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MINIMAXNESS PROPERTIES OF EXTENSION FUNCTORS OF LOCAL COHOMOLOGY MODULES

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Abstract. Let $\mathfrak{a}$ be an ideal of a commutative Noetherian ring R and M an R-module. In this paper, it is shown that if $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is $\mathfrak{a}$-minimax for all $i \geq 0$, then $M/\mathfrak{a}^n M$ is $\mathfrak{a}$-minimax for all $n \geq 0$. Several applications of this result are given. Among other things, we provide a proof of the equivalence of the $\mathfrak{a}$-minimaxness of the R-modules $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$, $\mathrm{Tor}^R_i(R/\mathfrak{a}, M)$ and $H^i(x_1, \ldots, x_t; M)$, for all $i \geq 0$, where $x_1, \ldots, x_t$ are generators for $\mathfrak{a}$. Using this, we show that, if $\mathfrak{b} \supseteq \mathfrak{a}$ is an ideal of R such that M is $\mathfrak{b}$-minimax and $\mathrm{cd}(\mathfrak{b}, R) = 1$, then for every finitely generated R-module L with $\mathrm{Supp}\, L \subseteq V(\mathfrak{b})$, the R-modules $\mathrm{Ext}^j_R(L, H^i_{\mathfrak{a}}(M))$ are $\mathfrak{b}$-minimax for all i and j. As a consequence, it follows that $H^i_{\mathfrak{a}}(M)/\mathfrak{b}^n H^i_{\mathfrak{a}}(M)$ are $\mathfrak{b}$-minimax R-modules for all i and n.

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1. Introduction

We continue the study of minimax modules with respect to an ideal $\mathfrak{a}$ of a commutative Noetherian ring R. In [6], H. Zöschinger, introduced the interesting class of minimax modules, and he has in [6] and [7] given many equivalent conditions for a module to be minimax. The R-module M is said to be a *minimax module*, if there is a finitely generated submodule N of M, such that M/N is Artinian. The concepts of $\mathfrak{a}$-minimax and $\mathfrak{a}$-cominimax modules were introduced in [1] as generalization of minimax and $\mathfrak{a}$-cofinite modules. We say that an R-module M is $\mathfrak{a}$-*minimax* if the $\mathfrak{a}$-relative Goldie dimension of any quotient module of M is finite. Recall that, an R-module M is said to have finite Goldie dimension (written $G\dim M < \infty$), if M does not contain an infinite direct sum of non-zero submodules, or equivalently the injective hull $E(M)$ of M decomposes as a finite direct sum of indecomposable (injective) submodules. Also, an R-module M is said to have finite $\mathfrak{a}$-relative Goldie

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dimension if the Goldie dimension of the $\mathfrak{a}$-torsion submodule $\Gamma_{\mathfrak{a}}(M)$ of M is finite. It is known that if M is $\mathfrak{a}$-torsion, then M is $\mathfrak{a}$-minimax if and only if M is minimax (see [1, Remark 2.2(ii)]). In addition, we say that an R-module M is $\mathfrak{a}$-*cominimax* if the support of M is contained in $V(\mathfrak{a})$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is $\mathfrak{a}$-minimax for all $i \geq 0$.

A brief summary of the contents of this paper will now be given. In Section 2, it is shown that if M is an $\mathfrak{a}$-cominimax R-module, then the R-modules $M/\mathfrak{a}^n M$ are $\mathfrak{a}$-minimax for all $n \in \mathbb{N}$ (see Theorem 2.3). Several applications of this result are given. Among other things, we provide a proof of the equivalence of the $\mathfrak{a}$-minimaxness of the R-modules $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$, $\mathrm{Tor}^R_i(R/\mathfrak{a}, M)$ and $H^i(x_1, \dots, x_t; M)$, for all $i \geq 0$, in Theorem 2.7, where $x_1, \dots, x_t$ are generators for $\mathfrak{a}$ and $H^i(x_1, \dots, x_t; M)$ is the i^{th} Koszul cohomology module of M with respect to $x_1, \dots, x_t$. This theorem is then used to deduce the change of rings principle for $\mathfrak{a}$-cominimax modules (see Theorem 2.10).

Moreover, in this section by using Theorems 2.3 and 2.7 we show that, if M is an $\mathfrak{a}$-cominimax R-module, then for any finitely generated R-module L with support in $V(\mathfrak{a})$, the R-modules $\mathrm{Ext}^i_R(L, M)$ and $\mathrm{Tor}^R_i(L, M)$ are $\mathfrak{a}$-minimax, for all i. Also, we give a sufficient condition for $\mathfrak{a}$-cominimax modules, and it is shown that if for an R-module M with $\mathrm{Supp}\, M \subseteq V(\mathfrak{a})$, there exists $x \in \sqrt{\mathfrak{a}}$ such that $0 :_M x$ and M/xM are both $\mathfrak{a}$-minimax, then M is $\mathfrak{a}$-cominimax. Finally, we prove that if $\mathfrak{b}$ is a second ideal of R with $\mathfrak{b} \supseteq \mathfrak{a}$, $\mathrm{cd}(\mathfrak{b}, R) = 1$, and M is a $\mathfrak{b}$-minimax R-module, then for every finitely generated R-module L with $\mathrm{Supp}\, L \subseteq V(\mathfrak{b})$, the R-modules $\mathrm{Ext}^j_R(L, H^i_{\mathfrak{a}}(M))$ are $\mathfrak{b}$-minimax for all i and j, and so the R-modules $H^i_{\mathfrak{a}}(M)/\mathfrak{b}^n H^i_{\mathfrak{a}}(M)$ are $\mathfrak{b}$-minimax for all i and n.

