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NOTE ON THE REGULARITY OF THE RADICAL OF IDEALS

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Abstract. Let $I = \langle f_1, \ldots, f_k \rangle$ be an ideal generated by homogeneous forms f_i of degree d_i for $i = 1, \ldots, k$ in the $\mathbb{Z}$-graded ring $\mathbb{K}[x_0, x_1, \ldots, x_n]$ where $\mathrm{Char}(\mathbb{K}) = 0$. It is well-known that there is an integer $e(I)$ called Noether exponent defined as $e(I) = \min\{\mu : \sqrt{I}^{\mu} \subset I\}$. In this paper, we estimate the regularity $\mathrm{reg}\left(\sqrt{I}^{e(I)}\right)$ in terms of $\mathrm{reg}\,(I)$ and $e(I)$ in certain cases.

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1. Introduction

In Chapter 14 of [15] Mumford introduced the concept of *regularity* for a coherent sheaf $\mathcal{F}$ on projective space $\mathbb{P}^n$: $\mathcal{F}$ is p-regular if, for all $i \geq 1$ we have vanishing for the twists

$$H^i(\mathbb{P}^n, \mathcal{F}(k)) = 0, \quad \text{for all } k + i = p.$$

This in turn implies the stronger condition of vanishing for $k + i \geq p$. Regularity was investigated later by several people, notably Bayer and Mumford [1], Bayer and Stillman [2], Eisenbud and Goto [9], and Ooishi [16]. Let $R = \mathbb{K}[x_0, ..., x_n]$ be the polynomial algebra in $n + 1$ variables over a field K, graded in the usual way. If M is a finitely generated graded R-module, then the local cohomology groups $H^i_{\mathfrak{m}}(M)$ with respect to the ideal $\mathfrak{m} = (x_0, ..., x_n)$ are graded in a natural way and we say that M is p-regular if

$$H^i_{\mathfrak{m}}(M)_k = 0 \quad \text{for all } k + i \geq p + 1.$$

If $\mathcal{F}$ is the coherent sheaf on $\mathbb{P}^n$ associated with M in the usual way, we have

$$H^{i+1}_{\mathfrak{m}}(M)_k = H^i(\mathbb{P}^n, \mathcal{F}(k)) \quad \text{for all } i \geq 1,$$

which shows the compatibility of these definitions.

An important result in this theory is the following: suppose $\mathbb{K}$ is a field and $I \subset R$ is a graded ideal, then I is p-regular if and only if the minimal free graded resolution of I has the form

$$0 \longrightarrow \bigoplus_{\alpha=1}^{r_s} Re_{\alpha,s} \longrightarrow \cdots \longrightarrow \bigoplus_{\alpha=1}^{r_0} Re_{\alpha,0} \longrightarrow I \longrightarrow 0$$

where $\deg(e_{\alpha,i}) \leq p+i$ for all $i \geq 0$. Hence, one can obtain the regularity of an ideal from the degree twists of its minimal free graded resolution, and vice versa.

In this short note we raise the general question: *Is there any way to compare the regularity of an ideal I with that of its radical $\sqrt{I}$?*

Here is an example: Let I be a monomial ideal in a polynomial ring $R = \mathbb{K}[x_1, \dots, x_n]$. Then the primary decomposition of I has the form $I = \cap_{i=1}^{s} \mathfrak{p}_i^{d_i}$, where each associated prime $\mathfrak{p}_i \in \operatorname{Ass}(R/I)$ is generated by a subset of the the elements $\{x_1, \dots, x_n\}$. Derksen and Sidman [8] proved that $\operatorname{reg}(\cap_{i=1}^{d} I_i) \leq d$ for any set of ideals $I_1, \cdots, I_d$ generated by linear forms, which had been conjectured previously by Sturmfels. Therefore, $\operatorname{reg}(\sqrt{I}) = \operatorname{reg}(\cap_{i=1}^{s} \mathfrak{p}_i) \leq s$, where s is the number of associated primes of R/I. Moreover, if I is a square-free monomial ideal, then by [Remark 15, [11]] the primary decomposition of I has the shape $I = \cap_{i=1}^{s} \mathfrak{p}_i^{d_i}$ with $\mathfrak{p}_i \not\subset \mathfrak{p}_j$. Thus, if I is a square-free monomial ideal, then $\operatorname{reg}(\sqrt{I}) = s$ where s is the number of associated primes of R/I.

