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DECOMPOSITION OF TENSOR PRODUCTS OF MODULAR IRREDUCIBLE REPRESENTATIONS FOR SL_3: THE $p \geq 5$ CASE

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Abstract. We study the structure of the indecomposable direct summands of tensor products of two *restricted* rational simple modules for the algebraic group $\mathrm{SL}_3(K)$, where K is an algebraically closed field of characteristic $p \geq 5$. We also give a characteristic-free algorithm for the decomposition of such a tensor product into indecomposable direct summands. The $p < 5$ case was studied in the authors' earlier paper [4]. We find that for characteristics $p \geq 5$ all the indecomposable summands are rigid, in contrast to the characteristic 3 case.

Mathematics Subject Classification (2010): 20C30
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Introduction

Let $G = \mathrm{SL}_3(K)$ where K is an algebraically closed field of characteristic $p \geq 5$. The purpose of this paper, which is a continuation of [4], is to describe the family $\mathfrak{F}$ of indecomposable direct summands of a tensor product $L \otimes L'$ of two simple G-modules L, L' of p-restricted highest weights. (All modules considered are rational.) We give a characteristic-free algorithm for the computation of the decomposition multiplicities of such a tensor product into indecomposable modules and give structural information about the indecomposable summands. Thanks to Steinberg's tensor product theorem, such information provides a first approximation toward a description of the indecomposable direct summands of a general tensor product of two (not necessarily restricted) simple G-modules. Similar questions were previously studied for characteristic $p < 5$ in [4], where sharper results were obtained. The current paper and its prequel [4] were motivated by [11], which studied the $\mathrm{SL}_2(K)$ case.

Our main results are summarized in Theorems A and B in Section 3. The aforementioned algorithm is given in 7.4 and 8.9. In contrast to what happens in characteristic $p = 3$, in characteristics $p \geq 5$ we find that all the indecomposable summands are rigid modules (socle series and radical series coincide). All of the indecomposable summands are in fact tilting modules, except for certain non-tilting simple modules and a certain family of non-highest weight modules, which had also been observed in the $p = 3$ case. The first examples of non-rigid tilting modules for algebraic groups were exhibited in [4]; further examples and a general positive rigidity result for tilting modules are now available in [2].

We had hoped that determining the indecomposable summands of $L \otimes L'$ in the restricted case would lead to their determination in general, by some sort of generalized tensor product result; e.g., see Lemma 1.1 in [4]. However, our results show this is not the case, and the general (unrestricted) decomposition problem remains open. Although our results do in principle give a partial decomposition in the unrestricted case, using formula (1.1.3) of [4], the summands there

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will not always be indecomposable, and the problem of finding all splittings of those summands remains in general unsolved.

1. Preliminaries

We adopt the notational conventions of [4]. Throughout this paper $G = \mathrm{SL}_3(K)$ where K is an algebraically closed field of characteristic $p \geq 5$. This means that $p \geq 2h-2$ where $h=3$ is the Coxeter number of the underlying root system, so general results on algebraic groups that are known to hold only for $p \geq 2h-2$ are available.

1.1. Weight notations. Sometimes we need to use both GL_3 and SL_3 weight notation for calculations. Any GL_3-module is an SL_3-module by restriction. A given SL_3-module M may be lifted to a GL_3-module in many ways, but all such lifts differ by a power of the determinant representation, and which lift we choose makes no difference for our results. It is sometimes convenient to work with GL_3-modules in order to apply, for example, the Littlewood–Richardson rule. We adopt the notation of II.1.21 of [16]. In particular, $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is the standard basis of $X(T_{\mathrm{GL}_3}) \simeq \mathbb{Z}^3$ where T_{GL_3} is the diagonal torus in GL_3. We identify a GL_3-weight $\chi = a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_3$ with the 3-tuple $((a_1, a_2, a_3)) \in \mathbb{Z}^3$. The inclusion $\mathrm{SL}_3 \subset \mathrm{GL}_3$ induces an embedding $T_{\mathrm{SL}_3} \subset T_{\mathrm{GL}_3}$ of their diagonal subgroups, which in turn induces a surjection $X(T_{\mathrm{GL}_3}) \twoheadrightarrow X(T_{\mathrm{SL}_3})$ given by restricting characters from T_{GL_3} to T_{SL_3}, with kernel generated by $\varepsilon_1 + \varepsilon_2 + \varepsilon_3$. This map sends $a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_3$ onto $(a_1 - a_2)\varpi_1 + (a_2 - a_3)\varpi_2$, where ϖ_1, ϖ_2 are the fundamental weights in $X(T_{\mathrm{SL}_3})$. We shall identify an SL_3-weight $\lambda_1\varpi_1 + \lambda_2\varpi_2$ in $X(T_{\mathrm{SL}_3}) \simeq \mathbb{Z}^2$ with the ordered pair (λ_1, λ_2). In terms of these identifications, the map $X(T_{\mathrm{GL}_3}) \twoheadrightarrow X(T_{\mathrm{SL}_3})$ is given by

$$((a_1, a_2, a_3)) \to (a_1 - a_2, a_2 - a_3).$$

We use double brackets $((\ ,\ ,\ ))$ versus single brackets $(\ ,\ )$ in order to easily distinguish between GL_3 and SL_3-weights. Given an element χ in $X(T_{\mathrm{GL}_3})$ we shall denote its image under the restriction map $X(T_{\mathrm{GL}_3}) \twoheadrightarrow X(T_{\mathrm{SL}_3})$ by $\overline{\chi}$. In particular, we shall need the SL_3-weights $\overline{\varepsilon_1} = (1,0)$, $\overline{\varepsilon_2} = (-1,1)$, and $\overline{\varepsilon_3} = (0,-1)$ coming from the GL_3-weights $\varepsilon_1 = ((1,0,0))$, $\varepsilon_2 = ((0,1,0))$, and $\varepsilon_3 = ((0,0,1))$ forming the standard basis of $X(T_{\mathrm{GL}_3})$. As usual, $\rho = \varpi_1 + \varpi_2 = (1,1)$ is one-half the sum of the positive roots for SL_3.

1.2. Until further notice we use only SL_3 notation for weights. Thus $X^+ = \{(a,b) : a, b \geq 0\}$ the set of dominant weights. The simple roots are $\alpha_1 = (2,-1)$, $\alpha_2 = (-1,2)$. As in [4], $T(\lambda)$ is the indecomposable tilting module of highest weight λ, $\Delta(\lambda)$ the Weyl module of highest weight λ, $\nabla(\lambda)$ its contravariant dual, and $L(\lambda)$ the simple head of $\Delta(\lambda)$. We also denote the Steinberg module $\Delta(p-1,p-1) = L(p-1,p-1)$ by St. In case $\nabla(\lambda) = L(\lambda)$ is simple, we have $T(\lambda) = \Delta(\lambda) = \nabla(\lambda) = L(\lambda)$; this applies in particular to the Steinberg module St, the natural module $E = \Delta(1,0)$ and its linear dual $E^* = \Delta(0,1)$. The tilting modules $T(\lambda)$ are always contravariantly self-dual, and a tensor product of two tilting modules is again tilting.

By $\mathcal{F}(\Delta)$ we mean the category of G-modules admitting a Δ-filtration, and, dually, by $\mathcal{F}(\nabla)$ we mean the category of G-modules admitting a ∇-filtration. We note that $T(\lambda)$ is the unique indecomposable module of highest weight λ in $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$.

1.3. It is useful to regard a given G-module as a module for some generalized Schur algebra $S(\pi)$, where π is an appropriate finite saturated poset of dominant weights; see [8] or Chapter II.A in [16] for details on generalized Schur algebras. The fact that $S(\pi)$ is quasi-hereditary is used repeatedly. For any $S(\pi)$ we let $P_\pi(\lambda)$ denote the projective cover of $L(\lambda)$ in the category of $S(\pi)$-modules. We may drop the subscript π in case the set π is fixed by the context. The contravariant dual of $P_\pi(\lambda)$ is isomorphic to the injective hull of $L(\lambda)$ in the category of $S(\pi)$-modules.

It is known that $P(\mu) \in \mathcal{F}(\Delta)$. Let $[P(\mu)\colon \Delta(\lambda)]$ denote the number of subquotients isomorphic to $\Delta(\lambda)$ in a Δ-filtration of $P(\mu)$; this number is known to be independent of the choice of Δ-filtration. For any finite dimensional $S(\pi)$-module M let $[M\colon L(\mu)]$ be the multiplicity of $L(\mu)$ in a composition series of M. See Proposition A2.2(iv) of [10] or Theorem 2.6 of [7] for the following basic reciprocity property, which will be used repeatedly.

Proposition 1. *Let $S(\pi)$ be the generalized Schur algebra determined by a finite saturated set $\pi \subset X^+$. Then $[P(\mu)\colon \Delta(\lambda)] = [\nabla(\lambda)\colon L(\mu)]$ for all $\lambda, \mu \in \pi$.*

This is sometimes called *Brauer–Humphreys reciprocity.* The Schur algebra setting also allows us to make use of the following refinement of Proposition 1 from [5]. Let $\operatorname{rad}_i P(\mu)$ be the ith radical layer of $P(\mu)$.

Proposition 2. *Let $S(\pi)$ be a generalized Schur algebra, where $\pi \subset X^+$ is a finite saturated set. Then for any $\lambda, \mu \in \pi$ we have:*

$$[\operatorname{rad}_i P(\mu) : L(\lambda)] = [\operatorname{rad}_i P(\lambda) : L(\mu)].$$

This reciprocity respects the Δ-filtration of the projective modules:

$$[\operatorname{rad}_i P(\mu) : \operatorname{head} \Delta(\lambda)] = [\operatorname{rad}_i \Delta(\lambda) : L(\mu)].$$

By $[\operatorname{rad}_i P(\mu) : \operatorname{head} \Delta(\lambda)]$ we mean the number of successive quotients $\Delta(\lambda_j)$ in a given *fixed* Δ-filtration of $P(\mu)$ such that $\lambda_j = \lambda$ and there is a surjection $\operatorname{rad}^i P(\mu) \to \Delta(\lambda)$ which carries the subquotient $\Delta(\lambda_j)$ onto $\Delta(\lambda)$. It is easily checked that this is independent of the choice of Δ-filtration. See 5.2 for an example.

Remark. The contravariant dual of the above theorem relates the socle layers of injective modules, and gives information about where ∇-modules occur in a ∇-filtration of an injective module. This will also be used where needed.

We will need the following basic result from the theory of quasi-hereditary algebras. This follows for instance from Proposition A2.2 of [10].

Proposition 3. *For any $\lambda, \mu \in X^+$, we have:*

(a) *If* $\operatorname{Ext}^1(\Delta(\lambda), \Delta(\mu)) \neq 0$ *then* $\mu > \lambda$.

(b) $\dim_K \operatorname{Hom}_G(M, N) = \sum_{\nu \in X^+} [M : \Delta(\nu)][N : \nabla(\nu)]$, *for any* $M \in \mathcal{F}(\Delta)$, $N \in \mathcal{F}(\nabla)$.

1.4. Let $X_1 = \{(a,b)\colon 0 \le a, b \le p-1\}$ be the set of restricted weights. Let w_0 be the longest element in the Weyl group W. By Jantzen [15] we have for any $\lambda \in X_1$ that

$$T(2(p-1)\rho + w_0\lambda) \cong P_\pi(\lambda), \tag{1}$$

an isomorphism of $S(\pi)$-modules, where $\pi = \{\lambda \in X^+\colon \lambda \le 2(p-1)\rho + w_0\lambda)\}$.

This fact will be used repeatedly in the proof of our results. Any projective tilting module for $S(\pi)$ is also injective, since tilting modules are contravariantly self-dual, so the above module is projective, injective, and tilting, for any $\lambda \in X_1$.

For $p \geq 5$, there is a twisted tensor product theorem for tilting modules, due to Donkin [9, (2.1) Proposition]. Every $\lambda \in X^+$ has a unique expression in the form

$$\lambda = \textstyle\sum_{j=0}^{m} a_j(\lambda)\, p^j \tag{2}$$

with $a_0(\lambda), \ldots, a_{m-1}(\lambda) \in (p-1)\rho + X_1$ and $\langle a_m(\lambda), \alpha^\vee \rangle < p-1$ for at least one simple root α. Given $\lambda \in X^+$, express λ in the form above. There is an isomorphism of G-modules

$$T(\lambda) \simeq \textstyle\bigotimes_{j=0}^{m} T(a_j(\lambda))^{[j]}. \tag{3}$$

2. Facets and alcoves for $G = \mathrm{SL}_3$

We now introduce a labelling scheme for keeping track of the various alcoves and facets needed in our calculations.

2.1. The Euclidean space associated to the root system of G is $X \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^2$. The Weyl group $W = \langle s_1, s_2 \rangle$ is isomorphic to the symmetric group on 3 letters. Here s_1, s_2 are reflections in lines orthogonal to the simple roots α_1, α_2 respectively.

2.2. The map $V \mapsto V^*$, where V^* is the linear dual of V, is an involution on the set of G-modules. If V is a highest weight module of highest weight λ, the highest weight of V^* is $-w_0(\lambda)$, so this involution on G-modules induces a corresponding involution $\lambda \mapsto -w_0(\lambda)$ on the set X^+, where $w_0 = s_1 s_2 s_1$ is the longest element of W. We refer to this involution as *symmetry*, and we generally will omit stating results that can be obtained 'by symmetry' from results already stated.

2.3. Let $\rho = \alpha_1 + \alpha_2 = (1, 1)$. Recall that the dot action of W_p on X is defined by the rule $w \cdot \lambda = w(\lambda + \rho) - \rho$. The bottom alcove C_1 is defined by

$$C_1 = \{v \in \mathbb{R}^2 : 0 < \langle v + \rho, \alpha^\vee \rangle < p \text{ for all positive roots } \alpha\}.$$

As depicted in Figure 1, C_1 is the interior of an equilateral triangle in $\mathbb{R}^2$ with one vertex at the point $-\rho$. The affine Weyl group W_p is generated by the reflections in the walls of C_1.

Any translate $w \cdot C_1$ of C_1 under the dot action of W_p is called an alcove. The closure of an alcove C_i will be denoted by $\overline{C}_i$. In Figure 1 below we number the alcoves, which we call *fundamental* alcoves, that arise in our study. Alcoves i and i' are in symmetry according to the involution of 2.2. We let $\mathcal{F}_{i|j} = \overline{C}_i \cap \overline{C}_j$ denote the wall between any pair C_i, C_j of adjacent alcoves; this wall is a facet in the sense of [16, II.6.2]. The reflection in the wall $\mathcal{F}_{i|j}$ will be denoted by $s_{i|j}$. In this notation the generators of W_p are $s_{1|2}$ along with the elements s_1, s_2 defined in 2.1.

2.4. The points at intersections of very light grid lines in Figure 1 are weights. The region of highest weights of composition factors that can occur in a p-restricted tensor product $L \otimes L'$ is the set of all dominant weights λ such that $\lambda \leq (2p-2)\rho$; this is the set of weights on or below the dashed lines in Figure 1. The alcoves in question are the numbered ones in the figure.

In case $p < 7$, not all of the labelled alcoves in Figure 1 actually appear in restricted tensor product decompositions. The degeneracies for $p < 7$ are caused by fewer points appearing in each alcove; to understand this the reader is encouraged to draw the analogue of Figure 1 for the smaller primes, after which the degeneracies are apparent.

