -]_R : R \times R \times R \longrightarrow R$ is a smooth mapping.

Note that this definition appears to extend both left and right Lie racks [1] to ternary operations.

Example 3.2. *Let G be a Lie group endowed with the operation*

$$[a,b,c]_G = acbc^{-1}a^{-1}.$$

It follows by Example 2.5 that G is a Lie $\mathfrak{c}$-triple rack.

Example 3.3. *Let $(H, \{-,-,-\})$ be a group endowed with an antisymmetric ternary operation and V an H-module. Define the ternary operation $[-,-,-]_R$ on $R := V \times H$ by*

$$[(a,A),(b,B),(c,C)]_R := (\{A,B,C\}b, ACBC^{-1}A^{-1}),$$

where $a,b,c \in V$ and $A,B,C \in H$. Then $(R, [-,-,-]_R, (0,1))$ is a Lie $\mathfrak{c}$-triple rack.

For a pointed $\mathfrak{c}$-triple rack R, consider

$$Aut(R) := \left\{\xi : R \to R,\ \xi \text{ smooth bijection}:\ \xi([a,b,c]_R) = [\xi(a),\xi(b),\xi(c)]_R\right\}$$

and let $\phi : R \times R \times R \to R$ be the mapping given by $\phi(a,b,c) = [a,b,c]_R$. We have as a consequence of the second axiom of Definition 2.1 that the map $D : R \times R \longrightarrow Aut(R)$, $(a,c) \mapsto D(a,c) = \phi_{(a,c)}$ where $\phi_{(a,c)}(x) = [a,x,c]_R$ for all $x \in R$, is well-defined differentiable map. Let $D_* : \mathfrak{g} \times \mathfrak{g} \longrightarrow gl(\mathfrak{g})$ be the induced map of tangent spaces, where $\mathfrak{g} := T_1R$ is the tangent space of R at the point 1 and $gl(\mathfrak{g})$ is the Lie algebra associated to $GL(\mathfrak{g})$. Define a trilinear bracket on $\mathfrak{g}$ by

$$[X,Y,Z]_{\mathfrak{g}} = D_*(X,Z)(Y).$$

Proposition 3.4. *Let $(R, [-,-,-]_R, 1)$ be a pointed $\mathfrak{c}$-triple rack. Then $D(a,c) \in Aut(R)$ for all $a,c \in R$.*

Proof.

$$\begin{aligned}
D(a,c)([x,y,x]_R) &= \phi(a,c)([x,y,z]_R)\\
&= [a,[x,y,z]_R,c]_R\\
&= [[a,x,c]_R,[a,y,c]_R,[a,z,c]_R]_R \text{ by Definition 2.1}\\
&= [\phi_{(a,c)}(x),\phi_{(a,c)}(y),\phi_{(a,c)}(z)]_R\\
&= [D(a,c)(x),D(a,c)(y),D(a,c)(z)]_R.
\end{aligned}$$

□

Remark 3.5. *Note that for all $a,c \in R$, R acts on itself (considered as a differentiable manifold) via the maps $\phi_{(a,c)}$ by Proposition 3.4. Also, $\phi_{(a,c)}(1) = [a,1,c]_R = 1$. So the tangent functor T_1 applied to $\phi_{(a,c)} : R \longrightarrow R$ yields a linear map $\phi_{(a,c)*} : T_1R \longrightarrow T_1R$. Since $\phi(a,c) \in Aut(R)$ by Proposition 3.4, it follows that $\phi_{(a,c)*} \in GL(T_1R)$. Now let $X \in T_1R$ and denote by $X_1 := \phi_{(a,c)*}(X)$ the vector field extension of X. Then X_1 is generated by a one-parameter family of* *diffeomorphisms* $\gamma_X : \mathbb{R} \to R$ *with initial point* $\gamma_X(0) = 1$ *and initial tangent vector* $d\gamma_X(0) = X$*. The corresponding exponential map (see* [12, Chapter 9]*) denoted* $exp_1 : T_1(R) \to R$ *is then defined by* $exp_1(X) = \gamma_X(1)$.

Theorem 3.6. *Let* $(R, [-,-,-]_R, 1)$ *be a Lie* $\mathfrak{c}$*-triple rack and* $\mathfrak{g} := T_1R$*. Then for all* $a, c \in R$*, the tangent mapping* $\phi_{(a,c)*} = T_1(\phi_{(a,c)})$ *is an automorphism of* $\mathfrak{g}$.