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity, and $\mathfrak{a}$ will be an ideal of R. The i^{th} local cohomology module of an R-module M with respect to $\mathfrak{a}$ is defined by

$$H^i_{\mathfrak{a}}(M) = \varinjlim_{n \geq 1} \mathrm{Ext}^i_R(R/\mathfrak{a}^n, M).$$

We refer the reader to [4] or [2] for the basic properties of local cohomology.

2. The results

The purpose of this section is to prove if $\mathfrak{a}$ is an ideal of a commutative Noetherian ring R and M is an $\mathfrak{a}$-cominimax module over R, then the R-modules $M/\mathfrak{a}^n M$ are $\mathfrak{a}$-minimax for all $n \in \mathbb{N}$ (see Theorem 2.3). Further, several applications of this result are given.

The following lemmas are needed in the proof of Theorem 2.3

Lemma 2.1. *Let M be an R-module such that* $\mathrm{Hom}_R(R/\mathfrak{a}, M)$ *is an* $\mathfrak{a}$*-minimax* R*-module. Then* $\mathrm{Hom}_R(R/\mathfrak{a}^n, M)$ *is* $\mathfrak{a}$*-minimax for all* $n \in \mathbb{N}$.

Proof. We use induction on n. When $n = 1$, there is nothing to prove. Now, let $n > 1$ and suppose that the result has been proved for $n - 1$. Consider the exact sequence

$$0 \longrightarrow 0 :_M \mathfrak{a} \longrightarrow 0 :_M \mathfrak{a}^n \stackrel{f}{\longrightarrow} a_1(0 :_M \mathfrak{a}^n) \oplus \cdots \oplus a_t(0 :_M \mathfrak{a}^n),$$

where $\mathfrak{a} = (a_1, \dots, a_t)$ and $f(x) = (a_1x, \dots, a_tx)$. As, $a_i(0 :_M \mathfrak{a}^n)$ is a submodule of $0 :_M \mathfrak{a}^{n-1}$, it follows from [1, Proposition 2.3] that $a_i(0 :_M \mathfrak{a}^n)$ is $\mathfrak{a}$-minimax for all $i = 1, \dots, t$. Now the result follows from [1, Corollary 2.4 and Proposition 2.3]. □

Lemma 2.2. *Let M be an R-module such that $M/\mathfrak{a}M$ is* $\mathfrak{a}$*-minimax. Then $M/\mathfrak{a}^nM$ is* $\mathfrak{a}$*-minimax for all* $n \in \mathbb{N}$.

Proof. We use induction on n. The case $n = 1$ is true by hypothesis. Now, let $n > 1$ and suppose that the result has been proved for $n - 1$. By [1, Corollary 2.4] and induction hypothesis, $(M/\mathfrak{a}^{n-1}M)^k$ is $\mathfrak{a}$-minimax, for all integers $k \geq 0$. Now consider the exact sequence

$$(M/\mathfrak{a}^{n-1}M)^t \stackrel{f}{\to} M/\mathfrak{a}^nM \stackrel{g}{\to} M/\mathfrak{a}M \to 0,$$

where $\mathfrak{a} = (a_1, \dots, a_t)$ and

$$f(m_1 + \mathfrak{a}^{n-1}M, \dots, m_t + \mathfrak{a}^{n-1}M) = a_1m_1 + \cdots + a_tm_t + \mathfrak{a}^nM.$$

Therefore, by [1, Proposition 2.3], $M/\mathfrak{a}^nM$ is $\mathfrak{a}$-minimax. □

Now, we are prepared to present the following theorem which plays a key role in this paper.

Theorem 2.3. *Let M be an R-module such that* $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ *is an* $\mathfrak{a}$*-minimax* R*-module for all $i \geq 0$. Then $M/\mathfrak{a}^nM$ is* $\mathfrak{a}$*-minimax for all* $n \in \mathbb{N}$.