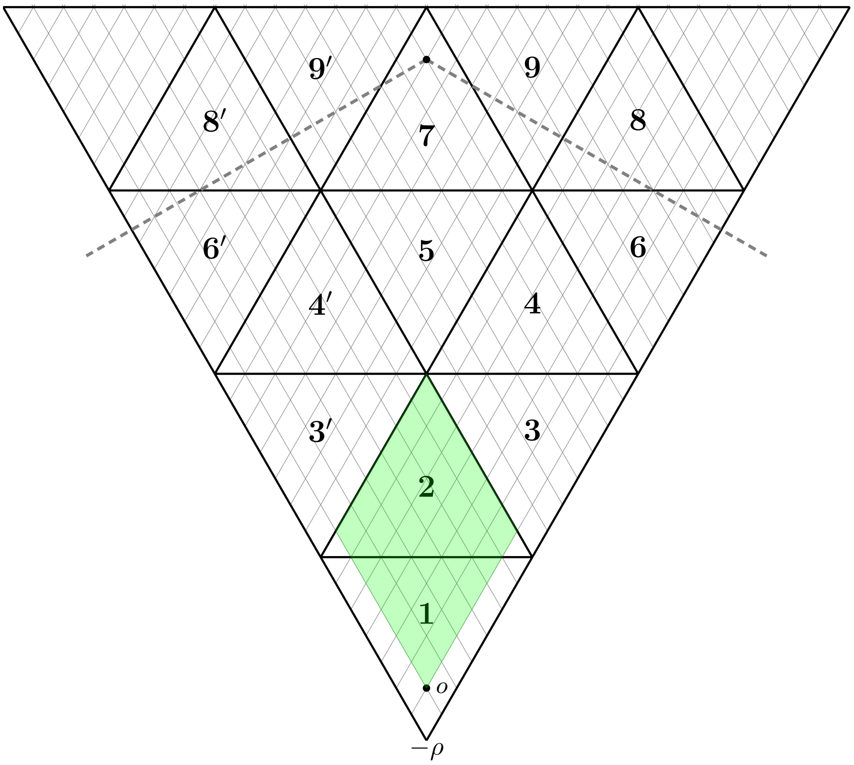


FIGURE 1. Labelled alcoves for $p = 7$. The southernmost vertex is $-\rho$. The origin $o = (0, 0)$ is the marked point directly above it in alcove 1. The unlabelled marked point in alcove 7 is $2(p-1)\rho$. The region of dominant weights λ with $\lambda \leq 2(p-1)\rho$ is bounded above by the two dashed lines emanating from that point. The labelled alcoves, and walls between them, contain all the points in the region. The coloured region of points is the restricted region X_1. The simple root α_1 (respectively α_2) is orthogonal to the wall of C_1 pointing northeast (respectively, northwest) from $-\rho$.

3. Main results

In this section we give the two main results of the paper. The first main result describes the members of the family $\mathfrak{F} = \mathfrak{F}(G)$ of indecomposable direct summands of a tensor product of two restricted simple G-modules. The second main result is a description of the structure of the modules in $\mathfrak{F}$, as far as we can deduce structural information by current methodology. The highest weight modules in $\mathfrak{F}$ are either simple modules $L(\lambda)$ or indecomposable tilting modules $T(\lambda)$ for various λ, however there are non highest weight modules $M(\lambda)$ which occur in $\mathfrak{F}$.

3.1. Notational conventions. We assume the reader is familiar with Jantzen's translation principle [16], which in particular implies an equivalence of module structure for highest weight modules of the form $L(\lambda)$, $\Delta(\lambda)$, $\nabla(\lambda)$, or $T(\lambda)$ belonging to the same facet. We will therefore adopt facet notation for highest weight modules whenever convenient. This replaces a highest weight $\lambda \in X^+$ by its corresponding alcove label j whenever $\lambda \in C_j$. Thus, for example, given $\lambda \in C_1 \cap X$, $L(\lambda)$ is denoted by $L(1)$, $\Delta(\lambda)$ is denoted by $\Delta(1)$, $T(\lambda)$ is denoted by $T(1)$, and so on. Furthermore, within module diagrams, for each label j the simple module $L(j)$ will be identified with its alcove label j. For p-singular weights $\lambda \in \mathcal{F}_{i|j}$ lying on the wall common to two alcoves i and j (but not a vertex of any alcove) we use the notation $i|j$ to denote the facet, and use the notation $L(i|j)$, $\Delta(i|j)$, $T(i|j)$ for the highest weight modules $L(\lambda)$, $\Delta(\lambda)$, $T(\lambda)$ respectively. Similar to the above, within module diagrams, for each facet label $i|j$ the simple module $L(i|j)$ will be identified with its facet label $i|j$.

3.2. Corresponding to each weight $\lambda \in C_2$, there is a unique indecomposable module $M(\lambda)$ in $\mathfrak{F}$ which is *not* generated by a single highest weight vector. For $\lambda \in C_2$, the module $M(\lambda)$ has a simple socle and head isomorphic to $L(\lambda)$, with the quotient $\operatorname{rad} M(\lambda)/\operatorname{soc} M(\lambda)$ of the radical by the socle a semisimple module isomorphic to $L(s_{2|3'} \cdot \lambda) \oplus L(s_{2|3} \cdot \lambda) \oplus L(s_{1|2} \cdot \lambda)$. The module $M(\lambda)$ may be constructed as a submodule of the tilting module $T(s_{3|4} \cdot s_{2|3} \cdot \lambda)$ or, symmetrically, as a submodule of the tilting module $T(s_{3'|4'} \cdot s_{2|3'} \cdot \lambda)$. We construct it in section 6.3 as a quotient of an appropriate generalized Schur algebra. Similar modules (with identical structure diagrams) already appeared in [4] for $p = 3$.

Making use of our convention (from 3.1) of replacing highest weights by their alcove or facet labels we often write $M(2)$ instead of $M(\lambda)$ for $\lambda \in C_2$. In this notation, $M(2)$ is isomorphic to a submodule of $T(4)$ or $T(4')$. The Alperin diagram of $M(2)$ is

$$M(2)= \quad \begin{array}{ccccc} & & 2 & & \\ & \diagup & | & \diagdown & \\ 3 & & 1 & & 3' \\ & \diagdown & | & \diagup & \\ & & 2 & & \end{array}$$

and this is a strong Alperin diagram. (Recall from [4] that a diagram is said to be strong if it determines the socle as well as radical series.) Although $M(2)$ is not a highest weight module, it still has simple head isomorphic to $L(2)$, so this notation should not cause confusion. The module $M(2)$ is rigid and self-dual under contravariant duality.

Let $\mathfrak{F} = \mathfrak{F}(\mathrm{SL}_3)$ be the family of isomorphism classes of indecomposable direct summands of a p-restricted tensor product $L \otimes L'$, where L, L' are p-restricted simple modules. We studied this family in [4] for $p = 2, 3$. For $p = 2$ we found [4, Prop. 3.2] that $\mathfrak{F}$ is precisely the set

$$\mathfrak{F}_R = \{T(\lambda) : 0 \le \langle \lambda + \rho, \alpha^\vee \rangle \le 2p - 2, \text{ for all simple roots } \alpha\}.$$

For $p = 3$, we found [4, Proposition 4.2] that $\mathfrak{F}_R \subset \mathfrak{F}$, but there are also a few other modules in $\mathfrak{F}$, which we called 'exceptional'. It turns out that $\mathfrak{F}_R$ is contained in $\mathfrak{F}$ for every characteristic p. We call the members of $\mathfrak{F}_R$ *regular* and all other members of $\mathfrak{F}$ *exceptional.* In the following, we have incorporated part of the results of [4] in order to summarize our results for all p.

Theorem A. *Let the characteristic $p > 0$ of K be arbitrary. Let $\mathfrak{F} = \mathfrak{F}(\mathrm{SL}_3)$ be the family of isomorphism classes of indecomposable direct summands of a p-restricted tensor product $L \otimes L'$, where L, L' are p-restricted simple modules. Then $\mathfrak{F}$ consists of the following indecomposable modules:*

(a) *The set $\mathfrak{F}_R$ of regular tilting modules.*

(b) *The exceptional modules $L(\lambda)$ and $M(\lambda)$, for each $\lambda \in C_2$.*

(c) *A finite list, depending on p, of exceptional tilting modules of the form $T(\lambda)$, for various λ not already listed in part (a). For $p = 2$ there are no exceptional tilting modules; for $p = 3$ there are precisely four (see* [4, Propositions 3.2 and 4.2]*) of highest weight lying on the boundary of $\overline{C}_6$; for larger p the number of exceptional modules grows with p with the* *highest weight of such modules lying in the region* $\overline{C}_6 \cup \overline{C}_8 \cup \overline{C}_9$ *(and those obtained by symmetry).*

Remarks. *(1) One can explicitly determine the decomposition of a tensor product of two p-restricted irreducible modules, by an algorithm described in Section 7.*

(2) We include results of [4] *for* $p = 2, 3$ *in the theorem, for completeness. In case* $p = 2$ *the alcove* C_2 *is empty, so part (b) of the theorem is vacuous, and thus for* $p = 2$ *the members of* $\mathfrak{F}$ *are just the tilting modules listed in part (a).*

For $p \geq 3$ there are three vertices (points common to the closure of six alcoves) in the admissible region of weights defined in Figure 1, and each of them gives the highest weight of a (simple) tilting member of $\mathfrak{F}$. These are the Steinberg module $\mathrm{St} = T((p-1)\rho) = L((p-1)\rho)$ and the two modules $T(p\varpi_1 + (p-1)\rho) = L(p\varpi_1 + (p-1)\rho) \cong E^{[1]} \otimes \mathrm{St}$, $T(p\varpi_2 + (p-1)\rho) = L(p\varpi_2 + (p-1)\rho) \cong (E^*)^{[1]} \otimes \mathrm{St}$.

Our second main result describes the structure of the other tilting members of $\mathfrak{F}$, of highest weight lying in alcoves or on walls between a pair of alcoves, assuming $p \geq 5$.

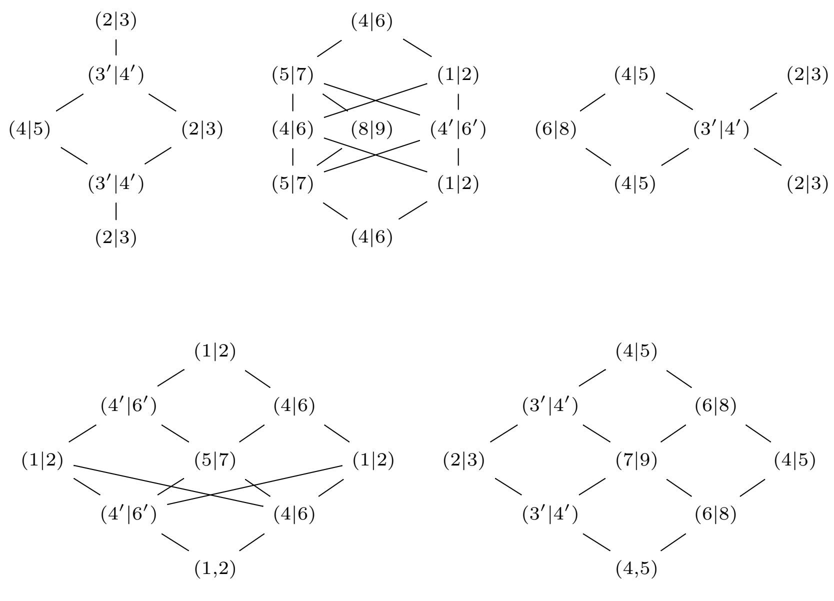
Theorem B. *Assume that the characteristic of* K *is* $p \geq 5$.

(a) The p-singular tilting modules in $\mathfrak{F} = \mathfrak{F}(\mathrm{SL}_3)$ *of highest weight lying on walls are all rigid, with structure as follows. The uniserial modules of highest weight lying on walls are:*

$$T(1|2) = [(1|2)]; \quad T(2|3) = [(2|3)];$$
$$T(3|4) = [(2|3'), (3|4), (2|3')]; \quad T(4|6) = [(1|2), (4|6), (1|2)]$$

along with their symmetric versions.

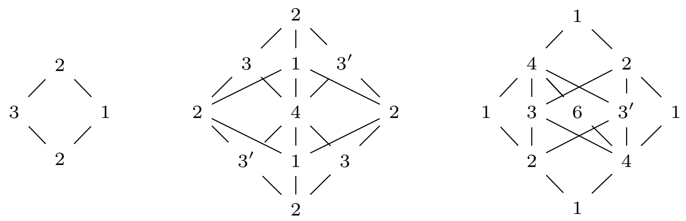
The non-uniserial modules of highest weight lying on walls are $T(4|5)$, $T(8|9)$, $T(6|8)$, $T(5|7)$ *and* $T(7|9)$, *with structure given by the following strong Alperin diagrams, respectively:*



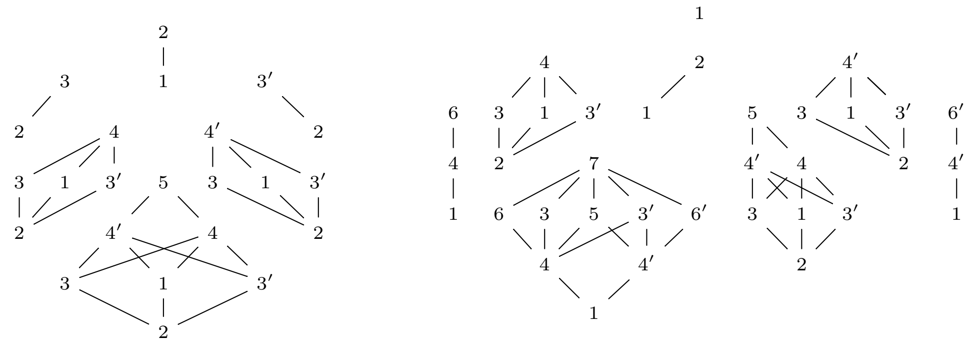
along with their symmetric versions.

(b) The p-regular tilting modules in $\mathfrak{F}$ *are all rigid. The uniserial ones have the following structure:* $T(1) = [1]$; $T(2) = [1, 2, 1]$.

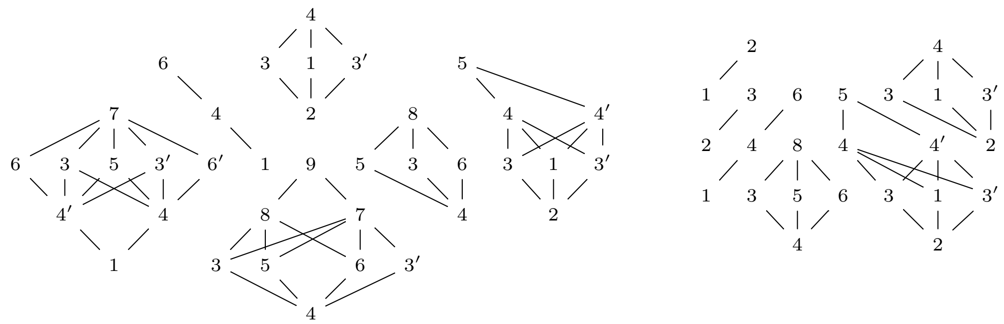
The non-uniserial ones for which the structure can be completely worked out are $T(3)$, $T(4)$, and $T(6)$ with the following strong Alperin diagrams, respectively:



along with their symmetric versions. In the larger cases we give only the Loewy structure of the tilting modules. We highlight the Weyl filtrations below for $T(5)$ and $T(7)$, respectively:



and below we provide the Weyl filtrations of $T(9)$ and $T(8)$, respectively:



where the symmetric versions of these modules are not listed, as usual.

Note that all members of $\mathfrak{F}$ have a simple p-restricted G_1T-socle (and head) except for $T(8)$ and $T(6|8)$ for $p \geq 5$. Therefore every direct summand in the decomposition (1.1.3) of [4] is indecomposable, unless it involves a factor of the form $T(\lambda)$, for $\lambda \in C_8 \cup \mathcal{F}_{6|8}$. Because of the upper bound constraint of $2(p-1)\rho$ on the highest weights of tilting members of $\mathfrak{F}$, when $p = 5$ there are no members $T(\lambda)$ in $\mathfrak{F}$ with $\lambda \in C_8 \cup C_9$, although such modules do appear for $p > 5$.

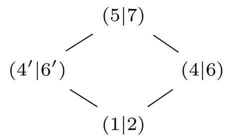
The rest of the paper is devoted to the proof of these results. The proof of Theorem B is given in Sections 5 and 6. The proof of Theorem A is given in Sections 7 and 8. First we need the structure of certain Weyl modules, which are summarized in the next section.

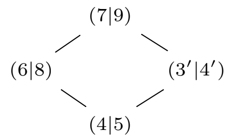
4. The structure of certain Weyl modules

In [18] the p-filtration structure of Weyl modules for $\mathrm{SL}_3(K)$ is determined for all primes. When these layers are semisimple this gives the radical structure of the Weyl modules. Therefore when $p \geq 5$ and we consider weights from the first p^2-alcove this re-derives the generic structures calculated in [12] and [14], which we shall recall below. The p-filtrations are semisimple for all but three of the Weyl modules considered in [4]. For a given prime these calculations can also be checked using the Weyl module GAP package available on the second author's web page.