Proof. Let $X, Y, Z \in \mathfrak{g}$ and let x, y, z be respectively the images of X, Y and Z by the exponential map exp_1 (see Remark 3.5). By the $\mathfrak{c}$-distributive property of $\mathfrak{c}$-triple racks, we have

$$\phi_{(a,c)}(\phi_{(x,z)}(y)) = \phi_{(\phi_{(a,c)}(x),\phi_{(a,c)}(z))}(\phi_{(a,c)}(y))$$

which when successively differentiated at $1 \in R$ with respect to the parameter γ_Y then γ_Z then γ_X yields

$$\phi_{(a,c)*}([X,Y,Z]_{\mathfrak{g}}) = [\phi_{(a,c)*}(X), \phi_{(a,c)*}(Y), \phi_{(a,c)*}(Z)]_{\mathfrak{g}} \quad (2).$$ □

Theorem 3.7. *Let* R *be a Lie* $\mathfrak{c}$*-triple rack and* $A, C \in \mathfrak{g} := T_1R$*. Let* a, c *be respectively the images of* A *and* C *by the exponential map* exp_1*.Then the mapping* $D_{(A,C)*} : \mathfrak{g} \longrightarrow gl(\mathfrak{g})$ *is a derivation of* $\mathfrak{g}$*. Moreover,* $D_{(A,C)*}$ *is exactly* $T_1(\Phi)$*, where* Φ *is the mapping* $\Phi : R \times R \longrightarrow GL(\mathfrak{g})$ *defined by* $\Phi(a,c) = \phi_{(a,c)*}$.

Proof. From the proof of Theorem 3.6, $\phi_{(a,c)*} \in GL(\mathfrak{g})$. In addition, we have $\phi_{(1,1)*} = I$, where I is the identity of $GL(\mathfrak{g})$. Now differentiating Φ at $(1,1)$ gives a map $T_{(1,1)}(\Phi) : T_1(R \times R) \longrightarrow gl(\mathfrak{g})$. Also differentiating the identity (2) above at $(1,1)$ with respect to $\gamma_{(A,C)}$ yields

$$\begin{aligned} D_{(A,C)*}(D_{(X,Z)*}(Y)) &= [A, [X,Y,Z]_{\mathfrak{g}}, C]_{\mathfrak{g}} \\ &= [[A,X,C]_{\mathfrak{g}}, Y, Z]_{\mathfrak{g}} + [X, [A,Y,C]_{\mathfrak{g}}, Z]_{\mathfrak{g}} + [X, Y[A,Z,C]_{\mathfrak{g}}]_{\mathfrak{g}} \\ &= D_{(D_{(A,C)*}(X),Z)*}(Y) + D_{(X,Z)*}(D_{(A,C)*}(Y)) \\ &\quad + D_{(X,D_{(A,C)*}(Z))*}(Y). \end{aligned}$$

Hence $D_{(A,C)*}$ is a derivation of $\mathfrak{g}$ and the map $T_1(\Phi)$ is exactly $D_{(A,C)*}$. □

From the calculations performed in the proofs of Theorem 3.6 and Theorem 3.7, we deduce that the ternary operation $[-,-,-]_{\mathfrak{g}}$ satisfies the identity (1). We then have the following result:

Corollary 3.8. *Let* R *be a Lie* $\mathfrak{c}$*-triple rack and* $\mathfrak{g} := T_1R$*. Then there exists a trilinear map* $[-,-,-]_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ *such that* $(\mathfrak{g}, [-,-,-]_{\mathfrak{g}})$ *is a gb-triple system.*

Remark 3.9. *Let G be the Lie $\mathfrak{c}$-triple rack of Example 3.2 and l the Lie algebra associated to the underlying group G. Then the bracket of the gb-triple system $\mathfrak{g} = T_1(R)$ can be written in terms of the bracket of the Lie algebra l as*

$$[X, Y, Z]_{\mathfrak{g}} = [Y, [X, Z]_l]_l.$$

To check that $[-,-,-]_{\mathfrak{g}}$ satisfies the identity (1), let $X, Y, A, B, C \in \mathfrak{g}$; we have on one hand

$$\begin{aligned}[X, Y, [A, B, C]_{\mathfrak{g}}]_{\mathfrak{g}} + [[A, X, C]_{\mathfrak{g}}, Y, B]_{\mathfrak{g}} &= [Y, [X, [A, B, C]_{\mathfrak{g}}]_l]_l + [Y, [[A, X, C]_{\mathfrak{g}}, B]_l]_l \\ &= \big[Y, [X, [B, [A, C]_l]_l]_l + [[X, [A, C]_l]_l, B]_l\big]_l \\ &= \big[Y, [[X, B]_l, [A, C]_l]_l\big]_l.\end{aligned}$$

On the other hand,

$$\begin{aligned}[A, [X, Y, B]_{\mathfrak{g}}, C]_{\mathfrak{g}} - [X, [A, Y, C]_{\mathfrak{g}}, B]_{\mathfrak{g}} &= \big[[X, Y, B]_{\mathfrak{g}}, [A, C]_l\big]_l - \big[[A, Y, C]_{\mathfrak{g}}, [X, B]_l\big]_l \\ &= \big[[Y, [X, B]_l]_l, [A, C]_l\big]_l - \big[[Y, [A, C]_l]_l, [X, B]_l\big]_l.\end{aligned}$$

The equality holds by the Jacoby identity.

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Guy Roger Biyogmam
Department of Mathematics
Southwestern Oklahoma State University
100 Campus Drive
Weatherford, OK 73096, USA
e-mail: guy.biyogmam@swosu.edu