Proof. In view of Lemma 2.2, it is enough to prove that $M/\mathfrak{a}M$ is $\mathfrak{a}$-minimax. To do this, let $\mathfrak{a} = (x_1, \dots, x_n)$. Then

$$M/\mathfrak{a}M \simeq H^n(x_1, \dots, x_n; M),$$

where $H^n(x_1, \dots, x_n; M)$ denotes the n^{th} Koszul cohomology module. Consider the co-Koszul complex $K^\bullet(\mathbf{x}, M)$ as the following:

$$0 \to \mathrm{Hom}_R(K_0(\mathbf{x}), M) \to \mathrm{Hom}_R(K_1(\mathbf{x}), M) \to \cdots \to \mathrm{Hom}_R(K_n(\mathbf{x}), M) \to 0.$$

Then $H^i(x_1, \ldots, x_n; M) = Z^i/B^i$, where B^i and Z^i are the modules of coboundaries and cocycles of the complex $K^\bullet(\mathbf{x}, M)$, respectively. Put

$$\mathcal{C} = \{N \mid \operatorname{Ext}^i_R(R/\mathfrak{a}, N) \text{ is } \mathfrak{a}\text{-minimax for all } i \geq 0\}.$$

By induction we claim that $B^j \in \mathcal{C}$ for all j. We have $B^0 = 0 \in \mathcal{C}$. Now, let $B^t \in \mathcal{C}$. Put $C^i = \operatorname{Hom}_R(K_i(\mathbf{x}), M)/B^i$. Since $K_t(\mathbf{x})$ is a finitely generated free R-module, it follows from [1, Corollary 2.4] that $\operatorname{Hom}_R(K_t(\mathbf{x}), M) \in \mathcal{C}$. Now, since $B^t \in \mathcal{C}$ and $\operatorname{Hom}_R(K_t(\mathbf{x}), M) \in \mathcal{C}$, we have $C^t \in \mathcal{C}$ by [1, Proposition 2.3]. Hence

$$0 :_{C^t} \mathfrak{a} \simeq \operatorname{Hom}_R(R/\mathfrak{a}, C^t)$$

is $\mathfrak{a}$-minimax. Because of $\mathfrak{a}H^t(x_1, \ldots, x_n; M) = 0$, it follows that

$$H^t(x_1, \ldots, x_n; M) \subseteq 0 :_{C^t} \mathfrak{a},$$

and so $H^t(x_1, \ldots, x_n; M)$ is $\mathfrak{a}$-minimax. Consequently, from the short exact sequence

$$0 \to H^t(x_1, \ldots, x_n; M) \to C^t \to B^{t+1} \to 0$$

and [1, Proposition 2.3] we deduce that $B^{t+1} \in \mathcal{C}$. Hence by induction we have proved that $B^j \in \mathcal{C}$ for all j. Now, since $B^n \in \mathcal{C}$ and $\operatorname{Hom}(K_n(\mathbf{x}), M) \in \mathcal{C}$, we obtain that $C^n \in \mathcal{C}$ by [1, Proposition 2.3]. Hence $0 :_{C^n} \mathfrak{a} \simeq \operatorname{Hom}_R(R/\mathfrak{a}, C^n)$ is $\mathfrak{a}$-minimax. Thus $H^n(x_1, \ldots, x_n; M) \subseteq 0 :_{C^n} \mathfrak{a}$ implies that $H^n(x_1, \ldots, x_n; M)$ is $\mathfrak{a}$-minimax. On the other hand, since $M/\mathfrak{a}M = H^n(x_1, \ldots, x_n; M)$, it follows that $M/\mathfrak{a}M$ is $\mathfrak{a}$-minimax. □

Remark 2.4. *We note that if* $\dim R = 0$*, then each* $\mathfrak{a}$*-cominimax* R*-module* M *is* $\mathfrak{a}$*-minimax. In fact, as* $\operatorname{Supp} M \subseteq V(\mathfrak{a})$ *and* R *is Artinian, it follows that* $M = 0 :_M \mathfrak{a}^n$*, and so* M *is* $\mathfrak{a}$*-minimax by Lemma* 2.1*.*

In general, we have the following.

Corollary 2.5. *Let* M *be an* $\mathfrak{a}$*-cominimax* R*-module. Then* $M/\mathfrak{a}^nM$ *is* $\mathfrak{a}$*-minimax for all* $n \in \mathbb{N}$*.*

Proof. The assertion follows from the definition and Theorem 2.3. □

Corollary 2.6. *Let* $\mathfrak{a}$ *be an ideal of* R*, and let* M *be an* R*-module such that* $H^i_{\mathfrak{a}}(M)$ *is* $\mathfrak{a}$*-cominimax for all* i*. Then* $M/\mathfrak{a}^nM$ *is* $\mathfrak{a}$*-minimax for all* $n \in \mathbb{N}$*.*

Proof. Since $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cominimax for all i, in view of [1, Proposition 3.7] the R-module $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is $\mathfrak{a}$-minimax for all i. Now the result follows from Theorem 2.3. □

The next theorem provides a proof of the equivalence of the $\mathfrak{a}$-minimaxness of the R-modules $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$, $\mathrm{Tor}^R_i(R/\mathfrak{a}, M)$ and $H^i(x_1, \dots, x_t; M)$, for all $i \geq 0$.