We remind the reader of the notational conventions of 2.3 and in particular that in diagrams we will identify simple modules with their facet label. The structure of the p-singular Weyl modules in question is given by the following strong Alperin diagrams, where as in [11], [4] we use the notation $[L_1, L_2, \ldots, L_s]$ to depict the structure of the unique uniserial module M with composition factors $L_1, \ldots, L_s$ arranged so that $\mathrm{rad}_i\, M \cong L_i$ for all i.

$$\begin{aligned} \Delta(1|2) &= [(1|2)], & \Delta(2|3) &= [(2|3)], \\ \Delta(3|4) &= [(3|4), (2|3')], & \Delta(4|6) &= [(4|6), (1|2)], \\ \Delta(4|5) &= [(4|5), (3'|4'), (2|3)], & \Delta(6|8) &= [(6|8), (4|5)], \\ \Delta(8|9) &= [(8|9), (5|7), (4|6)], \end{aligned}$$

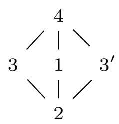
$\Delta(5|7) =$ 

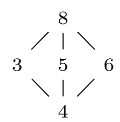
, $\Delta(7|9) =$ 

.

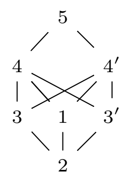
The structure of the p-regular Weyl modules we need is as follows, where once again each diagram is a strong Alperin diagram.

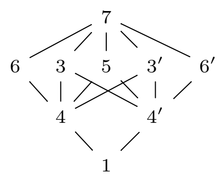
$$\Delta(1) = [1], \quad \Delta(2) = [2, 1], \quad \Delta(3) = [3, 2], \quad \Delta(6) = [6, 4, 1],$$

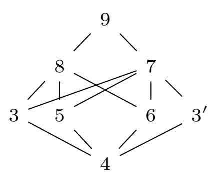
$\Delta(4) =$ 

, $\Delta(8) =$ 

,

$\Delta(5) =$ 

, $\Delta(7) =$ 

, $\Delta(9) =$ 

.

All of these Weyl modules, including the p-singular ones, are rigid.

5. The p-singular tilting modules

We now begin the proof of Theorem B. The characters of the tilting modules for $G = \mathrm{SL}_3$ are known [17,19] (see also [6]) so our task is just to prove the structural results in Theorem B. This proof is split over the next two sections. The present section considers only the p-singular case while the next considers the p-regular case.

5.1. It turns out that most tilting modules $T(\lambda)$ that we consider are projective for the generalized Schur algebra $S(\leq \lambda)$. To prove injectivity (and hence projectivity) one need only check that

(a) the socle of the tilting module is simple, and
(b) the character of the relevant projective module (given by Proposition 1) coincides with the tilting character.

Note that part (a) can usually be done by constructing an injection into a tilting module of the form $T(2(p-1)\rho + \varpi_0\lambda)$ for $\lambda \in X_1$ and applying 1.4.

We begin with the p-singular tilting modules, not only because their structures tend to be less complicated, but also because their images under translation functors provide useful filtrations of the p-regular tilting modules considered in the next section.

5.2. We shall build the tilting modules $T(2|3), T(3'|4')$ and $T(4|5)$ as modules for the Schur algebra $S(\pi)$ corresponding to the poset $\pi = \{(2|3) < (3'|4') < (4|5)\}$. We begin with $T(4|5)$. By 1.4 the tilting module $T(4|5) = P(2|3)$ is the projective cover of $L(2|3)$. By Proposition 1 and the Weyl module structure in Section 4, the projective module $P(2|3)$ has a Δ-filtration with Δ-factors $\Delta(2|3)$, $\Delta(3'|4')$, and $\Delta(4|5)$ each occurring with multiplicity one. Using Proposition 2 we can locate where the heads of the Δ-modules occur in a radical filtration of $T(4|5)$. The diagram below gives the radical structure of the module

$$\begin{tikzcd}[column sep=small, row sep=small]
& (2|3) & \\
& (3'|4') \arrow[dr, dash] & \\
(4|5) \arrow[dr, dash] & & (2|3) \\
& (3'|4') \arrow[d, dash] & \\
& (2|3) &
\end{tikzcd}$$

in which the connected components are the layers in the Δ-filtration. Since $\mathrm{rad}_1(\Delta(2|3)) = L(2|3)$, $\mathrm{rad}_2(\Delta(3'|4')) = L(2|3)$ and $\mathrm{rad}_3(\Delta(4|5)) = L(2|3)$, by Proposition 2 the heads of the Δ-modules $\Delta(2|3)$, $\Delta(3'|4')$ and $\Delta(4|5)$ appear in the first, second, and third layers of $P(2|3)$ respectively (as pictured above). Note that $\mathrm{rad}_4(P(2|3)) = L(3'|4')$, however this module is not the head of a Δ-module in a Δ-filtration. Considering also the ∇-filtration gives the Alperin diagram

$$\begin{tikzcd}[column sep=small, row sep=small]
& (2|3) \arrow[d, dash] & \\
& (3'|4') \arrow[dl, dash] \arrow[dr, dash] & \\
(4|5) \arrow[dr, dash] & & (2|3) \arrow[dl, dash] \\
& (3'|4') \arrow[d, dash] & \\
& (2|3) &
\end{tikzcd}$$

of $T(4|5)$ shown above. This is projective-injective and so Proposition 2 (and the subsequent remark) give both the radical and socle structure. Hence $T(4|5)$ is rigid and the above diagram is a strong Alperin diagram.

The above also gives us the structure of $T(3'|4')$, which appears as a quotient of $T(4|5)$. To see this, notice that the module $P(2|3) = T(4|5)$ has a uniserial quotient module isomorphic to $[(2|3), (3'|4'), (2|3)]$; this quotient has both Δ and ∇ filtrations, hence is tilting and isomorphic to $T(3'|4')$. Moreover, since $\Delta(2|3) = L(2|3) = \nabla(2|3)$, the module $T(2|3) = L(2|3)$ is a simple tilting module.

We now determine the structure of $T(6|8)$. The calculation is similar to that given above, so we only sketch it. We have that $\Delta(6|8) = [(6|8), (4|5)]$, so $\Delta(4|5)$ must appear at the top. Contravariant duality and our knowledge of the other Weyl modules in the linkage class give us the Δ-filtration

$$\begin{array}{ccccc}
 & (4|5) & & (2|3) & \\
 & & \diagdown & & \\
(6|8) & & (3'|4') & & \\
 & \diagdown & & \diagdown & \\
 & (4|5) & & (2|3) &
\end{array}$$

of $T(6|8)$ as shown above. Finally, consideration of the ∇-filtration gives us the full structure diagram as depicted in Theorem B.

We now consider $T(7|9)$. By (3) we have an isomorphism $T(7|9) \cong T(4'|5) \otimes E^{(1)}$, and by comparing characters (or composition factors) we see that $T(7|9) = P(4|5)$, as $S(\leq 7|9)$-modules. Therefore $T(7|9)$ is rigid by Proposition 2. The full Alperin diagram can then be deduced from the Δ and ∇-filtrations.

5.3. Coefficient quivers. To proceed further we need to use methods from the theory of finite dimensional algebras. This need will come up again in Section 6. Our approach will be similar to that of Ringel's Appendix to [4]. In particular, when writing quiver and relations we will switch to right modules instead of left ones, so that composite paths can be read from left to right. Recall that by Gabriel's theorem (see e.g., Proposition 4.1.7 in [3]) the basic algebra of any finite dimensional algebra is isomorphic to a suitable quotient of the path algebra of the ext-quiver of the algebra.

We recall the terminology of coefficient quivers. Let K be a field, let Q be a (finite) quiver and KQ the path algebra of Q over K. Recall that a representation N of Q over K associates to each vertex $i \in Q$ a vector space N_i and to each arrow $\alpha : i \to j$ a linear transformation $N_i \to N_j$. There is a natural correspondence between representations of Q and KQ-modules. See [21] or [3] for more details.

Given a representation N, let d_i be the dimension of N_i, and $d = \sum_i d_i$; d is called the dimension of N. A basis $\mathcal{B}$ of N is by definition a subset of the disjoint union of the various K-spaces N_i, such that for any vertex i the set $\mathcal{B}_i = \mathcal{B} \cap N_i$ is a basis of N_i. Let us assume that such a basis $\mathcal{B}$ of N is given. For any arrow $\alpha : i \to j$, we may express N_α as a $(d_j \times d_i)$-matrix whose rows are indexed by $\mathcal{B}_j$ and whose columns are indexed by $\mathcal{B}_i$. We denote by $N_{\alpha,\mathcal{B}}(b, b')$ the corresponding matrix coefficients, where $b \in \mathcal{B}_i$, $b' \in \mathcal{B}_j$, these matrix coefficients $N_{\alpha,\mathcal{B}}(b, b')$ are defined by $N_\alpha(b) = \sum_{b' \in \mathcal{B}} N_{\alpha,\mathcal{B}}(b, b')b'$. By definition, the coefficient quiver $\Gamma(N, \mathcal{B})$ of N with respect to $\mathcal{B}$ is the oriented graph with $\mathcal{B}$ as set of vertices, and there is an arrow $\alpha : b \to b'$ provided $N_{\alpha,\mathcal{B}}(b, b') \neq 0$. Usually that arrow would take α as label, but since we deal only with quivers without multiple arrows, the labels are omitted. We will always arrange our coefficient quivers so that arrows point downwards; then arrows can be replaced by edges. It should be noted that the choice of basis can affect the shape of the coefficient quiver.

5.4. We now consider the generalized Schur algebra $S(\leq 8|9) = S(\pi)$, or rather its basic algebra. From the structure diagrams of the Weyl modules in Section 4 it follows that the Ext^1-quiver P for $S(\leq 8|9)$ with indexing set $\pi = \{1|2, 4|6, 4'|6', 5|7, 8|9\}$ is as illustrated in Figure 2.

We label the idempotents corresponding to the nodes by $e_{1|2}$, $e_{4|6}$, $e_{4'|6'}$, $e_{5|7}$, and $e_{8|9}$. By Gabriel's theorem $S(\leq 8|9)$ is Morita equivalent to a quotient of the path algebra of P.

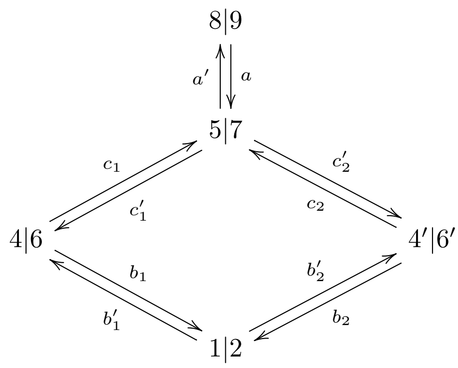


FIGURE 2. The quiver P.

Proposition 4. *The basic algebra of the Schur algebra $S = S(\leq 8|9)$ is isomorphic to the path algebra of P modulo the ideal generated by the following relations:*

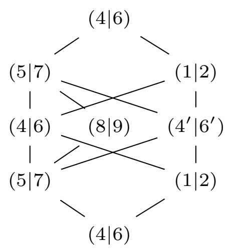
$$aa' = 0, ac_2' = 0, ac_1'b_1 = 0, ac_1'c_1 = 0, c_i'c_i = a'a, c_1'b_1 = c_2'b_2, c_1'b_1b_2' = 0,$$
$$b_1b_1' = c_1c_1', b_1b_2' = c_1c_2', c_2a' = 0, b_2b_1' = c_2c_1', b_2b_2' = (-1)c_2c_2', b_1'c_1 = b_2'c_2.$$

Proof. By Proposition 1, the projective indecomposable modules for $S(\leq 8|9)$ have the following Δ-filtrations (going downwards)

$P(1\|2)$	$\Delta(1\|2)\|\Delta(4\|6) \oplus \Delta(4'\|6')\|\Delta(5\|7)$
$P(4\|6)$	$\Delta(4\|6)\|\Delta(5\|7)\|\Delta(8\|9)$
$P(4'\|6')$	$\Delta(4'\|6')\|\Delta(5\|7)$
$P(5\|7)$	$\Delta(5\|7)\|\Delta(8\|9)$
$P(8\|9)$	$\Delta(8\|9)$.

By equation (1), we have the isomorphism $T(4|5) \cong P(2|3)$ and by equation (3) we have $T(8|9) \cong T(4|5) \otimes E^{[1]}$. This implies that $T(8|9)$ has simple head; it therefore appears as a quotient of $P(4|6)$. One can check that the character of $T(4|5) \otimes E^{[1]}$ is equal to that of $P(4|6)$ given above, and therefore $T(8|9) = P(4|6)$ for $S(\leq 8|9)$. By equation (1), $P(1|2)$ is isomorphic to the tilting module $T(5|7)$ for $S(\leq 8|9)$.

The Loewy layers of the module $T(8|9)$ are multiplicity-free and so its structure is given by



as determined by Proposition 2 for both the Δ and ∇-filtrations.

A tilting module $T(\lambda)$ has a unique submodule isomorphic to $\Delta(\lambda)$ and a unique quotient module isomorphic to $\nabla(\lambda)$. By definition (Section 4 of [4, Appendix]) the *core* of $T(\lambda)$ is $C(\lambda) = Q(\lambda)/R(\lambda)$, where $R(\lambda)$ is the radical of $\Delta(\lambda)$ and $Q(\lambda)$ is the kernel of the canonical quotient map $T(\lambda) \to \nabla(\lambda)/L(\lambda)$. It is easily checked that $R(\lambda) \subset Q(\lambda)$ and that $L(\lambda)$ is a direct summand of the quotient $C(\lambda) = Q(\lambda)/R(\lambda)$. From the structure of $T(8|9)$, we see that its core decomposes as a direct sum of $L(8|9)$ and the module pictured in Figure 3.

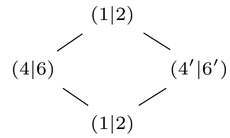
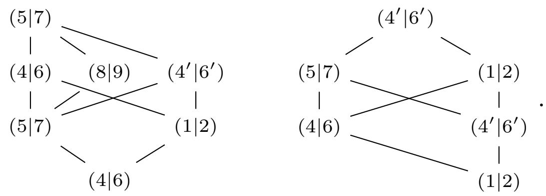


FIGURE 3. The non-simple direct summand of the core of $T(8|9)$.

Arguing by symmetry, we can obtain the structure for $T(8'|9')$ from that of $T(8|9)$ above. This equals $P_{\pi'}(4'|6')$ for the generalized Schur algebra $S(\pi') = S(\leq 8'|9')$. It has a unique submodule isomorphic to $\Delta(8'|9')$ and the corresponding quotient module $T(8'|9')/\Delta(8'|9')$ is isomorphic to $P(4'|6')$ for $S(\leq 8|9)$. This determines the structure of $P(4'|6')$. Furthermore, by Proposition 3 and considerations of the Loewy structure of $P(5|7)$ we deduce that $T(5|7)$ embeds in $T(8|9)$. Therefore, the coefficient quivers of $P(5|7)$ and $P(4'|6')$ are as follows

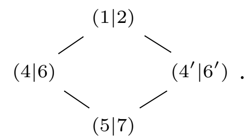


The diagrams allow us to immediately deduce when a coefficient in the quiver is equal to zero. The extensions in the diagram all contribute non-zero coefficients.

From $P(8|9) = \Delta(8|9) = [8|9, 5|7, 4|6]$, we have that $aa' = ac_2' = 0, ac_1'b_1 = ac_1'c_1 = 0$. From the coefficient quiver for $P(5|7)$ we deduce that $c_1'b_1 = c_2'b_2$, $c_1'b_1b_2' = 0$, and $c_1'c_1 = \alpha a'a$, $c_2'c_2 = \beta a'a$, for some non-zero constants $\alpha, \beta \in K$.

Consider the projective module $P(4|6) = T(8|9)$ (as pictured above). From the coefficient quiver for $P(4|6)$, we deduce that $b_1b_j' = \gamma_j c_1 c_j'$ for non-zero coefficients $\gamma_j \in K$ where $j = 1, 2$. By examining $P(4'|6')$ in a similar fashion, we deduce that $b_2b_j' = \delta_j c_2 c_j'$ for non-zero coefficients $\delta_j \in K$, but with the additional relation $c_2a' = 0$.