Many studies have been done to understand the relationship between $\operatorname{reg}(I)$ and $\operatorname{reg}(\sqrt{I})$. The paper by Ravi [17] proved that $\operatorname{reg}(\sqrt{I}) \leq \operatorname{reg}(I)$ if R/I is a Buchsbaum R-module, or if I is a monomial ideal, or in some cases $\sqrt{I}$ defines a non-singular curve in $\mathbb{P}^3$. But this is not true in general, for example, Chardin-D'Cruz [6] considered the family of complete intersection ideals

$$I_{m,n} = \langle x^m t - y^m z, z^{n+2} - xt^{n+1} \rangle \subset \mathbb{K}[x, y, z, t].$$

Then for all $m, n \geq 1$,

$$\operatorname{reg}(I_{m,n}) = m+n+2, \quad \operatorname{reg}(\sqrt{I_{m,n}}) = mn+2.$$

Hence regularity of the radical may be much larger than the regularity of the ideal itself. To our knowledge, there is not yet a complete general answer to this question.

It is well-known that there is an integer $e(I) = \min\{\mu : \sqrt{I}^{\mu} \subset I\}$ called Noether exponent, which is the smallest integer μ such that the $(\sqrt{I})^{\mu}$ is contained in I. There are recent papers by Kollár [14], Jelonek [12], Sombra [18] and others on effective versions of the Nullstellensatz that give quite good bounds on $e(I)$. An interesting question is to estimate the regularity of $\sqrt{I}^{e(I)}$ in terms of $\operatorname{reg}(I)$ and $e(I)$.

In this paper, our focus is to provide bound for the $\operatorname{reg}(\sqrt{I}^{e(I)})$ in terms of $\operatorname{reg}(I)$ and $e(I)$. Our approach is to study $\dim R/I$, the Krull dimension of R/I. In Section 2 we analyze the case when $\dim R/I \leq 1$. The case when $\dim R/I \geq 2$

is extremely complicated, hence we focus our attention to irreducible projective varieties of minimal degree in Section 3. Illustrative computational examples are provided.

2. The case for $\dim R/I \leq 1$

In this section, we study the case when the Krull dimension $\dim R/I \leq 1$. To do this, let us recall some useful definitions and results.

Recall basic definitions: An ideal $I \subset R$ is called a *primary* ideal, if whenever $ab \in I$ then either $a \in I$ or $b \in \sqrt{I}$ for all $a, b \in R$. An ideal I is a $\mathfrak{p}$-*primary* ideal, if $\sqrt{I} = \mathfrak{p}$ for a prime ideal $\mathfrak{p}$. Any ideal has an irredundant primary decomposition, $I = \bigcap_{i=1}^{k} \mathfrak{q}_i$, where $\mathfrak{q}_\mathfrak{i}$ is a $\mathfrak{p}_i$-primary ideal, and the $\mathfrak{p}_i$'s are called *associated primes of* I. Moreover, if $\mathfrak{p}_i$ does not properly contain any other associated prime, then it is called *a minimal associated primes of* I. The non-minimal associated primes are called *embedded associated primes*. For example, $I = \langle x^2, xy \rangle = \langle x \rangle \cap \langle x^2, y \rangle$ is a primary decomposition, where the minimal associated prime is $\langle x \rangle$ and the embedded associated prime is $\langle x, y \rangle$.