Theorem 2.7. *Let $\mathfrak{a} = (x_1, \dots, x_t)$ be an ideal of R, and let M be an R-module. Then the following statements are equivalent:*

(i) $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ *is an $\mathfrak{a}$-minimax R-module for all i.*
(ii) $\mathrm{Tor}^R_i(R/\mathfrak{a}, M)$ *is an $\mathfrak{a}$-minimax R-module for all i.*
(iii) *The Koszul cohomology modules $H^i(x_1, \dots, x_t; M)$ are $\mathfrak{a}$-minimax R-modules for all i.*

Proof. (i) $\Longrightarrow$ (ii) Let

$$\mathbb{F}_\bullet : \quad \cdots \to F_2 \to F_1 \to F_0 \to R/\mathfrak{a} \to 0$$

be a free resolution of finitely generated R-modules for $R/\mathfrak{a}$. Then it follows that $\mathrm{Tor}^R_i(R/\mathfrak{a}, M) = Z_i/B_i$, where B_i and Z_i are the modules of boundaries and cycles of the complex $\mathbb{F}_\bullet \otimes_R M$, respectively. Put

$$\mathcal{C} = \{N \mid \mathrm{Ext}^i_R(R/\mathfrak{a}, N) \ \text{ is } \ \mathfrak{a}\text{-minimax for all } \ i \geq 0\}.$$

By induction we claim that $Z_j \in \mathcal{C}$ for all j. We have $Z_0 = F_0 \otimes_R M \in \mathcal{C}$. Now let $Z_t \in \mathcal{C}$. Consider the exact sequence

$$0 \to C_{i+1} \to Z_i \to \mathrm{Tor}^R_i(R/\mathfrak{a}, M) \to 0, \qquad (\dagger)$$

where $C_i = (F_i \otimes_R M)/Z_i$. Hence we obtain the exact sequence

$$Z_i/\mathfrak{a}Z_i \to \mathrm{Tor}^R_i(R/\mathfrak{a}, M) \to 0.$$

Therefore, $\mathrm{Tor}^R_t(R/\mathfrak{a}, M)$ is a homomorphic image of $Z_t/\mathfrak{a}Z_t$. Now, since $Z_t \in \mathcal{C}$, it follows from Theorem 2.3 that $Z_t/\mathfrak{a}Z_t$ is $\mathfrak{a}$-minimax, and so $\mathrm{Tor}^R_t(R/\mathfrak{a}, M)$ is $\mathfrak{a}$-minimax. Hence, we deduce from ($\dagger$) that $C_{t+1} \in \mathcal{C}$, and so $Z_{t+1} \in \mathcal{C}$. Hence by induction we have proved that $Z_j \in \mathcal{C}$ for all j. It follows from Theorem 2.3 that $Z_i/\mathfrak{a}Z_i$ is $\mathfrak{a}$-minimax for all i, and so $\mathrm{Tor}^R_i(R/\mathfrak{a}, M)$ is $\mathfrak{a}$-minimax for all i.

To prove the implication (ii) $\Longrightarrow$ (iii), as

$$H^i(x_1, \dots, x_t; M) \simeq H_{n-i}(x_1, \dots, x_t; M),$$

it is sufficient to show that $H_i(x_1, \dots, x_t; M)$ is $\mathfrak{a}$-minimax for all i. Let $\mathbf{x} = x_1, \dots, x_n$. Consider the Koszul complex

$$K_\bullet(\mathbf{x}) : 0 \to K_n(\mathbf{x}) \to K_{n-1}(\mathbf{x}) \to \cdots \to K_1(\mathbf{x}) \to K_0(\mathbf{x}) \to 0.$$

Then $H_i(x_1,\dots,x_t;M)=Z_i/B_i$, where B_i and Z_i are the modules of boundaries and cycles of the complex $K_\bullet(\mathbf{x})\otimes_R M$, respectively. Put

$$\mathcal{C}=\{N \mid \operatorname{Tor}_i^R(R/\mathfrak{a},N) \ \text{ is } \ \mathfrak{a}\text{-minimax for all } \ i\geq 0\}.$$

Consider the exact sequence

$$0\to C_{i+1}\to Z_i\to H_i(x_1,\dots,x_t;M)\to 0,$$

where $C_i=(K_i(\mathbf{x})\otimes_R M)/Z_i$. Hence we obtain the exact sequence

$$Z_i/\mathfrak{a}Z_i\to H_i(x_1,\dots,x_t;M)\to 0.$$

Now, analogous to the proof of the implication (i) $\Longrightarrow$ (ii), $Z_i\in\mathcal{C}$ for all i. It follows that $Z_i/\mathfrak{a}Z_i=\operatorname{Tor}_0^R(R/\mathfrak{a},Z_i)$ is $\mathfrak{a}$-minimax for all i, and so $H_i(x_1,\dots,x_t;M)$ is $\mathfrak{a}$-minimax for all i.