Consider the structure of the quotient $\nabla(5|7)$ of $T(5|7) = P(1|2)$. By self duality, we have that $b_1'b_1 = \zeta b_2'b_2$ for a non-zero coefficient $\zeta \in K$. The module $\nabla(5|7)$ has coefficient quiver



Choosing a suitable basis of $\nabla(5|7)$, we can assume that at least 3 of the non-zero coefficients are equal to 1 and we look at the remaining coefficient, say that for the arrow c_2. It will be a non-zero scalar $k \in K$. Recall that we have started with a particular generator choice for the algebra $S(\leq 8|9)$ which we can change. If we replace the element c_2 by $(1/k)c_2$, then the coefficients needed for $\nabla(5|7)$ will all be equal to 1. Similarly, the coefficients α, β can to be chosen to be equal to 1.

Now consider the submodules of $P(1|2)$ generated by the copies of $L(1|2)$ in the third Loewy layer; we shall use the self-duality of $P(1|2)$ and the homomorphisms from other projective modules into $P(1|2)$, in order to deduce the values of the coefficients $\gamma_j, \delta_j \in K$ for $j = 1, 2$.

The module $P(4|6) = T(8|9)$ has a unique submodule isomorphic to $\Delta(8|9)$. We let $P'(4|6)$ denote the corresponding quotient module $P(4|6)/\Delta(8|9)$. (The notation reflects that we have trivially inflated the corresponding projective module for $S(\pi')$ for $\pi' = \{(1|2), (4|6), (4'|6'), (5|7)\}$). By Proposition 3 and considerations of Loewy structure, we deduce that there is an injective map f_1 (respectively f_2) from $P(4'|6')$ (respectively $P'(4|6)$) to a submodule of $P(1|2)$. In what follows we shall identify a simple composition factor of a projective module with the path in the quiver which terminates at the given simple composition factor. An example of how to pass between these two pictures (the coefficient quiver and the subspace lattice) is given in [4, Appendix page 217].

The injection f_1 (respectively, f_2) takes the simple head of $P(4'|6')$ (respectively, of $P'(4|6)$) which is labelled by the element $e_{4'|6'}$ (respectively $e_{4|6}$) to the simple composition factor $L(4'|6')$ in the second radical layer of $P(1|2)$, which is labelled by the path b_1' (respectively b_2'). Therefore f_1 (respectively f_2) takes the simple composition factor $L(1|2)$ in the second radical layer of $P(4'|6')$ (respectively $P'(4|6)$) labelled by b_1 (respectively b_2) to the corresponding simple factor $L(1|2)$ in the second radical layer of $P(1|2)$ labelled by $b_1'b_1$ (respectively $b_2'b_2$). From the diagram of $P(4'|6')$ (respectively $P'(4|6)$) we know that the simple composition factor labelled by b_2 (respectively b_1) generates the module with structure as given in Figure 3. We therefore deduce that the copy of $L(1|2)$ in the third Loewy layer of $P(1|2)$ labelled by the path $b_1'b_1$ (respectively $b_2'b_2$) generates a submodule isomorphic to the module given in Figure 3.

By the self-duality of $P(1|2)$ we know that $[(1|2), (4|6), (1|2)]$ and $[(1|2), (4'|6'), (1|2)]$ must also appear as submodules of $P(1|2)$ and are therefore generated by diagonal embeddings of $L(1|2)$ into the third radical layer of $P(1|2)$, these are labelled by linear combinations of the paths $b_1'b_1$ and $b_2'b_2$. Rescaling the generators if necessary, we may choose $b_1'b_1 + b_2'b_2$ (respectively $b_1'b_1 - b_2'b_2$) as the path labelling the diagonal copy of $L(1|2)$ which generates the submodule $[(1|2), (4|6), (1|2)]$ (respectively $[(1|2), (4'|6'), (1|2)]$).

The submodule of $P(1|2)$ generated by the copy of $L(1|2)$ labelled by $b_1'b_1$ has composition factors labelled by the following paths

$$\begin{aligned}
&b_1'b_1\\
&b_1'(b_1b_1') = \gamma_1 b_1'c_1c_1'\\
&b_1'(b_1b_2') = \gamma_2 b_1'c_1c_2'\\
&b_1'(b_1b_1')b_1 = \gamma_1 b_1'c_1c_1'b_1,
\end{aligned}$$

and the submodule generated by the copy of $L(1|2)$ labelled by $b_2'b_2$ has composition factors labelled by the following paths

$$\begin{aligned}
&b_2'b_2\\
&b_2'b_2b_1' = \delta_1 b_2'c_2c_1' = \delta_1\gamma_1 b_1'c_1c_1'\\
&b_2'b_2b_2' = \delta_2 b_2'c_2c_2' = \delta_2\gamma_2 b_1'c_1c_2'\\
&(b_2'b_2)(b_2'b_2) = \zeta^2(b_1'b_1)(b_1'b_1) = \zeta^2 b_1'(b_1b_1')b_1 = \zeta^2\gamma_1 b_1'c_1c_1'b_1.
\end{aligned}$$

From our discussion of the embeddings f_1 and f_2, we deduce that all of these paths are non-zero.

The diagonal copy of $L(1|2)$ labelled by the path $b_1'b_1 + b_2'b_2$ generates the submodule $[(1|2), (4|6), (1|2)]$. Therefore, we require that the coefficients satisfy the following identities

$$\gamma_1 + \delta_1\gamma_1 \neq 0 \qquad \gamma_2 + \delta_2\gamma_2 = 0\ .$$

Here, the left hand side of these (in)equalities is the coefficient of the path labelling the composition factor $L(4|6)$ (respectively $L(4'|6')$) in the submodule generated by the diagonal copy of $L(1|2)$ labelled by the path $b_1'b_1 + b_2'b_2$. We require this coefficient to be non-zero (respectively zero) in order for the submodule generated by the copy of $L(1|2)$ labelled by the path $b_1'b_1+b_2'b_2$ to be of the form $[(1|2),(4|6),(1|2)]$.

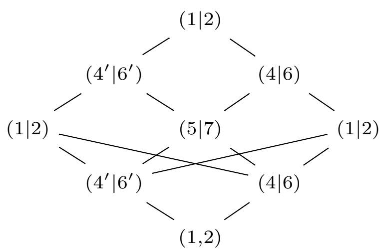
Similar considerations in the case $b_1'b_1 - b_2'b_2$ imply the (in)equalities

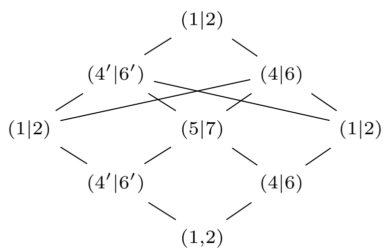
$$\gamma_1 - \delta_1\gamma_1 = 0 \qquad \gamma_2 - \delta_2\gamma_2 \neq 0\ .$$

This implies that the coefficients satisfy $\delta_1 = 1$ and $\delta_2 = -1$. Rescaling if necessary, we may choose to take $\gamma_1 = \gamma_2 = 1$. This completes the proof. $\square$

Remark. Using the "Quivers and Path Algebras" package [20] *for GAP* [13]*, we have checked that the algebra described by quiver and relations in Proposition 4 does have the indicated dimension. We have also checked that the dimensions and Loewy layers of the indecomposable projectives computed by the package agree with our results.*

We provide the two most symmetric coefficient quivers of $T(5|7)$ below.





The only choice to be made is which basis to take for the 2-dimensional space $L(1|2) \oplus L(1|2)$ in the third Loewy layer. The left coefficient quiver corresponds to the basis $N = \{b_1'b_1, b_2'b_2\}$ and the right one corresponds to the basis $N' = \{b_1'b_1 + b_2'b_2, b_1'b_1 - b_2'b_2\}$.

6. The p-regular tilting modules

Having determined the structure of the p-singular tilting modules, we now turn to the p-regular ones. This will complete the proof of Theorem B. The structure of the tilting modules $T(1)$, $T(2)$, and $T(3)$ is easily verified, and is left to the reader, but $T(4)$ is more complicated.

6.1. We first consider the module $T(6)$ as this will be helpful in determining the structure of $T(4)$. This module can be seen to be projective-injective for $S(\leq 6)$ by application of translation functors to the embedding $T(4|6) \hookrightarrow T(5|7)$ (and use of Proposition 1 and 1.4). It is therefore equal to the projective cover of $L(1)$. The diagram of $T(6)$ in Theorem B is obtained using the equalities $\dim_K \mathrm{Ext}^1_G(L(2), L(1)) = 1 = \dim_K \mathrm{Ext}^1_G(L(4), L(1))$ coming from the structure of the Weyl modules, and by reconciling the Δ-filtration with the corresponding ∇-filtration. The Loewy length of this module is 5 and the radical structure is symmetric about the middle, in other words $\mathrm{rad}_i(T(6)) = \mathrm{rad}_{4-i}(T(6))$. Identical statements hold for the the socle series of $T(6)$ by way of the remark following Proposition 2. We therefore conclude that the module is rigid, i.e. $\mathrm{rad}_i(T(6)) = \mathrm{soc}_{4-i}(T(6))$.

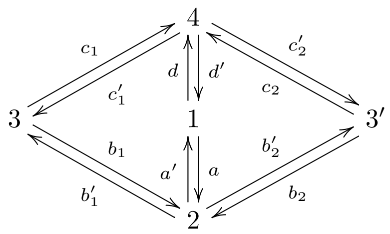


FIGURE 4. The quiver Q

6.2. To determine the structure of $T(4)$ we will study the generalized Schur algebra $S(\leq 4)$. From the structure diagrams of the Weyl modules in Section 4 it follows that the Ext^1-quiver Q for $S(\leq 4) = S(\pi)$ for $\pi = \{1, 2, 3, 3', 4\}$ is as illustrated in Figure 4. By Gabriel's theorem (see e.g. Proposition 4.1.7 in [3]) we have that $S(\leq 4)$ is a quotient of the path algebra of the quiver Q. Notice that applying appropriate translation functors to the embedding $T(3|4) \hookrightarrow T(4'|5)$ produces an embedding $T(4) \hookrightarrow T(5)$; see [16, II.E.11]. By Propositions 1 and 3 it follows that $T(4)$ is the projective cover of $L(2)$ as an $S(\leq 4)$-module.

Before considering the defining relations of $S(\pi) = S(\leq 4)$ we first consider the simpler question of describing the Schur algebra $S' = S(\pi')$ for $\pi' = \{1, 2, 3, 3'\}$, which is a quotient algebra of $S(\leq 4)$, by quiver and relations. From the Weyl module structure, the Ext^1-quiver for S' is the full subquiver $Q' = Q(1, 2, 3, 3')$ of Q obtained by removing the vertex 4 and the arrows $c_1, c_1', c_2, c_2', d, d'$ starting or terminating at that vertex. It will soon become necessary to compare projective indecomposables for S' with those for S. Our notational convention is to use $P(j) = P_\pi(j)$ for the projective cover of $L(j)$ in the algebra $S = S(\pi)$, and $P'(j) = P_{\pi'}(j)$ for the corresponding projective cover in $S' = S(\pi')$. We have the following description of the algebra S'.

Proposition 5. *The basic algebra of the Schur algebra $S' = S(\pi')$ is isomorphic to the path algebra of Q' modulo the ideal generated by the following relations:*

$$ab_i' = 0, \quad aa'a = 0, \quad b_ia' = b_ib_i' = b_ib_j' = 0, \quad a'a = b_1'b_1 + b_2'b_2.$$

where $i \neq j$.

Proof. By Proposition 1, the projective indecomposable modules for S' have the following Δ-factors (going downwards)

$$\begin{array}{cc}
P'(1) & \Delta(1)|\Delta(2) \\
P'(2) & \Delta(2)|\Delta(3) \oplus \Delta(3') \\
P'(3) & \Delta(3) \\
P'(3') & \Delta(3').
\end{array}$$

From the structure of the Weyl modules and Proposition 2 it follows that $P'(1)$, $P'(3)$, and $P'(3')$ are uniserial with structure $P'(1) = [1, 2, 1]$, $P'(3) = \Delta(3) = [3, 2]$, $P'(3') = \Delta(3') = [3', 2]$. This implies the first three relations in the proposition.

We now address the remaining relation $a'a = b_1'b_1 + b_2'b_2$. For this we need structural information about $P'(2)$. Proposition 2 gives the Loewy structure of $P'(2)$ as pictured below

$$\begin{array}{ccc}
 & 2 & \\
 & | & \\
3 & 1 & 3' \\
| & & | \\
2 & & 2
\end{array}$$

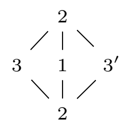
which has a Δ-filtration with subquotients isomorphic to $\Delta(3)$, $\Delta(3')$ and $\Delta(2)$. It is immediate that $a'a = \beta_1 b_1' b_1 + \beta_2 b_2' b_2$ where β_1, β_2 are scalars, since otherwise the independence of the paths $a'a$, $b_1'b_1$, $b_2'b_2$ would force $[P'(2) \colon L(2)] > 3$, which is a contradiction.

By Proposition 3 we have $\dim_K \operatorname{Hom}_G(P'(2), T(3)) = 2$. The Loewy structure of $P'(2)$ implies that one of the two homomorphisms is given by projection of the head of $P'(2)$ onto the socle of $T(3)$, and the other homomorphism is a surjection of $P'(2)$ onto $T(3)$. Therefore $T(3)$ is a quotient module of $P'(2)$ and $\beta_1 \neq 0$. We also have that $\beta_2 \neq 0$ by symmetry. Finally, fixing our choice of a, a', b_1, b_2 and adjusting our choice for b_1', b_2' if necessary, we can pick β_1 and β_2 to both be 1. This concludes the proof. $\square$

6.3. *Definition.* Let $M(2)$ denote the quotient module defined by

$$M(2) = P'(2)/(b_1'b_1 - b_2'b_2).$$

It is clear that $M(2)$ has strong Alperin diagram

$M(2)=$ 

as already mentioned in 3.2. For any $\lambda \in C_2$ we therefore have a module $M(\lambda)$ with similar structure, as described in 3.2.

We are interested in the structure of $T(4)$ and hence wish to consider the modules which appear as both quotients *and* submodules of $T(4)$. This leads us to consider the quotient modules of $P'(2)$ which have simple socle isomorphic to $L(2)$. These are given by taking the quotients corresponding to setting $b_2' = 0$, $b_1' = 0$, $b_1'b_1 - b_2'b_2 = 0$, $b_1'b_1 + b_2'b_2 = 0$ or $a' = 0$. The resulting modules have coefficient quivers depicted in Figure 5. Each of these five modules has a corresponding coefficient quiver, with basis $\mathcal{B}_i$ for $i = 1, \ldots, 5$ respectively.

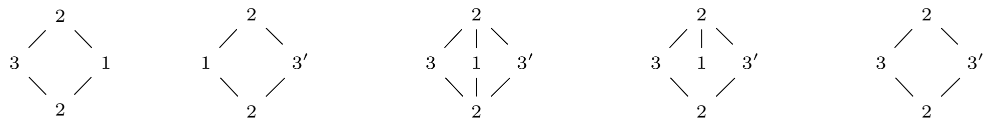
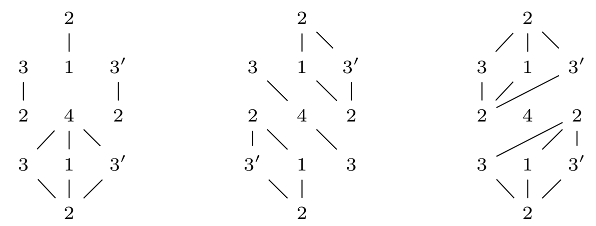


FIGURE 5. The coefficient quivers of quotient modules of $P'(2)$ corresponding $b_2' = 0$, $b_1' = 0$, $b_1'b_1 - b_2'b_2 = 0$, $b_1'b_1 + b_2'b_2 = 0$ or $a' = 0$ respectively. Numbering from left to right, let $\mathcal{B}_i$ denote the basis of the ith coefficient quiver for $i = 1, \ldots, 5$. The non-trivial coefficients are $N_{a,\mathcal{B}_3}(a', b_1'b_1) = 2$, $N_{b_2,\mathcal{B}_4}(b_2', b_1'b_1) = -1$, and $N_{b_2,\mathcal{B}_5}(b_2', b_1'b_1) = -1$. All other coefficients may be chosen to be equal to 1.