Definition 2.1. Let $I \subset R = \mathbb{K}[x_0, \ldots, x_n]$, be a graded ideal. Then the saturation of the ideal I is defined as

$$I^{\text{sat}} = \{r \in R \mid \mathfrak{m}^k r \subset I \text{ for some } k\}.$$

Theorem 2.2. *Let $R = \mathbb{K}[x_0, \ldots, x_n]$, and I be a homogeneous graded ideal. If the Krull dimension* $\dim R/I = 0$, *then* $e(I) = \operatorname{reg} I$, *and* $\operatorname{reg} (\sqrt{I})^{e(I)} = \operatorname{reg} I$.

Proof. First, we note that if Krull dimension $\dim R/I = 0$, then $\mathbb{V}(I) = \emptyset$ in $\mathbb{P}^n(\overline{\mathbb{K}})$ ($\overline{\mathbb{K}}$ = algebraic closure of $\mathbb{K}$), and $\sqrt{I} = \mathfrak{m} = \langle x_0, \ldots, x_n \rangle$, i.e., I is a graded $\mathfrak{m}$-primary ideal. It is well-known that if $\dim R/I = 0$, then $I_i = R_i$ for all $i \geq p$ if and only if $\operatorname{reg} I = p$. By definition, $e(I) = \min\{\mu \in \mathbb{Z} \mid (\sqrt{I})^\mu = \mathfrak{m}^\mu \subset I\}$. Since $\mathfrak{m}^\mu$ is generated by R_μ we see that $e(I) = \min\{\mu \in \mathbb{Z} \mid R_\mu = I_\mu\} = \operatorname{reg} I$. Moreover, $(\sqrt{I}^{e(I)})_\mu = (\mathfrak{m}^{e(I)})_\mu = R_\mu$ for $\mu \geq e(I) = \operatorname{reg} I$, but $(\sqrt{I}^{e(I)})_\mu = (\mathfrak{m}^{e(I)})_\mu = 0$ for $\mu < e(I) = \operatorname{reg} I$. Therefore, $\operatorname{reg} (\sqrt{I})^{e(I)} = \operatorname{reg} I$. This completes the proof of the theorem.

For convenience we recall the proof of the implication: $\dim R/I = 0$, then $I_i = R_i$ for all $i \geq p$ if and only if $\operatorname{reg} I = p$. First note that $\mathfrak{m}^k \subseteq I$ for some positive integer k, and for any $r \in R$ we have $r\mathfrak{m}^\mu \subseteq I$ for some μ. Hence $I^{\text{sat}} = R$. If $\dim R/I = 0$, then $H^i_\mathfrak{m}(R/I) = 0$ for $i \geq 1$. Consider the following exact sequence:

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$

The cohomology sequence will be:

$$H^{i-1}_{\mathfrak{m}}(R/I) \longrightarrow H^i_{\mathfrak{m}}(I) \longrightarrow H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(R/I).$$

If $i \geq 2$, then $H^i_{\mathfrak{m}}(I) = H^i_{\mathfrak{m}}(R)$. If $i = 1$, then $H^1_{\mathfrak{m}}(I) = I^{\mathrm{sat}}/I = R/I$. If $i = 0$, then $H^0_{\mathfrak{m}}(I) = 0$. Since R is 0-regular, we have $H^i_{\mathfrak{m}}(I)_k = H^i_{\mathfrak{m}}(R)_k = 0$ for all $k \geq 0$. I is p-regular, $H^1_{\mathfrak{m}}(I)_k = R_k/I_k = 0$ for all $k \geq p$. Therefore, we have the following:

$$H^i_{\mathfrak{m}}(I)_k = 0,$$

for all $k \geq p$. Thus, I is p-regular. □

Example 2.3. *Let $R = \mathbb{K}[x, y, z, t]$, and $I = \langle x^2, y, z, t\rangle$. Then*

$$\mathbb{V}(I) = \emptyset \subset \mathbb{P}^3,\ \ \sqrt{I} = \mathfrak{m},\ \ e(I) = \operatorname{reg}(I) = 2,\ \ \operatorname{reg}(\sqrt{I})^{e(I)} = 2.$$

We also observe that $(\sqrt{I})^{e(I)}_{\mu} = (\sqrt{I})^2_{\mu} = R_{\mu}$, for all $\mu \geq 2$.