Finally, to prove the implication (iii) $\Longrightarrow$ (i), let

$$\mathbb{F}_\bullet: \quad \cdots\to F_2\to F_1\to F_0\to R/\mathfrak{a}\to 0$$

be a free resolution of finitely generated R-modules for $R/\mathfrak{a}$. Then it follows that $\operatorname{Ext}_R^i(R/\mathfrak{a},M)=Z^i/B^i$, where B^i and Z^i are the modules of coboundaries and cocycles of the complex $\operatorname{Hom}_R(\mathbb{F}_\bullet,M)$, respectively. Put

$$\mathcal{C}=\{N \mid H^i(x_1,\dots,x_t;N) \ \text{ is } \ \mathfrak{a}\text{-minimax for all } \ i\geq 0\}.$$

Consider the short exact sequence

$$0\to \operatorname{Ext}_R^i(R/\mathfrak{a},M)\to C^i\to B^{i+1}\to 0,$$

where $C^i=\operatorname{Hom}_R(F_i,M)/B^i$. Then in view of the proof of Theorem 2.3, $B^i\in\mathcal{C}$ for all i. Thus $C^i\in\mathcal{C}$ for all i. Now, since

$$\operatorname{Ext}_R^i(R/\mathfrak{a},M)\subseteq 0:_{C^i}\mathfrak{a}\simeq \operatorname{Hom}_R(R/\mathfrak{a},C^i)\simeq H^0(x_1,\dots,x_t;C^i)$$

and $H^0(x_1,\dots,x_t;C^i)$ is $\mathfrak{a}$-minimax, we see that $\operatorname{Ext}_R^i(R/\mathfrak{a},M)$ is $\mathfrak{a}$-minimax for all i. □

The following result is an extension of Theorem 2.3.

Theorem 2.8. *Let M be an R-module such that $\operatorname{Ext}_R^i(R/\mathfrak{a},M)$ is an $\mathfrak{a}$-minimax R-module for all $i\geq 0$. Then for any finitely generated R-module L with support in $V(\mathfrak{a})$, the R-modules $\operatorname{Ext}_R^i(L,M)$ and $\operatorname{Tor}_i^R(L,M)$ are $\mathfrak{a}$-minimax for all i.*

Proof. Since $V(\operatorname{Ann}_R L) \subseteq V(\mathfrak{a})$, there exists $n \in \mathbb{N}$ such that $\mathfrak{a}^n L = 0$. Hence $\mathfrak{a}^n \operatorname{Ext}^i_R(L, M) = 0$ and $\mathfrak{a}^n \operatorname{Tor}^R_i(L, M) = 0$ for all i. Let

$$\mathbb{F}_\bullet : \quad \cdots \to F_2 \to F_1 \to F_0 \to L \to 0$$

be a free resolution of finitely generated R-modules for L. Then $\operatorname{Ext}^i_R(L, M) = Z^i/B^i$, where B^i and Z^i are the modules of coboundaries and cocycles of the complex $\operatorname{Hom}_R(\mathbb{F}_\bullet, M)$, respectively. Put

$$\mathcal{C} = \{N \mid \operatorname{Ext}^i_R(R/\mathfrak{a}, N) \text{ is } \mathfrak{a}\text{-minimax for all } i \geq 0\},$$

and consider the short exact sequence

$$0 \to \operatorname{Ext}^i_R(L, M) \to C^i \to B^{i+1} \to 0,$$

where $C^i = \operatorname{Hom}_R(F_i, M)/B^i$. Then in view of the proof of Theorem 2.3 and Lemma 2.1, we have that $B^i \in \mathcal{C}$ for all i. (Note that $\operatorname{Ext}^i_R(L, M) \subseteq 0 :_{C^i} \mathfrak{a}^n$.) Thus $C^i \in \mathcal{C}$ for all i. Hence $0 :_{C^i} \mathfrak{a}$ is $\mathfrak{a}$-minimax for all i, and so it follows from Lemma 2.1 that $0 :_{C^i} \mathfrak{a}^n$ is $\mathfrak{a}$-minimax for all i. Now, as $\operatorname{Ext}^i_R(L, M) \subseteq 0 :_{C^i} \mathfrak{a}^n$, it follows that $\operatorname{Ext}^i_R(L, M)$ is $\mathfrak{a}$-minimax for all i.