6.4. We now turn our attention to computing the defining relations for $S(\leq 4) = S(\pi)$ with $\pi = \{1, 2, 3, 3', 4\}$. The algebra $S(\leq 4)$ is the direct sum of its projective indecomposables $P(1)$, $P(2)$, $P(3)$, $P(3')$ and $P(4)$ and the corresponding projective indecomposables in S' are homomorphic images of these. By character considerations or otherwise it is easy to see that $P(2) = T(4)$. We will soon need the following fact.

Lemma 1. *The tilting module $T(4)$ has three filtrations as depicted in the diagrams below*



in which the connected components in each diagram identify successive subquotients of the filtrations.

Proof. The first filtration, depicted in the leftmost diagram above, is a Δ-filtration whose structure is determined by Proposition 2.

To obtain the second filtration, first pick a weight $\nu \in \mathcal{F}_{(3|4)}$ so that $\nu + \bar{\varepsilon}_1 \in C_4$. (See 1.1 for the notation $\bar{\varepsilon}_1$.) In what follows, fix all alcove weights to be elements of the linkage classes $\nu + \bar{\varepsilon}_j$ for $j = 1, 2, 3$ (There are only two such linkage classes involved, as $\nu + \bar{\varepsilon}_1$ is linked to $\nu + \bar{\varepsilon}_3$.) Note that $T(3|4) = [(2|3'), (3|4), (2|3')]$ is uniserial. Therefore the p-regular linkage class of $E \otimes T(3|4)$ has a filtration with layers given by the p-regular linkage classes of $E \otimes L(2|3')$, $E \otimes L(3|4)$ and $E \otimes L(2|3')$. Since $L(2|3')$ is tilting, it follows by character considerations that the p-regular linkage class of $E \otimes L(2|3')$ is equal to $T(3')$. This justifies the top and bottom connected components in the middle diagram above. It remains to show that the p-regular linkage class of $E \otimes L(3|4)$ is uniserial. It is enough to show that

$$\dim_K(\mathrm{Hom}(L(3), E \otimes L(3|4))) = 1.$$

This is equivalent to showing that $\dim_K(\mathrm{Hom}(L(3) \otimes E^*, L(3|4))) = 1$, which is easily seen to hold, as $L(3) \otimes E^*$ has exactly one composition factor isomorphic to $L(3|4)$.

To obtain the third filtration, we switch temporarily to highest weight notation, and consider for example the tensor product $T(p-2, p-2) \otimes T(p, 0)$, which is tilting, hence a direct sum of indecomposable tilting modules, with $T(2p-2, p-2) = T(4)$ occurring with multiplicity one. Since $T(p-2, p-2) = [K, L(p-2, p-2), K]$ we see that the tensor product has a filtration with layers

$$\begin{array}{c}
T(p,0) \\ \hline
L(p-2,p-2) \otimes L(p-2,1) \\ \hline
L(p-2,p-2) \otimes (L(p,0) \oplus L(p-3,0)) \\ \hline
L(p-2,p-2) \otimes L(p-2,1) \\ \hline
T(p,0)
\end{array}$$

where every layer is contravariantly self-dual (as it is a tensor product of contravariantly self-dual modules). The simple module $L(2p-2, p-2) = L(p-2, p-2) \otimes L(p, 0)$ of highest weight appears as a direct summand of the third layer; it must extend both above and below to result in a module isomorphic to $T(2p-2, p-2)$, i.e. there must exist modules N_1 and N_2 which are direct summands of $T(p, 0)$ and $L(p-2, p-2) \otimes L(p-2, 1)$ respectively such that $N_1|N_2|L(4)|N_2|N_1$ is a filtration of $T(4)$. We do not insist that both N_1 and N_2 are non-zero.

By assumption, N_1 is either equal to $T(p, 0)$ or zero. Assume that $N_1 = T(p, 0)$ for a contradiction. Then (by character considerations) we have that $N_2 = L(0, 2p-3)$, this results in a filtration of $T(2p-2, p-2)$ with layers given by

$$T(p,0)|L(0,2p-3)|L(2p-2,p-2)|L(0,2p-3)|T(p,0).$$

Now, setting $V = L(p-2, p-2)$ to ease the notation, we have

$$\mathrm{Hom}(V \otimes L(p-2,1), L(0,2p-3)) = \mathrm{Hom}(V, L(1,p-2) \otimes L(0,p-3) \otimes (E^*)^{[1]}). \qquad (4)$$

We shall show that the right hand side in the above is equal to zero and thus arrive at a contradiction.

By Lemma 4 of Section 7, the tensor product $L(1,p-2) \otimes L(0,p-2)$ decomposes as a direct sum of indecomposable tilting modules labelled by highest weights in alcoves $1, 2, 3, 4, 4'$ along with a number of p-restricted simple modules. (The proof of Lemma 4 is independent of this subsection.) We have constructed injections of all of these modules into modules of the form $T(2(p-1)\rho + w_0\lambda)$ for $\lambda \in X_1$ and therefore by Donkin's tilting tensor product theorem none of the summands of $(L(1,p-2) \otimes L(0,p-2)) \otimes L(0,1)^{[1]}$ have a p-restricted simple module in their socle. Therefore, the right hand side of (4) is zero, as claimed. Therefore $N_1 \neq T(p,0)$.

This leaves us in the case that $N_1 = 0$, and it remains to deduce the structure of N_2. We know that N_2 is a self-dual quotient of $P'(2)$ and therefore has head and socle isomorphic to $L(2)$. Therefore N_2 appears in the list of modules in Figure 5. There are only two modules in Figure 5 with the correct character: namely the third and fourth. But only the former module is contravariantly self-dual, so we conclude that it is N_2. This shows that $N_2 = M(p-3,1)$. Therefore $T(2p-2, p-2)$ does possess the required third filtration. One may conclude that the desired filtration exists for any $T(\lambda)$ such that $\lambda \in C_4$, by Jantzen's translation principle. This completes the proof. □

Remark. The proof of Lemma 1 shows that the module $M(\nu)$ for $\nu \in C_2$ sometimes appears as a direct summand of some tensor product of the form $L(\lambda) \otimes L(\mu)$ for $\lambda, \mu \in X_1$. We shall see later that this is the case for all $\nu \in C_2$ and that these are the only non-simple, non-tilting indecomposable modules which can appear as a direct summand in such a tensor product.

Proposition 6. *Let $\pi = \{1, 2, 3, 3', 4\}$. The basic algebra of the Schur algebra $S = S(\pi) = S(\leq 4)$ is isomorphic to the path algebra of Q modulo the ideal generated by the following relations:*

$$c_i'c_i = d'd = 0, c_i'b_i = d'a, d'ab_i' = d'aa' = 0, aa'a = 0, ab_1' = -dc_1', ab_2' = dc_2', b_ib_i' = 0,$$
$$b_1a' = c_1d', b_1b_2' = c_1c_2', b_2a' = -c_2d', b_2b_1' = -c_2c_1', b_1'b_1 + b_2'b_2 = a'a, b_i'c_i = a'd,$$

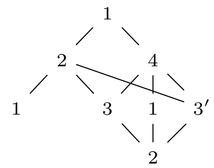
where $i \neq j$.

Proof. By Proposition 1, the projective indecomposable modules for S have the following Δ-factors (going downwards)

$P(1)$	$\Delta(1)\lvert\Delta(2)\rvert\Delta(4)$
$P(2)$	$\Delta(2)\lvert\Delta(3) \oplus \Delta(3')\rvert\Delta(4)$
$P(3)$	$\Delta(3)\lvert\Delta(4)$
$P(3')$	$\Delta(3')\lvert\Delta(4)$
$P(4)$	$\Delta(4)$.

The structure of $P(4) = \Delta(4)$ ensures that the relations $c_i'c_i = d'd = 0, d'ab_i' = d'aa' = 0$, $d'ab_i' = d'aa' = 0$ all hold. We also see that $c_i'b_i$, $d'a$ are all equal up to scalar multiplication, and we may choose to take these coefficients to be equal to 1.

The tilting module $T(6)$ is projective for the generalized Schur algebra $S(\sigma)$ for $\sigma = \{1, 2, 3, 3', 4, 6\}$, and its structure has already been calculated. We let e_π denote the idempotent corresponding to the subset of weights $\pi \subset \rho$ in which we are interested. Applying the idempotent truncation map we see that $e_\pi T(6) = P(1)$. Hence $P(1)$ has the following coefficient quiver

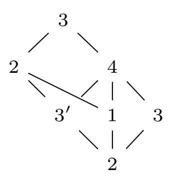
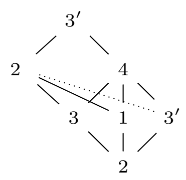


Therefore $aa'a = 0$, $ab'_1 = \alpha_1 dc'_1, ab'_2 = \alpha_2 dc'_2$ where the $\alpha_i \in K$ are non-zero constants which are yet to be determined.

The coefficients we need to understand the structure of $P(3)$ are $\beta_1, \beta_2, \beta_3 \in K$ where $b_1 b'_1 = \beta_1 c_1 c'_1$, $b_1 a' = \beta_2 c_1 d'$, $b_1 b'_2 = \beta_3 c_1 c'_2$. We have seen in Lemma 1 that there exists a uniserial module of the form $[3, 4, 3]$. Therefore this module occurs as a quotient of the projective $P(3)$, this implies that $\beta_1 = 0$. By the structure of $P(1)$, we know that there does not exist a uniserial module of the form $[1, 4, 3]$ and therefore neither does there exist a module of the form $[3, 4, 1]$; therefore $\beta_2 \in K$ is a non-zero constant that is yet to be determined.

We now show that $\beta_3 \neq 0$. The module $P(3)$ has two quotient modules with socle $L(2) = \mathrm{soc}(T(4))$, namely $[3, 2]$ and $P(2)$ itself, and $\dim_K \mathrm{Hom}(P(3), T(4)) = 2$ by Proposition 3. Therefore $P(2)$ embeds into $T(4)$. In what follows we shall identify a simple composition factor of a projective module with the path in the quiver which terminates at the given simple composition factor. The injection f_1 takes the simple head of $P(3)$ which is labelled by the element e_3 to the simple composition factor $L(3)$ in the second radical layer of $P(2)$ labelled by the path b'_1. Therefore f_1 takes the simple composition factor $L(2)$ in the second radical layer of $P(3)$ labelled by b_1 to the simple composition factor $L(2)$ in the third radical layer of $P(2)$ labelled by $b'_1 b_1$. The module $P(2) = T(4)$ is contravariantly self-dual and so each copy of $L(2)$ in the third layer must extend at least one of the $L(3)$ and $L(3')$ in the fourth layer. We therefore deduce that the simple composition factor $L(2)$ in the second Loewy layer of $P(3)$ labelled by the path b_1 generates a submodule of $P(3)$ with either an $L(3)$ or $L(3')$ as a composition factor. We have already seen that $L(3')$ is not a composition factor of this module, therefore we conclude that $L(3)$ is a composition factor and hence $\beta_3 \neq 0$.

Dual arguments hold for all but one of the above statements (allowing us to make conclusions about $P(3')$ and the corresponding coefficients $\gamma_1, \gamma_2, \gamma_3 \in K$). The statement which has no dual comes from the fact that we have not constructed a uniserial module $[3', 4, 3']$, i.e. we do not know if there does or does not exist a uniserial module of the form $[3', 4, 3']$. Therefore the projective modules $P(3)$ and $P(3')$ have the following coefficient quivers (where the extension corresponding to the dashed line may or may not exist),

where the non-zero coefficients $\beta_2, \beta_3, \gamma_2, \gamma_3$ are yet to be determined. The coefficient γ_1, corresponding to the dashed line, will later be shown to be equal to zero.

We now consider the final projective module $P(2) = T(4)$. A ∇-filtration of the module $T(4)$ has $\nabla(4)$ at the top, and so the $b_i c'_i$, $a'd$ are non-zero and span a 1-dimensional space. We may pick the corresponding coefficients to be equal to 1.

Arguing as in the proof of Proposition 5, we conclude that $a'a = \zeta_1 b_1' b_1 + \zeta_2 b_2' b_2$. This is the final path of length two from vertex 2 to itself. At this point, we make a non-trivial choice by setting $\zeta_1 = \zeta_2 = 1$, it is this choice that determines the remaining coefficients. In particular, the quivers of all the self-dual proper quotient of $P(2)$ are given in Lemma 1.

Now consider the coefficients for $P(1)$. There is no submodule of $P(1)$ that is isomorphic to $M(2)$. This can easily be seen as $aa'a = 0$. Therefore the submodule of $P(1)$ generated by the simple module $L(2)$ labelled by the path a is isomorphic to the module $P'(2)/\langle b_1'b_1 + b_2'b_2\rangle$ depicted in Figure 5. By our assumption on the coefficients above, this implies that $ab_1'b_1 = (-1)ab_2'b_2$. This implies that $\alpha_1 = -\alpha_2$.

We now consider the submodule of $T(4)$ generated by the copy of $L(2)$ in the third Loewy layer of $T(4)$ labelled by the path $b_1'b_1 - b_2'b_2$ (respectively $b_1'b_1 + b_2'b_2$). We have already seen in Lemma 1 that $T(4)$ has a filtration $[M(2), L(4), M(2)]$; we have chosen our coefficients so that the submodule isomorphic to $M(2)$ is generated by the copy of $L(2)$ labelled by $b_1'b_1 - b_2'b_2$. This submodule has basis $\mathcal{B} = \{b_1'b_1 - b_2'b_2, a'dc_1', a'dc_2', a'dd', a'dd'a\}$ with coefficient quiver

$$\begin{aligned}
N_{b_1',\mathcal{B}}(b_1'b_1 - b_2'b_2, a'dc_1') &= -\gamma_1 \\
N_{a',\mathcal{B}}(b_1'b_1 - b_2'b_2, a'da') &= \beta_2 - \gamma_2 \\
N_{b_2',\mathcal{B}}(b_1'b_1 - b_2'b_2, a'dc_2') &= \beta_3 - \gamma_3.
\end{aligned}$$

We deduce that $-\gamma_1 = \beta_3 - \gamma_3 = \sigma$ and $\beta_2 - \gamma_2 = 2\sigma$ for some choice of $\sigma \in K$.

We now study the submodule generated by the copy of $L(2)$ labelled by $b_1'b_1 + b_2'b_2$; to do this we study the image of a homomorphism from $P(1)$ to $P(2) = T(4)$. The homomorphism in which we are interested is an injection of $P(1)/\langle aa'\rangle$ into $T(4)$. This takes the simple head of $P(1)$ labelled by the path e_1 to the copy of $L(1)$ in the second Loewy layer of $P(2)$ labelled by a'. It hence takes the simple composition factor $L(2)$ in the second radical layer of $P(1)$ labelled by the path a to the simple composition factor $L(2)$ in the third radical layer of $P(2)$ labelled by the path $a'a = b_1'b_1 + b_2'b_2$. From the structure of $P(1)$ we deduce that the simple composition factor $L(2)$ in the third radical layer of $P(1)$ labelled by the path $b_1'b_1 + b_2'b_2$ generates a submodule as in the rightmost diagram in Figure 5. Therefore $\beta_2 + \gamma_2 = 0$ and and we may choose the coefficients so that $\rho = (\beta_1 + \gamma_1) = -(\beta_3 + \gamma_3)$ for some non-zero constant $\rho \in K$.

The unique solution to this set of relations is $\beta_1 = 0, \beta_2 = \beta_3 = k$ and $\gamma_1 = \gamma_2 = -k$, $\gamma_3 = 0$ for $k = (\sigma + \rho)/2$. We may now fix $k = 1$ and we are done. ☐

Remark. The authors have used the "Quivers and Path Algebras" package [20] *for GAP* [13] *to verify that the algebra defined by the quiver and relations of Proposition 6 has the correct dimension and that the projective modules have the correct dimension and Loewy series structure.*

We wish to consider the possible bases of coefficient quivers for the tilting module $T(4)$. The only choice to be made is which basis to take for the 2-dimensional space $L(2) \oplus L(2)$ in the third Loewy layer. The most obvious choice of basis is given by

$$\mathcal{B} = \{e_2, b_1', a', b_2', b_1'b_1, a'd, b_2'b_2, a'dc_2', a'dd', a'dc_1', a'dd'a\}$$

with respect to this basis the coefficient quiver is the leftmost quiver in Figure 6. The non-trivial coefficients in this coefficient quiver are given by $N_{b_1',\mathcal{B}}(b_2'b_2, a'dc_1') = -1$ and $N_{a',\mathcal{B}}(b_2'b_2, a'dd') = -1$. Here we have taken $N = \{b_1'b_1, b_2'b_2\}$ as the basis for the 2-dimensional space $L(2) \oplus L(2)$ in the third Loewy layer.