Remark 2.4. *I^{sat} is the largest ideal that defines the same closed subscheme of $\mathbb{P}^n$ as I does. Every closed subscheme of $\mathbb{P}^n$ is defined by some homogeneous ideal, and there is a bijection between closed subschemes of $\mathbb{P}^n$ and homogeneous saturated ideals. A radical ideal is saturated. Furthermore, given a homogeneous ideal $I \subset R$, the subscheme of $\mathbb{P}^n$ defined by I is reduced if and only if I^{sat} is a radical ideal.*

Remark 2.5. *Let $I \subset R = \mathbb{K}[x_0, \ldots, x_n]$ be a graded ideal such that the Krull dimension $\dim R/I = 1$. Then I has an irredundant primary decomposition, $I = (\cap_{i=1}^s \mathfrak{q}_i) \bigcap J$, where the $\sqrt{\mathfrak{q}_i} = \mathfrak{p}_i$ correspond to the distinct points in $\mathbb{P}^n$, and $\sqrt{J} = \mathfrak{m} = \langle x_0, \ldots, x_n\rangle$.*

Lemma 2.6. *Let $R = \mathbb{K}[x_0, \ldots, x_n]$, and I a homogeneous graded ideal. If the Krull dimension $\dim R/I = 1$, then $\sqrt{I} = I^{\mathrm{sat}}$.*

Proof. Since I^{sat} is the largest ideal that defines the same closed subscheme of $\mathbb{P}^n$ as I does, both I^{sat} and $\sqrt{I}$ define the same subset of $\mathbb{P}^n(\overline{\mathbb{K}})$, we have $\sqrt{I} \subset I^{\mathrm{sat}}$ because $\sqrt{I}$ is the largest ideal defining $\mathbb{V}(I)$. On the other hand $r \in I^{\mathrm{sat}}$ implies that

$$\mathfrak{m}^k r \subset I \subset \sqrt{I} = \sqrt{\cap_{i=1}^s \mathfrak{q}_i} \bigcap \sqrt{J} = \cap_{i=1}^s \mathfrak{p}_i \bigcap \mathfrak{m} = \cap_{i=1}^s \mathfrak{p}_i.$$

We claim that $r \in \cap_{i=1}^s \mathfrak{p}_i$, otherwise, if $r \notin \mathfrak{p}_j$ for some j, then $rx_i^k \in \mathfrak{p}_j$ for $i = 0, \ldots, n$ implies that $x_i^k \in \mathfrak{p}_j$ for $i = 0, \ldots, n$, hence $\mathfrak{m} \subset \mathfrak{p}_j$, contradicting the condition that each $\mathfrak{p}_i$ for $i = 1, \ldots, s$ corresponds to point in $\mathbb{P}^n$, i.e., $\mathrm{ht}(\mathfrak{p}_j) = n-1$. Therefore, $r \in \cap_{i=1}^s \mathfrak{p}_i = \sqrt{I}$. Thus, $\sqrt{I} = I^{\mathrm{sat}}$ as claimed. □

Theorem 2.7. *Let $R = \mathbb{K}[x_0, \ldots, x_n]$, and $I = \langle f_1, \ldots, f_k \rangle$ a homogeneous graded ideal generated by f_i with $d_1 \geq \cdots \geq d_k$ where $d_i = \deg f_i$ for $i = 1, \ldots, k$. If the Krull dimension $\dim R/I = 1$ with irredundant primary decomposition of the form $I = (\cap_{i=1}^{s} \mathfrak{q}_i) \bigcap J$ where $\sqrt{J} = \mathfrak{m} = \langle x_0, \ldots, x_n \rangle$, then*

$$\operatorname{reg}(\sqrt{I})^{e(I)} \leq e(I) \cdot s, \text{ where } e(I) \leq \max\{e(\mathfrak{q}_1), \ldots, e(\mathfrak{q}_s), e(J)\},$$

and s is the number of the associated primes of I.