Also, we have $\operatorname{Tor}^R_i(L, M) = Z_i/B_i$, where B_i and Z_i are the modules of boundaries and cycles of the complex $\mathbb{F}_\bullet \otimes_R M$, respectively. Put

$$\mathcal{C}' = \{N \mid \operatorname{Tor}^R_i(R/\mathfrak{a}, N) \text{ is } \mathfrak{a}\text{-minimax for all } i \geq 0\}.$$

In view of Theorem 2.7 and assumption, $M \in \mathcal{C}'$. Consider the exact sequence

$$0 \to C_{i+1} \to Z_i \to \operatorname{Tor}^R_i(L, M) \to 0,$$

where $C_i = (F_i \otimes_R M/Z_i)$. As $\mathfrak{a}^n \operatorname{Tor}^R_i(L, M) = 0$ for all i, we obtain the exact sequence

$$Z_i/\mathfrak{a}^n Z_i \to \operatorname{Tor}^R_i(L, M) \to 0.$$

Now, using the proof of Theorem 2.7((i) $\Rightarrow$ (ii)) and Lemma 2.2, we see that $Z_i \in \mathcal{C}$ for all i. Therefore, it follows from Lemma 2.2 that $Z_i/\mathfrak{a}^n Z_i$ is $\mathfrak{a}$-minimax for all i, and so $\operatorname{Tor}^R_i(L, M)$ is $\mathfrak{a}$-minimax for all i. □

To prove the change of rings principle for cominimaxness, we need to the following lemma. Before presenting it, recall that (cf. [3]), for any ideal $\mathfrak{a}$ of R and any R-module M, the $\mathfrak{a}$*-relative Goldie dimension of* M is defined as

$$G\dim_{\mathfrak{a}} M := \sum_{\mathfrak{p} \in V(\mathfrak{a})} \mu^0(\mathfrak{p}, M),$$

where $\mu^0(\mathfrak{p}, M)$ denotes the 0-th Bass number of M with respect to prime ideal $\mathfrak{p}$.

Lemma 2.9. *Let the ring T be a homomorphic image of R, and let M be an T-module. Then*

$$G\dim_{\mathfrak{a}T} M = G\dim_{\mathfrak{a}} M.$$

In particular, M is an $\mathfrak{a}T$-minimax T-module if and only if M is an $\mathfrak{a}$-minimax R-module.

Proof. Assume that $T = R/I$ for some ideal I of R. Then

$$\operatorname{Ass}_T M \cap V(\mathfrak{a}T) = \{\mathfrak{p}/I \mid \mathfrak{p} \in \operatorname{Ass}_R M \cap V(\mathfrak{a})\}.$$

On the other hand, for any $\mathfrak{p} \in \operatorname{Ass}_R M \cap V(\mathfrak{a})$ we have

$$\operatorname{Hom}_{T_{\bar{\mathfrak{p}}}}(k(\mathfrak{p}), M_{\bar{\mathfrak{p}}}) \cong \operatorname{Hom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), M_{\mathfrak{p}})$$

as $k(\mathfrak{p})$-vector spaces, where $\bar{\mathfrak{p}} = \mathfrak{p}/I$ and $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Therefore $\mu^0(\mathfrak{p}, M) = \mu^0(\mathfrak{p}/I, M)$ and this completes the proof. □

We are now ready to state and prove the change of rings principle for cominimaxness of modules.

Theorem 2.10. *Let the ring T be a homomorphic image of R, and let M be an T-module. Then M is an $\mathfrak{a}T$-cominimax as a T-module if and only if M is an $\mathfrak{a}$-cominimax as an R-module.*

Proof. Assume that $T = R/I$ for some ideal I of R. Then we have

$$\operatorname{Supp}_T M = \{\mathfrak{p}/I \mid \mathfrak{p} \in \operatorname{Supp}_R M\}.$$

Therefore, $\operatorname{Supp}_T M \subseteq V(\mathfrak{a}T)$ if and only if $\operatorname{Supp}_R M \subseteq V(\mathfrak{a})$. Let $\mathfrak{a} = (x_1, \ldots, x_t)$ and let $\varphi : R \to T$ be the natural epimorphism. As $\mathfrak{a}T = (\varphi(x_1), \ldots, \varphi(x_t))$, it follows from Theorem 2.7 that $\operatorname{Ext}^i_T(T/\mathfrak{a}T, M)$ is an $\mathfrak{a}T$-minimax T-module for all i if and only if the Koszul cohomology modules $H^i(\varphi(x_1), \ldots, \varphi(x_t); M)$ are $\mathfrak{a}T$-minimax T-modules for all i. But, in view of Lemma 2.9, $H^i(\varphi(x_1), \ldots, \varphi(x_t); M)$ is $\mathfrak{a}T$-minimax if and only if $H^i(\varphi(x_1), \ldots, \varphi(x_t); M)$ is $\mathfrak{a}$-minimax. Now the result follows from

$$H^i(\varphi(x_1), \ldots, \varphi(x_t); M) \cong H^i(x_1, \ldots, x_t; M).$$

and Theorem 2.7. □

Theorem 2.11. *Let $f : M \to N$ be an R-homomorphism such that $\operatorname{Ext}^i_R(R/\mathfrak{a}, \operatorname{Ker} f)$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, \operatorname{Coker} f)$ are both $\mathfrak{a}$-minimax for all i. Then $\operatorname{Ker} \operatorname{Ext}^i_R(\mathrm{id}_{R/\mathfrak{a}}, f)$ and $\operatorname{Coker} \operatorname{Ext}^i_R(\mathrm{id}_{R/\mathfrak{a}}, f)$ are also $\mathfrak{a}$-minimax for all i.*