An alternative basis for the coefficient quiver is given by substituting $N' = \{b_1'b_1 + b_2'b_2, b_1'b_1 - b_2'b_2\}$ as the basis for the 2-dimensional space $L(2) \oplus L(2)$ in the third Loewy layer. This is depicted in the rightmost diagram in Figure 6.

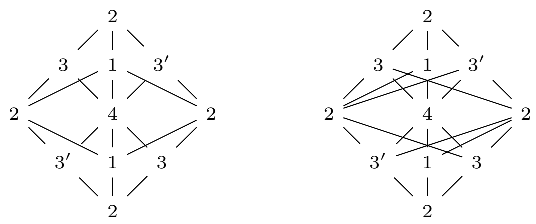


FIGURE 6. Two coefficient quivers for the module $T(4)$.

6.5. By [17, Proposition 4.2], the p-regular linkage class of $E \otimes T(6|8)$ is the module $T(8)$. Arguing as we did for the second filtration in the proof of Lemma 1, the head of $T(8)$ can be seen to be $L(2) \oplus L(4)$.

We know that $\Delta(3)$ must extend $\Delta(2)$ by Proposition 3 and therefore these subquotients are correctly placed within the diagram in Theorem B(b). We know the character of $P(4)$ by Proposition 1. We now consider $\mathrm{Hom}_{S(\leq 8)}(P(4), T(8))$. Since $P(4)$ has a Δ-filtration and $T(8)$ has a ∇-filtration we may apply Proposition 3 to see that $\dim_K \mathrm{Hom}_{S(\leq 8)}(P(4), T(8)) = 4$. This allows us to place the other Weyl modules within the structure diagram of $T(8)$ using Proposition 2.

6.6. It follows from 1.4 that $T(5)$ and $T(7)$ are projective-injective for suitable generalized Schur algebras. The projectivity of $T(9)$ can be seen by appealing to Proposition 1. Therefore Proposition 2 gives the Loewy structures of $T(5)$, $T(7)$ and $T(9)$. The Loewy structures are symmetric about the middle and so the modules are rigid, as claimed in Theorem B.

Since the radical layers of these tilting modules are not multiplicity-free, determining their full structure would be quite complicated by these methods, so we do not pursue this further.

7. Restricted tensor product decompositions: one or both factors tilting

We now turn to the proof of Theorem A, which is split over this section and the next. We need to show that each indecomposable direct summand in a decomposition of a tensor product of two p-restricted simple modules must have one of the the following forms:

(a) a tilting module of highest weight λ such that $\lambda \leq (2p-2)\rho$;

(b) a simple module (which is not tilting) of highest weight in C_2;

(c) a module of the form $M(\lambda)$, for $\lambda \in C_2$.

By highest weight considerations, it is easy to see that each of these possibilities actually occurs in some restricted tensor product, hence the above list provides a complete description of the isomorphism classes of indecomposable summands of restricted tensor products for $G = \mathrm{SL}_3$ (for $p \geq 5$). This will prove Theorem A.

We will see that there is an algorithm for the computation of the multiplicities of the indecomposable direct summands of any p-restricted tensor product. In this section we freely switch between alcove and highest weight notation depending on our needs. Highest weights will usually be written using SL_3 notation but we shall sometimes find it convenient to use GL_3 weight notation in certain calculations; our conventions for such transitions are laid out in 1.1.

Consider the set of p-restricted simple modules $L(\lambda)$ for SL_3. If $\lambda \notin C_2$ then $L(\lambda)$ is tilting; otherwise not. These two cases therefore guide the calculation. The present section considers the indecomposable direct summands of $L(\lambda) \otimes L(\mu)$ in case one or both of the factors is tilting. The more difficult case, in which both factors are not tilting, is considered in the next section.

7.1. For convenience, we work in the representation ring $\mathcal{R} = \mathrm{Rep}_k(\mathrm{SL}_3)$, which is the quotient of the free abelian group on the set $[L(\lambda)]$, as λ varies over X^+, by the subgroup generated by all expressions of the form $[M] - [M'] - [M'']$ such that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of finite-dimensional G-modules.

We have a ring homomorphism from $\mathcal{R}$ into the character ring $\mathbb{Z}[X]^W$, for either case $X = X(T_{\mathrm{GL}_3})$ or $X = X(T_{\mathrm{SL}_3})$, defined by sending $[M]$ for any module M to its formal character $\operatorname{ch} M \in \mathbb{Z}[X]^W$. This homomorphism is injective; i.e., $\operatorname{ch} M = \operatorname{ch} N$ implies $[M] = [N]$ for any finite-dimensional modules M, N. From highest weight considerations we know that any of the sets

$$\{[L(\lambda)] \colon \lambda \in X^+\}, \quad \{[\Delta(\lambda)] \colon \lambda \in X^+\}, \quad \{[\nabla(\lambda)] \colon \lambda \in X^+\}, \quad \{[T(\lambda)] \colon \lambda \in X^+\}$$

is a $\mathbb{Z}$-basis for $\mathcal{R}$. By highest weight theory, if M is a highest weight module of highest weight λ, then in $\mathcal{R}$ we have $[M] = \sum_{\mu \leq \lambda} m_\mu [L(\mu)]$.

7.2. Both factors are tilting. Since the tensor product of two tilting modules is tilting, any tensor product $L(\lambda) \otimes L(\mu)$ of two p-restricted simples such that $\lambda, \mu \notin C_2$ is tilting, and thus decomposes as a direct sum of indecomposable tilting modules. Furthermore, in this case the modules $L(\lambda) = \Delta(\lambda)$, $L(\mu) = \Delta(\mu)$ are also Weyl modules, therefore the (non-negative) coefficients $c^\nu_{\lambda,\mu}$ in the expression

$$[L(\lambda) \otimes L(\mu)] = [\Delta(\lambda) \otimes \Delta(\mu)] = \sum_{\nu \in X^+} c^\nu_{\lambda,\mu} [\Delta(\nu)] \tag{5}$$

are determined by the Littlewood–Richardson rule. We know the characters of the tilting modules by [17,19]; they also appear implicitly in Section 5. Thus we know the coefficients in the expression

$$[T(\nu)] = \sum_{\nu' \in X^+} t_{\nu,\nu'} [\Delta(\nu')]. \tag{6}$$

Note that $t_{\nu,\nu} = 1$ since the highest weight space of any indecomposable tilting module is known to have dimension 1, and furthermore $t_{\nu,\nu'} = 0$ unless $\nu' \leq \nu$.

This allows us to determine the multiplicities of the indecomposable direct summands of $L(\lambda) \otimes L(\mu)$ by highest weight theory, as follows: choose any ν which is maximal among the set of all ν' such that $c^{\nu'}_{\lambda,\mu} \neq 0$ in the finite sum in the right hand side of (5). Then $T(\nu)$ must occur exactly $c^\nu_{\lambda,\mu}$ times as a direct summand of $L(\lambda) \otimes L(\mu)$. Thus we subtract $c^\nu_{\lambda,\mu}[T(\nu)] = \sum_{\nu' \in X^+} c^\nu_{\lambda,\mu} t_{\nu,\nu'} [\Delta(\nu')]$ from the expression in the right hand side of (5), and repeat the procedure on the difference. The process terminates when the expression becomes zero, and termination after a finite number of such steps is guaranteed.

Example. Let $p = 5$ *and consider* $L(1,1) \otimes L(4,0)$. *By applying the Pieri rule for* GL_3 *to the pair of partitions* $((2,1,0))$ *and* $((4))$ *and restricting to* SL_3 *(see 1.1 for the notation and conventions)* *we see that* $[L(1,1)\otimes L(4,0)] = [\Delta(1,1)\otimes\Delta(4,0)] = [\Delta(5,1)]+[\Delta(3,2)]+[\Delta(4,0)]+[\Delta(2,1)]$. *From the known* Δ*-filtration multiplicities of the tilting modules, it follows that*

$$L(1,1)\otimes L(4,0)\simeq T(5,1)\oplus T(4,0)\oplus T(2,1)$$

as $[T(5,1)=[\Delta(5,1)]+[\Delta(3,2)]$, $[T(4,0)]=[\Delta(4,0)]$, *and* $[T(2,1)]=[\Delta(2,1)]$.

7.3. Only one factor is tilting. We now consider the tensor product of $L(\lambda)$ for $\lambda\in C_2$ with any p-restricted simple tilting module $L(\mu)$. Thus, μ is a p-restricted dominant weight belonging to C_1 or one of the walls $\mathcal{F}_{1|2}$, $\mathcal{F}_{2|3}$, $\mathcal{F}_{2|3'}$ of alcove C_2 (see Figure 1), or $L(\mu)=\mathrm{St}$ is the Steinberg module.

Since $L(\mu)$ is tilting, highest weight theory guarantees that it is isomorphic to a direct summand of the tilting module $E^{\otimes\mu_1}\otimes(E^*)^{\otimes\mu_2}$, where $\mu=(\mu_1,\mu_2)$. Therefore we consider the tensor products $L(\lambda)\otimes E$ and $L(\lambda)\otimes E^*$.

The following describes the Weyl filtration of a Weyl module tensored by E or E^*.

Lemma 2. *Let* $\lambda\in X^+$ *be any dominant weight. Recall that* $\overline{\varepsilon}_j$ *for* $j=1,2,3$ *are the weights of* E*. The weights of* E^* *are* $-\overline{\varepsilon}_j$ *for* $j=1,2,3$*. In the representation ring* $\mathcal{R}$ *we have:*

(a) $[\Delta(\lambda)\otimes E]=\sum_{j=1}^{3}[\Delta(\lambda+\overline{\varepsilon}_j)]$;

(b) $[\Delta(\lambda)\otimes E^*]=\sum_{j=1}^{3}[\Delta(\lambda-\overline{\varepsilon}_j)]$

with the stipulation that in the right hand side of each equality, we omit any summand $[\Delta(\lambda\pm\overline{\varepsilon}_j)]$ *of non-dominant highest weight* $\lambda\pm\overline{\varepsilon}_j$*.*

Proof. Let $\lambda=(\lambda_1,\lambda_2)$. We regard λ as arising from the partition $((\lambda_1+\lambda_2,\lambda_2,0))$ written as a GL_3-weight with three components for the sake of emphasis. Tensoring by E we apply the Pieri rule to get the decomposition with Δ-factors of highest GL_3-weight the partitions $((\lambda_1+\lambda_2+1,\lambda_2,0))$, $((\lambda_1+\lambda_2,\lambda_2+1,0))$, and $((\lambda_1+\lambda_2,\lambda_2,1))$ except the last one does not occur if $\lambda_2=0$ and the second one doesn't appear if $\lambda_1=0$. Part (a) then follows by passing to SL_3-weight notation.

Given (a), we can apply it to decompose the tensor product $\Delta(-w_0(\lambda))\otimes E$, which gives

$$[\Delta(-w_0(\lambda))\otimes E]=\sum_{j=1}^{3}[\Delta(-w_0(\lambda)+\overline{\varepsilon}_j)]$$

with the stated stipulation. Now formula (b) follows after applying the symmetry involution $-w_0$ to the weights of the above decomposition. Note that $-w_0(\overline{\varepsilon}_j)=-\overline{\varepsilon}_{4-j}$ for each $j=1,2,3$. Thus $-w_0(-w_0(\lambda)+\overline{\varepsilon}_j)=\lambda-\overline{\varepsilon}_{4-j}$ for each j, and (b) follows. $\square$

Lemma 3. *For any* $\lambda\in C_2$ *we have the decompositions*

(a) $$L(\lambda)\otimes E\simeq\begin{cases}L(\lambda+\overline{\varepsilon}_1)\oplus L(\lambda+\overline{\varepsilon}_2)\oplus L(\lambda+\overline{\varepsilon}_3), & \text{if } \lambda+\overline{\varepsilon}_3\notin\mathcal{F}_{1|2}\\ L(\lambda+\overline{\varepsilon}_1)\oplus L(\lambda+\overline{\varepsilon}_2) & \text{if } \lambda+\overline{\varepsilon}_3\in\mathcal{F}_{1|2}.\end{cases}$$

(b) $$L(\lambda)\otimes E^*\simeq\begin{cases}L(\lambda-\overline{\varepsilon}_1)\oplus L(\lambda-\overline{\varepsilon}_2)\oplus L(\lambda-\overline{\varepsilon}_3), & \text{if } \lambda-\overline{\varepsilon}_1\notin\mathcal{F}_{1|2}\\ L(\lambda-\overline{\varepsilon}_2)\oplus L(\lambda-\overline{\varepsilon}_3) & \text{if } \lambda-\overline{\varepsilon}_1\in\mathcal{F}_{1|2}.\end{cases}$$

Proof. For any $\lambda\in C_2$ we have $[L(\lambda)]=[\Delta(\lambda)]-[\Delta(s_{1|2}\cdot\lambda)]$. Thus we can compute $[L(\lambda)\otimes E]$ and $[L(\lambda)\otimes E^*]$ by the preceding lemma. The stated decompositions now follow from the linkage principle, which guarantees that the simple constituents of the tensor products cannot extend one another. $\square$

We remark that the lemma holds for all primes $p \geq 3$, although we will need it only in characteristics $p \geq 5$.

Examining the direct summands on the right hand side of either decomposition (a) or (b) in Lemma 3, we observe (since $p \geq 5$) that at most one of them can be tilting. In case (a) this happens if and only if $\lambda + \bar{\varepsilon}_1 \in \mathcal{F}_{2|3}$ or $\lambda + \bar{\varepsilon}_2 \in \mathcal{F}_{2|3'}$, and in case (b) it happens if and only if $\lambda - \bar{\varepsilon}_2 \in \mathcal{F}_{2|3}$ or $\lambda - \bar{\varepsilon}_3 \in \mathcal{F}_{2|3'}$. Furthermore, all the non-tilting direct summands are of highest weight belonging to the alcove C_2. So if we now tensor by another E or E^* then, applying Lemma 3 again to the non-tilting summands, we see that the module can be written as a direct sum of simple modules of highest weight in C_2, with one additional summand which is either a tilting module or zero.

By induction on μ_1 and μ_2 it follows that the tensor product $L(\lambda) \otimes E^{\mu_1} \otimes (E^*)^{\mu_2}$ can be decomposed into a direct sum of simple modules of highest weight belonging to C_2, modulo tilting summands. Since $L = L(\mu)$ is a direct summand of $E^{\mu_1} \otimes (E^*)^{\mu_2}$, the module $L(\lambda) \otimes L(\mu)$ is a direct summand of $L(\lambda) \otimes E^{\mu_1} \otimes (E^*)^{\mu_2}$, and it follows that $L(\lambda) \otimes L(\mu)$ is isomorphic to a direct sum of simple modules of highest weight belonging to C_2, modulo tilting summands. Let us record these observations.

Lemma 4. *For any $\lambda \in C_2$ and any p-restricted $\mu = (\mu_1, \mu_2)$ such that $L(\mu) = T(\mu)$ we have:*

(a) $L(\lambda) \otimes E^{\otimes \mu_1} \otimes (E^*)^{\mu_2}$ is a direct sum of simple modules of highest weight in C_2 modulo tilting summands.

(b) The same statement applies to $L(\lambda) \otimes L(\mu)$.

Next we need to analyze the highest weights of the indecomposable tilting summands that can occur. The main point is that they all have highest weight some ν such that $\nu \notin C_1$.

Lemma 5. *If $T(\nu)$ is an indecomposable tilting summand of the tensor product in (a) or (b) of the preceding lemma then $\nu \notin C_1$.*

Proof. By induction it is enough to consider $T(\lambda) \otimes E$ and $T(\lambda) \otimes E^*$. Considering the characters of the modules E and E^* it follows that if $\lambda \in X^+$ then

$$T(\lambda) \otimes E = \sum_{j=1}^{3} \mathrm{pr}_{\lambda + \bar{\varepsilon}_j}(T(\lambda) \otimes E); \quad T(\lambda) \otimes E^* = \sum_{j=1}^{3} \mathrm{pr}_{\lambda - \bar{\varepsilon}_j}(T(\lambda) \otimes E^*). \tag{7}$$

It should be noted that the above sums are not always direct, as it can happen that two or more summands coincide. However, by definition of the functor pr_μ it follows that two summands must be equal whenever they have non-trivial intersection.