Proof. First, we note that it is proved by Chandler [4] and Chardin [5] that $\operatorname{reg}(I^k) \leq k \cdot \operatorname{reg} I$ if $\dim R/I \leq 1$. By Lemma 2.6, we have that $\sqrt{I} = \cap_{i=1}^{s} \mathfrak{p}_i$ is a saturated ideal, and each $\mathfrak{p}_i$ is generated by linear forms. By Derksen and Sidman [8],

$$\operatorname{reg}(\sqrt{I}) = \operatorname{reg}(\cap_{i=1}^{s} \mathfrak{p}_i) \leq \sum_{i=1}^{s} \operatorname{reg}(\mathfrak{p}_i) = s,$$

where s is the number of the associated primes of I. Since we are assuming $\dim R/I \leq 1$, the above inequality is an equality. This follows from the main result of [11] already mentioned in the introduction to this paper, since there are no inclusion relations among the ideals $\mathfrak{p}_i$.

Therefore,

$$\operatorname{reg}(\sqrt{I})^{e(I)} \leq e(I) \cdot \operatorname{reg} \sqrt{I} = e(I) \cdot s, \text{ where } e(I) \leq \max\{e(\mathfrak{q}_1), \ldots, e(\mathfrak{q}_s), e(J)\},$$

where the last inequality follows easily from the definition of the Noether exponent. □

Example 2.8. *Let $R = \mathbb{K}[x, y, z, t]$, $I = \langle x^2, xt, y, z \rangle$. Then*

$$I = \langle x, y, z \rangle \cap \langle x^2, y, z, t \rangle, \ \sqrt{I} = \langle x, y, z \rangle \cap \langle x, y, z, t \rangle = \langle x, y, z \rangle,$$

and

$$e(I) = 2, \ \operatorname{reg}(\sqrt{I})^{e(I)} \leq e(I) \cdot \operatorname{reg}(\sqrt{I}) = 2 \cdot 1 = 2.$$

We also observe that $\sqrt{I} = I^{\text{sat}}$.

3. The case for $\dim R/I \geq 2$

The results for $\dim R/I \leq 1$ are no long true in higher dimensions. When $I \subset R$ is a homogeneous ideal, it is known from the work of Cutkosky, Herzog, Trun [7] and Kodiyalam [13] that the regularity of I^k is asymptotically a linear function in k. Many authors have studied the the function $\operatorname{reg}(I^k)$ from various perspectives. When I is generated by forms of a given degree, say d, and all its powers have a linear resolution, which implies $\operatorname{reg}(I^k) = dk$ for all k, we say that this ideal is an *ideal with linear powers*. Similarly, we say that a projective variety has linear powers when its defining ideal has linear powers.

Assume a variety $V \subset \mathbb{P}^n$ is irreducible. It is well-known that $\deg(V) \geq n - \dim V + 1$, and when $\deg(V) = n - \dim +1$, we obtain the irreducible varieties of minimal degrees. It is known that the irreducible varieties of minimal degree are the rational normal scrolls, the quadric hypersurfaces and the cone over the Veronese surface in $\mathbb{P}^5$ (see [Theorem 2, [19]]).

Theorem 3.1. *Let I be an ideal such that $\mathbb{V}(I)$ is an irreducible variety in $\mathbb{P}^n$ of minimal degree, and spanning $\mathbb{P}^n$. Then* $\operatorname{reg}(\sqrt{I})^k = 2k$.

Proof. It is proven in [3] that the ideal $\mathbb{I}(V)$ of an irreducible variety V of minimal degree has linear powers. By the Nullstellensatz, we are assuming that $\sqrt{I} = \mathbb{I}(V)$ for a minimal variety V. As mentioned above, this shows that $\operatorname{reg}(\sqrt{I})^k = dk$ for all k, where d is the common degree of the generators of $\sqrt{I}$. From the classification of varieties of minimal degree, we have $d = 2$. This shows in particular that $\operatorname{reg}(\sqrt{I}) = 2$, which also follows from the results in [10]. □

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