Proof. The exact sequences

$$0 \to \mathrm{Ker} f \to M \overset{g}{\to} \mathrm{Im} f \to 0 \quad \text{and} \quad 0 \to \mathrm{Im} f \overset{\iota}{\to} N \to \mathrm{Coker} f \to 0,$$

where $\iota \circ g = f$, provides the following two exact sequences

$$\cdots \to \mathrm{Ext}^i_R(R/\mathfrak{a}, \mathrm{Ker}\, f) \to \mathrm{Ext}^i_R(R/\mathfrak{a}, M) \to \mathrm{Ext}^i_R(R/\mathfrak{a}, \mathrm{Im}\, f) \to \cdots \quad (\dagger)$$

and

$$\cdots \to \mathrm{Ext}^i_R(R/\mathfrak{a}, \mathrm{Im}\, f) \to \mathrm{Ext}^i_R(R/\mathfrak{a}, N) \to \mathrm{Ext}^i_R(R/\mathfrak{a}, \mathrm{Coker}\, f) \to \cdots . \quad (\ddagger)$$

Now, since $\mathrm{Ext}^{i+1}_R(R/\mathfrak{a}, \mathrm{Ker}\, f)$ is $\mathfrak{a}$-minimax, it follows from the exact sequence (†) that $\mathrm{Coker}\,\mathrm{Ext}^i_R(id_{R/\mathfrak{a}}, g)$ and $\mathrm{Ker}\,\mathrm{Ext}^{i+1}_R(id_{R/\mathfrak{a}}, g)$ are both $\mathfrak{a}$-minimax for all i. Also, as $\mathrm{Ext}^i_R(R/\mathfrak{a}, \mathrm{Coker}\, f)$ is $\mathfrak{a}$-minimax, the exact sequence (‡) implies that the R-modules $\mathrm{Coker}\,\mathrm{Ext}^i_R(id_{R/\mathfrak{a}}, \iota)$ and $\mathrm{Ker}\,\mathrm{Ext}^{i+1}_R(id_{R/\mathfrak{a}}, \iota)$ are $\mathfrak{a}$-minimax for all i. Now, the assertion follows from the exact sequences

$$0 \to \mathrm{Ker}\,\mathrm{Ext}^i_R(id_{R/\mathfrak{a}}, g) \to \mathrm{Ker}\,\mathrm{Ext}^i_R(id_{R/\mathfrak{a}}, f) \to \mathrm{Ker}\,\mathrm{Ext}^i_R(id_{R/\mathfrak{a}}, \iota)$$

$$\mathrm{Coker}\,\mathrm{Ext}^i_R(id_{R/\mathfrak{a}}, g) \to \mathrm{Coker}\,\mathrm{Ext}^i_R(id_{R/\mathfrak{a}}, f) \to \mathrm{Coker}\,\mathrm{Ext}^i_R(id_{R/\mathfrak{a}}, \iota) \to 0. \quad \square$$

Corollary 2.12. *Let M be an R-module with* $\mathrm{Supp}\, M \subseteq V(\mathfrak{a})$. *Suppose that $x \in \mathfrak{a}$ such that $0 :_M x$ and M/xM are both $\mathfrak{a}$-cominimax. Then M is also $\mathfrak{a}$-cominimax.*

Proof. Put $f = x1_M$. Then $\mathrm{Ker}\, f = 0 :_M x$ and $\mathrm{Coker}\, f = M/xM$. Hence in view of Theorem 2.11, the R-module $\mathrm{Ker}\,\mathrm{Ext}^i_R(1_{R/\mathfrak{a}}, f)$ is $\mathfrak{a}$-minimax. Now, it follows from $\mathrm{Ext}^i_R(1_{R/\mathfrak{a}}, f) = 0$ that $\mathrm{Ker}\,\mathrm{Ext}^i_R(1_{R/\mathfrak{a}}, f) = \mathrm{Ext}^i_R(R/\mathfrak{a}, M)$. This completes the proof. $\square$

Corollary 2.13. *Let M be an R-module. Suppose that $x \in \sqrt{\mathfrak{a}}$ such that $0 :_M x$ and M/xM are both $\mathfrak{a}$-minimax. Then $\mathrm{Ext}^i_R(R/\mathfrak{a}, \Gamma_{Rx}(M))$ is also $\mathfrak{a}$-minimax for all i.*