Now assume that λ is a dominant weight not in alcove 1. There are three cases to consider: (1) either λ is a vertex (intersection point of two walls), (2) λ is a weight on a wall which is not a vertex, or (3) λ is a p-regular weight (and hence lies in the interior of an alcove). These cases are clearly mutually exclusive.

If λ is a vertex, then $T(\lambda) \otimes E$ and $T(\lambda) \otimes E^*$ have no indecomposable tilting summands of p-regular highest weight. Hence there can be no tilting summands of highest weight in alcove 1.

If λ is on a wall but is not a vertex, then Theorem 4.2 of [17] shows that the unique p-regular tilting summand of either tensor product $T(\lambda) \otimes E$ or $T(\lambda) \otimes E^*$ is indecomposable. Its highest weight is $\lambda + \bar{\varepsilon}_1$ and $\lambda - \bar{\varepsilon}_3$ respectively, and thus cannot lie in alcove 1.

Finally, suppose that λ lies in the interior of its alcove. If $\lambda \pm \bar{\varepsilon}_j$ is p-regular then it lies in the interior of the same alcove. By (7) and the translation principle, in this case $\mathrm{pr}_{\lambda + \bar{\varepsilon}_j}(T(\lambda) \otimes E) = T(\lambda + \bar{\varepsilon}_j)$ and $\mathrm{pr}_{\lambda - \bar{\varepsilon}_j}(T(\lambda) \otimes E^*) = T(\lambda - \bar{\varepsilon}_j)$ and neither highest weight $\lambda \pm \bar{\varepsilon}_j$ lies in alcove 1.

Note that whenever $\lambda \pm \bar{\varepsilon}_j$ is p-singular its corresponding linkage component in (7) cannot produce any tilting module of highest weight in alcove 1. □

7.4. Decomposition algorithm, I. Lemma 5 implies that we can compute the multiplicities of the indecomposable direct summands of $L(\lambda) \otimes L(\mu)$ by the following algorithm:

(a) Express $[L(\lambda) \otimes L(\mu)]$ in terms of the $[\Delta(\nu)]$-basis. This can be done using two applications of the Littlewood–Richardson rule (as in the proof of Lemma 3) as follows

$$[L(\lambda) \otimes L(\mu)] = [\Delta(\lambda) \otimes \Delta(\mu)] - [\Delta(s_{1|2} \cdot \lambda) \otimes \Delta(\mu)] = \sum_{\nu \in X^+} (c^{\nu}_{\lambda,\mu} - c^{\nu}_{s_{1|2}\cdot\lambda,\mu})[\Delta(\nu)].$$

(b) Express $[L(\lambda) \otimes L(\mu)]$ in terms of the $[L(\nu)]$-basis; i.e., compute the composition factor multiplicities in both filtrations and their difference. This produces an expression of the form

$$[L(\lambda) \otimes L(\mu)] = \textstyle\sum_{\nu} d^{\lambda,\mu}_{\nu}[L(\nu)]$$

in which each $d^{\lambda,\mu}_{\nu} \geq 0$.

(c) If $\nu \notin C_1 \cup C_2$ is maximal such that $d^{\lambda,\mu}_{\nu} > 0$, then subtract $d^{\lambda,\mu}_{\nu}[T(\nu)]$. Repeat on the difference, until there do not exist any $\nu \notin C_1 \cup C_2$ appearing in the expression.

(d) At this point, only terms of the form $[L(\nu)]$ for $\nu \in C_1 \cup C_2$ will remain. So we are dealing with an expression of the form $\sum_{\nu \in C_1 \cup C_2} b_\nu[L(\nu)]$. Each term of the form $b_\nu[L(\nu)]$ for $\nu \in C_1$ must, by Lemma 5, be a composition factor of some tilting module of highest weight in C_2. Since $[T(2)] = [L(2)] + 2[L(1)]$ it follows that each b_ν for $\nu \in C_1$ is even, and $b_{s_{1|2}\cdot\nu} \geq \frac{b_\nu}{2}$. Subtract $\frac{b_\nu}{2}[T(s_{1|2} \cdot \nu)]$ from the expression for each such $\nu \in C_1$. The remaining expression is a linear combination of various $[L(\nu)]$ for $\nu \in C_2$, and each of these simples appears as a summand of the decomposition according to its multiplicity.

To summarise step (d): we have shown that for $\nu \in C_1$ representing a given linkage class, the multiplicity of $T(s_{1|2} \cdot \nu)$ in $L(\lambda) \otimes L(\mu)$ is $\frac{b_\nu}{2}$ and the multiplicity of $L(\nu)$ is $b_{s_{1|2}\cdot\nu} - \frac{b_\nu}{2}$. In other words, the multiplicities of $T(s_{1|2} \cdot \nu)$ and $L(\nu)$ in the tensor product are given by the matrix product

$$\begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix} \begin{pmatrix} b_\nu \\ b_{s_{1|2}\cdot\nu} \end{pmatrix}.$$

We note that the above matrix is simply the inverse of

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix},$$

the change of basis matrix expressing the basis $\{[T(2)], [L(2)]\}$ in terms of the basis $\{[L(1)], [L(2)]\}$. A similar issue arises in Section 8.8.

Example. Suppose the characteristic is $p = 5$. Take the tensor product $L(2,2) \otimes L(1,1)$. The Littlewood–Richardson rule gives the character

$$[L(3,3)] + [L(1,4)] + [L(4,1)] + [L(2,2)],$$

and this can be seen to give a direct sum decomposition by the linkage principle.

8. Restricted tensor product decompositions: neither factor is tilting

It remains to describe the indecomposable direct summands of restricted tensor products $L(\lambda) \otimes L(\mu)$ in which neither factor is tilting (i.e., both $\lambda, \mu \in C_2$). Analysis of this remaining case will complete the proof of Theorem A.

8.1. To do this we will need to consider the set of weights in C_2 along the strip

$$\{\nu \in C_2 : \langle (\alpha_1+\alpha_2)^\vee, \nu+\rho\rangle = p-1\}$$

an example of which is pictured in Figure 7. We refer to this set as the set of *minimal* weights in C_2. We first consider a tensor product of the form $L(\sigma)\otimes L(\tau)$, where σ is the unique minimal weight of the form $\lambda - s\bar{\varepsilon}_1$ and τ the unique minimal weight of the form $\mu - t\bar{\varepsilon}_1$. It follows from Lemma 4 that $L(\lambda)$ and $L(\mu)$ are direct summands of $E^{\otimes s}\otimes L(\sigma)$ and $E^{\otimes t}\otimes L(\tau)$, respectively. Therefore $L(\lambda)\otimes L(\mu)$ is a direct summand of

$$E^{\otimes(s+t)}\otimes L(\sigma)\otimes L(\tau).$$

This will allow us to prove results for the decomposition of arbitrary tensor products of the form $L(\lambda)\otimes L(\mu)$ for $\lambda,\mu\in C_2$ by induction.

Example. For example, take $p=7$, $\lambda=(3,5)$ and $\mu=(5,4)$. In this case $s=2$, $t=3$ and so $\sigma=(1,5)$ and $\tau=(2,4)$. Therefore, $L(\lambda)\otimes L(\mu)$ is a direct summand of

$$E^{\otimes 5}\otimes L(1,5)\otimes L(2,4).$$

One can see this in Figure 7.

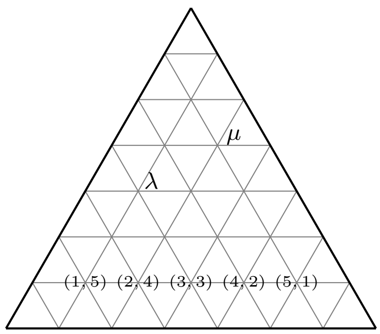


FIGURE 7. The closure of the alcove C_2, for $p=7$. The minimal weights in C_2, $\lambda=(3,5)$ and $\mu=(5,4)$ are at the labelled points.

8.2. We will eventually show that $L(\lambda)\otimes L(\mu)$, for $\lambda,\mu\in C_2$, decomposes as a direct sum of tilting modules and modules of the form $M(\nu)$ for $\nu\in C_2$. Sections 8.2 through 8.6 focus on minimal tensor products. Section 8.7 will deal with the general case.

Let σ,τ be any two minimal weights. There exist $0\le a,b\le p-3$, such that $\sigma=(p-2-a,a+1)$ and $\tau=(p-2-b,b+1)$. We let

$$\sigma' = \begin{cases} s_{2|3}\cdot\sigma \text{ if } a+b\le p-3 \\ s_{2|3'}\cdot\sigma \text{ if } a+b> p-3 \end{cases} \quad\text{and } \tau' = \begin{cases} s_{2|3}\cdot\tau \text{ if } a+b\le p-3 \\ s_{2|3'}\cdot\tau \text{ if } a+b> p-3. \end{cases}$$

Note that in the case that $0\le a+b\le p-3$, σ' and τ' are of the form $(p+a,0)$ and $(p+b,0)$, respectively (the other case is obtained by symmetry). It is clear from Section 4 that we get injections $L(\sigma)\hookrightarrow\Delta(\sigma')\hookrightarrow T(\sigma')$. We will consider the images of the injective homomorphisms

$$L(\sigma)\otimes L(\tau)\hookrightarrow \Delta(\sigma')\otimes\Delta(\tau')\hookrightarrow T(\sigma')\otimes T(\tau'). \tag{8}$$

The tensor product $T(\sigma')\otimes T(\tau')$ has a Δ-filtration and decomposes as a direct sum of tilting modules. We can bound the highest weights of the Δ-modules in such a filtration as illustrated in Figure 8. In the $0 \leq a+b \leq p-3$ case,

$$[T(\sigma')] = [\Delta(p+a,0)] + [\Delta(p-2-a,a+1)]$$
$$[T(\tau')] = [\Delta(p+b,0)] + [\Delta(p-2-b,b+1)]$$

and so the highest weights that appear are bounded by the α_2-string through $(p+a,0)+(p+b,0) = (2p+a+b,0) \in C_6$. The other case, $p-3 < a+b \leq 2(p-3)$, is similar.

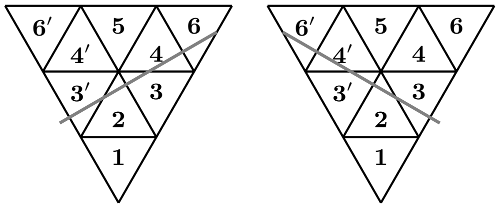


FIGURE 8. A typical example of the highest weights of Δ-modules in a filtration of $T(\sigma') \otimes T(\tau')$ for $0 \leq a+b \leq p-3$ and $p-3 < a+b \leq 2(p-3)$ respectively. The lines cutting across the alcoves are the α_j-strings through $(2p+a+b)\varpi_i$ which bound the weights above (for $i \neq j$).

Having bounded the character of the tensor product, we may now conclude that $T(\sigma') \otimes T(\tau')$ is a direct sum of tilting modules of highest weights in $C_1, C_2, C_3, C_3', C_4, C_6, \mathcal{F}_{2|3'}, \mathcal{F}_{2|3}, \mathcal{F}_{3|4}, \mathcal{F}_{4|6}$ when $0 \leq a+b \leq p-3$ (and those obtained by symmetry in the case $p-3 < a+b \leq 2(p-3)$).

By (8), $L(\sigma) \otimes L(\tau)$ appears as a submodule of such a tilting module. The simple modules are themselves contravariantly self-dual and therefore the tensor product $L(\sigma) \otimes L(\tau)$ is also contravariantly self-dual. Finally, the weights, λ, in $L(\sigma) \otimes L(\tau)$ satisfy the inequality $\langle (\alpha_1 + \alpha_2)^\vee, \lambda + \rho \rangle \leq 2p-2$; therefore the simple modules in the tensor product $L(\sigma) \otimes L(\tau)$ have highest weights in $C_1, C_2, C_3, C_3', \mathcal{F}_{2|3}, \mathcal{F}_{2|3'}, \mathcal{F}_{3|4}$ (or the set obtained by symmetry).

We shall focus on the $0 \leq a+b \leq p-3$ case, as the other case is obtained by symmetry.

8.3. We now focus on the modules $\Delta(\sigma') \otimes \Delta(\tau')$, in order to study $L(\sigma') \otimes L(\tau')$. In the representation ring, the decomposition of $\Delta(\sigma') \otimes \Delta(\tau')$ is given by the Littlewood–Richardson rule as follows

$$[\Delta(p+a,0) \otimes \Delta(p+b,0)] = \sum_{0 \leq j \leq p+b} [\Delta(2p+a+b-2j,j)]. \tag{9}$$

All these weights appear along the α_2-string through $(2p+a+b,0)$, as illustrated in Figure 8. The projection of $\Delta(\sigma') \otimes \Delta(\tau')$ onto any linkage class has a Δ-filtration. When we project onto a linkage class there are six distinct cases which can occur. These are summarised in the table below. We label each case by the highest weight in the linkage class.

character	highest weight	condition
$[\Delta(2\|3)]$	$(p-1, \frac{1}{2}(p+1+a+b))$	$a+b$ is even
$[\Delta(4\|6)]$	$(2p-1, \frac{1}{2}(a+b+1))$	$a+b$ is odd
$[\Delta(3\|4)] + [\Delta(2\|3')]$	$(2p-4-a-b, 2+a+b)$	for all a, b
$[\Delta(3)] + [\Delta(2)]$	$(2p+a+b-2j, j)$	$\frac{1}{2}(p+1+a+b) \leq j < 2+a+b$
$[\Delta(6) + [\Delta(4)]$	$(2p+a+b-2j, j)$	$a < 2j < a+b+1$
$[\Delta(6)] + [\Delta(4)] + [\Delta(3')]$	$(2p+a+b-2j, j)$	$2j \leq a$

8.4. For each possible linkage class of $\Delta(\sigma')\otimes\Delta(\tau')$ in the table in 8.3, we wish to calculate the image

$$L(\sigma)\otimes L(\tau)\hookrightarrow \Delta(\sigma')\otimes\Delta(\tau'). \tag{10}$$

In this section, we shall deal with the first five cases. We shall see that in these five cases, all possible summands are tilting. By Section 4, the character of the image can easily be seen to be given by

$$\begin{aligned}[L(\sigma)\otimes L(\tau)] &= [\Delta(\sigma')\otimes\Delta(\tau')]-[L(\sigma')\otimes L(\tau)]-[L(\sigma)\otimes L(\tau')]\\ &= [\Delta(\sigma')\otimes\Delta(\tau')]-[L(1,0)^{[1]}\otimes L(a,0)\otimes L(\tau)]-[L(\sigma)\otimes L(1,0)^{[1]}\otimes L(b,0)].\end{aligned}$$

Note that, by the Steinberg tensor product theorem, none of the subtracted terms have p-restricted composition factors. Therefore, $L(\sigma)\otimes L(\tau)$ is a submodule of $\Delta(\sigma')\otimes\Delta(\tau')$ which

(a) is contravariantly self-dual;

(b) has simple composition factors whose highest weights satisfy the inequality

$$\langle(\alpha_1+\alpha_2)^\vee,\lambda+\rho\rangle\le 2p-2;$$

(c) satisfies $[\Delta(\sigma')\otimes\Delta(\tau'):L(\nu)]=[L(\sigma)\otimes L(\tau):L(\nu)]$ for any $\nu\in X_1$.

We proceed case by case through the table in Section 8.3. In the first case described in the table, we see for $a+b$ even, that $\Delta(2|3)=L(2|3)$ appears as a direct summand of $\Delta(\sigma')\otimes\Delta(\tau')$. By condition (c) this implies that $L(2|3)$ appears as a direct summand of $L(\sigma)\otimes L(\tau)$.

In the second case, $\Delta(4|6)=[L(4|6),L(1|2)]$ appears as a summand of $\Delta(\sigma')\otimes\Delta(\tau')$. By conditions (b) and (c) this implies that $L(1|2)$ appears as a direct summand of $L(\sigma)\otimes L(\tau)$.

In the third case, we see that N appears as a summand of $\Delta(\sigma')\otimes\Delta(\tau')$, where N is an extension of the form

$$0\to\Delta(3|4)\to N\to\Delta(2|3')\to 0.$$

The issue is whether or not this splits. By (a) and (c), either the extension is split and $L(2|3')\oplus L(2|3')$ is a direct summand of $L(\sigma')\otimes L(\tau')$; or the sequence is the unique non-split extension, $N\cong T(3|4)$, and $T(3|4)$ is a direct summand of $L(\sigma')\otimes L(\tau')$. In either case, the result is a sum of tilting modules (note that $L(2|3')=T(2|3')$).

In the fourth case, we have a direct summand, N, of $\Delta(\sigma')\otimes\Delta(\tau')$ where N is an extension of the form

$$0\to\Delta(3)\to N\to\Delta(2)\to 0.$$

By Section 4, we know that $L(1)$ appears exactly once as a composition factor of N and that $\Delta(2)$ is a non-split extension of $L(2)$ by $L(1)$, which, by (c) is preserved under the embedding of (10). Therefore if N is a split extension, then there exists no submodule of N satisfying properties (a) and (c), which is a contradiction. Hence N is the unique non-split extension and isomorphic to $T(3)$. Now, notice that the only submodule of $T(3)$ satisfying properties (a), (b) and (c) is $T(3)$ itself.

In the fifth case, we have a direct summand, N, of $\Delta(\sigma')\otimes\Delta(\tau')$ where N is an extension of the form

$$0\to\Delta(6)\to N\to\Delta(4)\to 0.$$

Note that $L(2)$ appears with multiplicity one in N and it extends a p-restricted simple module. One can now argue as above by noting that if this extension is split we arrive at a contradiction. Therefore, N is the unique non-split extension. The only possible contravariantly self-dual submodule of N satisfying (c) is $T(2)$.

8.5. We now deal with the final case in the table in Section 8.3. Our aim in this section is to show that in the final case, the summand of $\Delta(\sigma')\otimes\Delta(\tau')$ is isomorphic to $\Delta(6)\oplus N$ where N is a non-split extension of the form

$$0\to\Delta(4)\to N\to\Delta(3)\to 0,$$

and that $L(1)\oplus M(2)$ is a direct summand of $L(\sigma)\otimes L(\tau)$. The character of the tensor product $T(\sigma')\otimes T(\tau')$ is given by

$$[T(\sigma')\otimes T(\tau')]=[\Delta(\sigma')\otimes\Delta(\tau')]+[\Delta(\sigma)\otimes\Delta(\tau')]+[\Delta(\sigma')\otimes\Delta(\tau)]+[\Delta(\sigma)\otimes\Delta(\tau)]. \quad (11)$$

We shall focus on linkage classes appearing along the α_2-root string through $(2p+a+b,0)$. Calculation of any of the tensor product decompositions along this α_2-root string is easily done using the Littlewood–Richardson rule (it is identical to the SL_2 case). Let $(c_1,c_2)=(a_1+b_1-2i,a_2+b_2+i)$ for $i\leq\frac{1}{2}(a_1+b_1)$. Then

$$[\Delta(a_1,a_2)\otimes\Delta(b_1,b_2):\Delta(c_1,c_2)]=\begin{cases}1 & \text{for } i\leq\min\{a_1,b_1\},\\ 0 & \text{otherwise.}\end{cases}$$

Applying this to the four terms in the right-hand side of equation (11) and projecting onto the linkage class with highest weight $(2p+a+b-2j,j)$, for $2j\leq a$, we get $[T(\sigma')\otimes T(\tau'):\Delta(6)]=1$, $[T(\sigma')\otimes T(\tau'):\Delta(4)]=3$ and $[T(\sigma')\otimes T(\tau'):\Delta(3')]=2$.

Highest weight theory tells us that the linkage class of $T(\sigma')\otimes T(\tau')$ in which we are interested therefore has $T(6)\oplus 2T(4)$ as a direct summand (using the method highlighted in Section 7.2). From the characters of $T(6)$ and $T(4)$, we deduce that each of the two copies of $\Delta(3')$ which occur in a Weyl filtration of $T(\sigma')\otimes T(\tau')$ must occur in one of the summands isomorphic to $T(4)$. Therefore $T(3')$ is not a direct summand of the linkage class .

We now turn our attention to the submodule $\Delta(\sigma')\otimes\Delta(\tau')\hookrightarrow T(\sigma')\otimes T(\tau')$. The character of the projection of $\Delta(\sigma')\otimes\Delta(\tau')$ onto the linkage class in which we are interested is $[\Delta(6)]+[\Delta(4)]+[\Delta(3')]$. The corresponding module appears as a submodule of $T(\sigma')\otimes T(\tau')$ and so is isomorphic to $\Delta(6)\oplus N$, where N is the unique non-split extension

$$0\to\Delta(4)\to N\to\Delta(3')\to 0,$$

as we have shown that any $\Delta(3')$ must appear in a $T(4)$.

Finally, $L(\sigma)\otimes L(\tau)$ has character

$$\begin{aligned}[L(\alpha)\otimes L(\beta)]=&[\Delta(p+a,0)\otimes\Delta(p+b,0)]-[L(p+a,0)\otimes L(p+b,0)]\\&-[L(p+a,0)\otimes L(p-2-b,1+b)]-[L(p+b,0)\otimes L(p-2-a,1+a)].\end{aligned}$$

By the Steinberg tensor product theorem, neither of the latter two terms contains an $L(3)$ or an $L(3')$. Considering the second term, we have that

$$\begin{aligned}L(p+a,0)\otimes L(p+b,0)&\cong(L(1,0)\otimes L(1,0))^{[1]}\otimes(L(a,0)\otimes L(b,0))\\&\cong(L(2,0)\oplus L(0,1))^{[1]}\otimes(L(a,0)\otimes L(b,0)).\end{aligned}$$

By the Steinberg tensor product theorem, $L(2,0)^{[1]}\otimes(L(a,0)\otimes L(b,0))$ does not contain an $L(3)$ or an $L(3')$. However, $L(3')$ does appear in $L(0,1)^{[1]}\otimes(L(a,0)\otimes L(b,0))$ with multiplicity equal to 1.

To summarise, we now know that $L(\sigma)\otimes L(\tau)$ satisfies properties (a), (b) and (c) of Section 8.4 and that $[L(\sigma)\otimes L(\tau):L(3')]=[\Delta(\sigma')\otimes\Delta(\tau'):L(3')]-1$ and $[L(\sigma)\otimes L(\tau):L(3)]=$

$[\Delta(\sigma') \otimes \Delta(\tau') : L(3)]$. There is a unique possible submodule which obeys all these properties, given as follows

$$L(1) \oplus M(2) \hookrightarrow \Delta(6) \oplus N \hookrightarrow T(6) \oplus T(4).$$

This follows from the fact that $M(2)$ is the only contravariantly self-dual submodule of $T(4)$ with the correct character (noting that $L(1)$ must occur as a direct summand as it is a submodule of $T(6)$).

8.6. By the above, the projection of $L(\sigma) \otimes L(\tau)$ onto the linkage class containing the highest weight $(2p+a+b-2j, j)$, for $2j \leq a$, $a+b \leq p-3$ is isomorphic to $M(2) \oplus L(1)$.

For $0 \leq 2c \leq p-3$, consider the tensor product $L(\lfloor c/2 \rfloor, 0) \otimes L(\lceil c/2 \rceil, 0)$. All weights in C_6 on the α_2-string through $(2p+c, 0)$ are of the form $(2p+c-2j, j)$ for $0 \leq 2j \leq \lfloor c/2 \rfloor$, and so they all label summands of $L(\lfloor c/2 \rfloor, 0) \otimes L(\lceil c/2 \rceil, 0)$ isomorphic to $M(2) \oplus L(1)$ (the symmetric version also holds). Letting c range over $0 \leq 2c \leq p-3$ (respectively $(p-3) \leq 2c \leq 2(p-3)$) we get that all weights in region B (respectively, region A) of C_6 (respectively C_6') in Figure 9 are of the form $(2p+c-2j, j)$ for some $0 \leq 2j \leq \lfloor c/2 \rfloor$.

Figure 9 illustrates that any weight in C_2 is linked to such a weight; more precisely, weights in region A' are linked to those in region A and similarly for regions B and B'. Therefore all $M(\lambda)$ for $\lambda \in C_2$ appear as direct summands of a minimal tensor product.

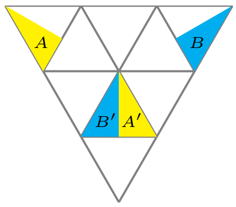


FIGURE 9. Regions A and B contain weights of the form $(2p+c-2j, j)$ and $(j, 2p+c-2j)$, respectively, for $0 \leq 2j \leq \lfloor c/2 \rfloor$. They are linked to regions A' and B' respectively.

8.7. We have now shown that $L(\sigma) \otimes L(\tau)$ is a direct sum of tilting modules and modules of the form $M(\nu)$, for $\nu \in C_2$. By Section 8.1, any tensor product of the form $L(\lambda) \otimes L(\mu)$ for $\lambda, \mu \in C_2$ is a direct summand of $E^{\otimes r} \otimes L(\sigma) \otimes L(\tau)$ such that $0 \leq r \leq 2(p-3)$.

Finally, it remains to show that $E^{\otimes r} \otimes M(2)$ is a direct sum of tilting modules and modules of the form $M(\nu)$ for $\nu \in C_2$. Any p-regular linkage class of $E \otimes M(\nu)$ is of the form $M(\nu')$ for $\nu' \in C_2$, by translation. It remains to check that a p-singular direct summand of $E \otimes M(\nu)$ is tilting. Such a tensor product involves one or two p-singular linkage classes: their highest weights are in $\mathcal{F}_{1|2}$, $\mathcal{F}_{3|4}$ or $\mathcal{F}_{3'|4'}$. A direct summand with highest weight in $\mathcal{F}_{1|2}$ is immediately seen to be a simple tilting module.

It is easy to see that the characters of the other linkage components of $E \otimes M(\nu)$ are equal to the corresponding tilting characters. Let $\gamma = \nu + \bar{\varepsilon}_1$ be a weight in $\mathcal{F}_{3|4}$ or $\mathcal{F}_{3'|4'}$. We have that $L(\gamma) \cong E^{[1]} \otimes L'$ where L' has highest weight in alcove C_1. Therefore $L(\gamma) \otimes E^* \cong E^{[1]} \otimes (L' \otimes E^*)$ has no p-restricted composition factors. The head of $M(\nu)$ is p-restricted, therefore

$$\mathrm{Hom}(M(\nu) \otimes E, L(\gamma)) \cong \mathrm{Hom}(M(\nu), L(\gamma) \otimes E^*) = 0.$$

Therefore $L(3|4)$ is not in the head of $M(\nu) \otimes E$. By the self-duality of $M(\nu) \otimes E$, we conclude that $L(3|4)$ is not in the socle of $M(\nu) \otimes E$. It follows that the linkage component of $E \otimes M(\nu)$ is the uniserial tilting module $[L(2|3'), L(3|4), L(2|3')]$ or its symmetric cousin.

8.8. We let $M'(2)$ denote any direct summand of the form $M(2) \oplus L(1)$ appearing in a minimal tensor product, $L(\sigma) \otimes L(\tau)$. We have seen in our case by case analysis, that a simple module $L(1)$ can appear as a summand of such a tensor product only if it appears as a summand of some $M'(2)$.

We have seen in Section 8.1 that $L(\lambda) \otimes L(\mu)$, for $\lambda, \mu \in C_2$, appears as a direct summand of $E^{\otimes r} \otimes L(\sigma) \otimes L(\tau)$ for $0 \leq r \leq 2(p-3)$. By Lemma 5 a simple module of the form $L(1)$ appears in the tensor product $L(\lambda) \otimes L(\mu)$ as a direct summand if and only if it appears as a direct summand of some $E^{\otimes r} \otimes M'(2)$.

Fixing some linkage class, recall that the characters of the modules $M'(2)$, $T(3), T(3')$ and $T(2)$ in that linkage class, are of the form

$$\begin{aligned}
[M'(2)] &= [L(3)] + [L(3')] + 2[L(2)] + 2[L(1)] \\
[T(3)] &= [L(3)] + 2[L(2)] + [L(1)] \\
[T(3')] &= [L(3')] + 2[L(2)] + [L(1)] \\
[T(2)] &= [L(2)] + 2[L(1)],
\end{aligned}$$

where each module on the right-hand side is the unique simple module in the given linkage class. Note that these characters are linearly independent as the transition matrix,

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 \end{pmatrix},$$

is non-singular. Therefore the decomposition of a tensor product $L(\lambda) \otimes L(\mu)$ is uniquely determined by its character.

8.9. Decomposition algorithm, II. It follows from the above that we can calculate the multiplicities of the indecomposable direct summands of $L(\lambda) \otimes L(\mu)$, for the case $\lambda, \mu \in C_2$, as follows:

(a) Express $[L(\lambda) \otimes L(\mu)]$ in terms of the $[\Delta(\nu)]$-basis. This can be done using three applications of the Littlewood–Richardson rule as follows

$$[L(\lambda) \otimes L(\mu)] = \sum_{\nu \in X^+} (c^{\nu}_{\lambda,\mu} - c^{\nu}_{s_{1|2}\cdot\lambda,\mu} - c^{\nu}_{\lambda,s_{1|2}\cdot\mu} + c^{\nu}_{s_{1|2}\cdot\lambda,s_{1|2}\cdot\mu})[\Delta(\nu)]$$

(b) Express $[L(\lambda) \otimes L(\mu)]$ in terms of the $[L(\nu)]$-basis; i.e., compute the composition factor multiplicities in both filtrations and their difference. This produces an expression of the form

$$[L(\lambda) \otimes L(\mu)] = \textstyle\sum_{\nu} d^{\lambda,\mu}_{\nu}[L(\nu)]$$

in which each $d^{\lambda,\mu}_{\nu} \geq 0$.

(c) If $\nu \notin C_1 \cup C_2 \cup C_3 \cup C_{3'}$ is maximal such that $d^{\lambda,\mu}_{\nu} > 0$, then subtract $d^{\lambda,\mu}_{\nu}[T(\nu)]$. Repeat on the difference, until there do not exist any $\nu \notin C_1 \cup C_2 \cup C_3 \cup C_{3'}$ appearing in the expression.

(d) At this point, only terms of the form $[L(\nu)]$ for $\nu \in C_1 \cup C_2 \cup C_3 \cup C_{3'}$ will remain. Let $\nu \in C_2$ be a representative of a linkage class in the above and consider the projection onto that linkage class. Then we are dealing with an expression of the form

$$b_\nu[L(\nu)] + b_{s_{1|2}\cdot\nu}[L(s_{1|2}\cdot\nu)] + b_{s_{2|3}\cdot\nu}[L(s_{2|3}\cdot\nu)] + b_{s_{2|3'}\cdot\nu}[L(s_{2|3'}\cdot\nu)].$$

Therefore the multiplicities of $M'(\nu), T(s_{2|3}\cdot\nu), T(s_{2|3'}\cdot\nu)$ and $T(\nu)$ in the tensor product are given by the matrix product

$$\begin{pmatrix} 3/4 & 3/4 & -1/2 & 1/4 \\ 1/4 & -3/4 & 1/2 & -1/2 \\ -3/4 & 1/4 & 1/2 & -1/4 \\ -1/2 & -1/2 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} b_{s_{2|3}\cdot\nu} \\ b_{s_{2|3'}\cdot\nu} \\ b_\nu \\ b_{s_{1|2}\cdot\nu} \end{pmatrix}.$$

The 4×4 matrix in the above product is obtained by inverting the transition matrix above. The resulting multiplicities must be non-negative integers.

Example. Suppose the characteristic is $p = 5$. Consider the tensor product $L(3,1) \otimes L(3,1)$. The Littlewood–Richardson rule gives the character

$$[L(6,2)] + 2[L(2,4)] + [L(4,3)] + [L(7,0)] + [L(0,5)] + 2[L(1,3)] + 2[L(0,2)].$$

The p-singular characters $[L(4,3)]$ and $[L(6,2)] + 2[L(2,4)]$ are both tilting. This leaves us with a linkage class component with character $[L(7,0)] + [L(0,5)] + 2[L(1,3)] + 2[L(0,2)]$. This is the character of $M'(1,3)$. Therefore $L(3,1) \otimes L(3,1) = M(1,3) \oplus T(0,2) \oplus T(6,2) \oplus T(4,3)$.

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