Proof. We have $x^n \in \mathfrak{a}$ for some $n \in \mathbb{N}$. Put $f = x^n 1_{\Gamma_{Rx}(M)}$. Then, we have

$$\mathrm{Ker}\, f = 0 :_{\Gamma_{Rx}(M)} x^n = 0 :_M x^n,$$

and $\mathrm{Coker}\, f = \Gamma_x(M)/x^n\Gamma_x(M)$. Now, it follows from the exact sequence

$$0 \longrightarrow \mathrm{Coker}\, f \longrightarrow M/x^n M,$$

and Lemma 2.2 that $M/x^n M$ is $\mathfrak{a}$-minimax. Thus $\mathrm{Coker}\, f$ is also $\mathfrak{a}$-minimax. Therefore, in view of [1, Corollary 2.5] and Theorem 2.11, $\mathrm{Ker}\,\mathrm{Ext}^i_R(1_{R/\mathfrak{a}}, f)$ is $\mathfrak{a}$-minimax. But $x \in \sqrt{\mathfrak{a}}$ implies that $\mathrm{Ext}^i_R(1_{R/\mathfrak{a}}, f) = 0$, and so

$$\mathrm{Ker}\,\mathrm{Ext}^i_R(1_{R/\mathfrak{a}}, f) = \mathrm{Ext}^i_R(R/\mathfrak{a}, \Gamma_{Rx}(M)).$$

This completes the proof. □

Corollary 2.14. *Let M be an R-module with support in $V(\mathfrak{a})$. Suppose that $x \in \sqrt{\mathfrak{a}}$ such that $0 :_M x$ and M/xM are both $\mathfrak{a}$-minimax. Then M is $\mathfrak{a}$-cominimax.*

Proof. The result follows from the Corollary 2.13. □

Before bringing the next result we recall that, for an R-module M, the *cohomological dimension of M with respect to an ideal* $\mathfrak{a}$ of R is defined as

$$\mathrm{cd}(\mathfrak{a}, M) = \sup\{i \in \mathbb{Z} \mid H^i_{\mathfrak{a}}(M) \neq 0\}.$$

Lemma 2.15. *Let $\mathrm{cd}(\mathfrak{a}, R) = 1$, and let M be an $\mathfrak{a}$-minimax R-module. Then $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cominimax for all i.*

Proof. Since $H^0_{\mathfrak{a}}(M)$ is a submodule of M, it follows that $H^0_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cominimax. Also, $\mathrm{cd}(\mathfrak{a}, R) = 1$ implies that $H^i_{\mathfrak{a}}(M) = 0$ for all $i > 1$. Therefore, the result follows from [1, Corollary 3.9]. □

Lemma 2.16. *Let $\mathfrak{b}$ be an ideal of R with $\mathfrak{b} \supseteq \mathfrak{a}$, $\mathrm{cd}(\mathfrak{b}, R) = 1$, and let M be an R-module with $\Gamma_{\mathfrak{a}}(M) = 0$. Then*

$$H^j_{\mathfrak{b}}(H^i_{\mathfrak{a}}(M)) \cong \begin{cases} H^1_{\mathfrak{b}}(M), & \text{if } \ \mathrm{j} = 0, \mathrm{i} = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The assertion follows from the proof of [5, Proposition 3.15]. □

Corollary 2.17. *Let $\mathfrak{b}$ be an ideal of R with $\mathfrak{b} \supseteq \mathfrak{a}$, $\mathrm{cd}(\mathfrak{b}, R) = 1$, and M a $\mathfrak{b}$-minimax R-module. Then $H^j_{\mathfrak{b}}(H^i_{\mathfrak{a}}(M))$ is $\mathfrak{b}$-cominimax for all i and j.*

Proof. Since $\mathrm{cd}(\mathfrak{b}, R) = 1$, it follows from Lemma 2.15 that $H^j_{\mathfrak{b}}(\Gamma_{\mathfrak{a}}(M))$ is $\mathfrak{b}$-cominimax for all j. Now, let $i > 0$. As $H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M))$, we may therefore assume that $\Gamma_{\mathfrak{a}}(M) = 0$. Thus, the result follows from Lemmas 2.15 and 2.16. □

Corollary 2.18. *Let $\mathfrak{b}$ be an ideal of R with $\mathfrak{b} \supseteq \mathfrak{a}$, $\mathrm{cd}(\mathfrak{b}, R) = 1$, and M a $\mathfrak{b}$-minimax R-module. Then for every finitely generated R-module L with $\mathrm{Supp}\, L \subseteq V(\mathfrak{b})$, the R-modules $\mathrm{Ext}^j_R(L, H^i_{\mathfrak{a}}(M))$ are $\mathfrak{b}$-minimax for all i and j. In particular, the R-modules $H^i_{\mathfrak{a}}(M)/\mathfrak{b}^n H^i_{\mathfrak{a}}(M)$ are $\mathfrak{b}$-minimax for all i and n.*

Proof. By Corollary 2.17, $H^j_{\mathfrak{b}}(H^i_{\mathfrak{a}}(M))$ is $\mathfrak{b}$-cominimax for all i and j. Therefore, it follows from [1, Proposition 3.7] that the R-modules $\mathrm{Ext}^j_R(R/\mathfrak{b}, H^i_{\mathfrak{a}}(M))$ are $\mathfrak{b}$-minimax for all i and j. Thus, the result follows from Theorems 2.7 and 2.3. □

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