HOMOLOGICAL DIMENSION OF SMASH PRODUCT OVER QUASITRIANGULAR WEAK HOPF ALGEBRA

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ABSTRACT. Let (H, R) be a quasitriangular weak Hopf algebra, and A a quantum commutative weak H-module algebra. We establish the relationship of homological dimensions between weak smash product algebra $A\#H$ and A under some conditions. As an application, we consider the case of twisted weak Hopf algebra.

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1. Introduction

Weak Hopf algebras were introduced by Böhm et al. in [3], as a generalized notation of Hopf algebras and groupoid algebras. Roughly speaking, a weak Hopf algebra is an object which has both algebra and coalgebra structures with some relations between them and that possesses an antipode which does not necessarily satisfy the usual convolution equalities with the identity morphism.

The main difference between ordinary and weak Hopf algebras is that the comultiplication of the latter is no longer required to preserve the unit, or equivalently, the counit is not required to be an algebra homomorphism. The motivations to study weak Hopf algebras mainly come from their connection with the theory of algebra extensions, the applications in the study of dynamical twists of Hopf algebras and their link with quantum field theories and operator algebras (see [13]).

The notion of quasitriangular weak Hopf algebras was introduced by Nikshych and Vainerman in [13] and consequently studied in [12], where it was shown that quasitriangular weak Hopf algebras play an important role in the quantum group theory, particularly in knot theory. It is known that the R-matrix in a quasitriangular weak Hopf algebra can generate a solution for the quantum Yang-Baxter

equation $R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$ which frequently appears in many contexts of mathematical physics. From categorical point of view quasitriangular Hopf algebras are characterized by the fact that its representation categories (categories of modules having finite dimensional over ground field) are braided rigid tensor categories.

The relationship of homological dimensions between an algebra and its some subalgebra have been investigated in many papers up to now. For example, Auslander proved in [1] that $gl.dim(\Lambda G) = gl.dim(\Lambda)$ for a skew group algebra ΛG with the order of G invertible in Λ. In [20] Yang established the relationship of homological dimension between smash product algebra $A\#H$ and quantum commutative H-module algebra A over a Hopf algebra H. It was shown that $gl.dim(A\#H) = n$ if and only if $gl.dim(A) = n$ and there exists an element $c \in A$ such that $t \cdot c = 1_A$, where $t \in H$ is a left integral. Jia and Li proved that for a quantum commutative weak H-module algebra A over a semisimple quasitriangular weak Hopf algebra H, the global dimension of weak smash product $A\#H$ equals that of A in [8]. Let H be a semisimple weak Hopf algebra, A a weak H-comodule algebra and B the coinvariant subalgebra of A. If the algebra extension A/B is H-Galois, then it was proved by Zhou in [22] the global dimension of A is no more than that of B.

Inspired by [20], in this paper we shall investigate the relationship of homological dimensions between a quantum commutative weak H-module algebra A over a quasitriangular weak Hopf algebra H and that of weak smash product $A\#H$. As a main result of Section 2, we show that $gl.dim(A\#H) = n$ if and only if $gl.dim(A) = n$ and there exists an element $c \in A$ such that $t \cdot c = 1_A$, where $t \in H$ is a left integral. As an application, we consider the case of twisted weak Hopf algebra in Section 3.

Throughout this paper, we always work over a fixed field k. Any unexplained definitions and notations of algebras, coalgebras, modules and comodules may be found in [10] or [16].

Definition 1.1. [3] Let H be both an algebra and a coalgebra. Then H is called a *weak bialgebra* if it satisfies the following conditions:

$$\Delta(xy) = \Delta(x)\Delta(y), \tag{1}$$

$$\varepsilon(xyz) = \Sigma\varepsilon(xy_1)\varepsilon(y_2z) = \Sigma\varepsilon(xy_2)\varepsilon(y_1z), \tag{2}$$

$$\Delta^2(1_H) = (\Delta(1_H) \otimes 1_H)(1_H \otimes \Delta(1_H)) \tag{3}$$

$$= (1_H \otimes \Delta(1_H))(\Delta(1_H) \otimes 1_H), \tag{4}$$

for any $x, y, z \in H$, where $\Delta(1_H) = 1_1 \otimes 1_2$ and $\Delta^2 = (\Delta \otimes id_H) \circ \Delta$.

Moreover, if there exists a k-linear map $S : H \to H$, called antipode, satisfying the following axioms for all $h \in H$:

$$\Sigma h_1 S(h_2) = \Sigma \varepsilon(1_1 h) 1_2, \quad \Sigma S(h_1) h_2 = \Sigma \varepsilon(h 1_1) 1_2, \quad \Sigma S(h_1) h_2 S(h_1) = S(h), \quad (5)$$

then the weak bialgebra H is called a *weak Hopf algebra*.

Definition 1.2. [12] A *quasitriangular weak Hopf algebra* is a pair (H, R), where H is a weak Hopf algebra and $R = R^{(1)} \otimes R^{(2)} \in \Delta^{op}(1)(H \otimes H)\Delta(1)$ satisfies the following conditions:

$$\Delta^{op}(h)R = R\Delta(h), \tag{6}$$

$$\Sigma \Delta(R^{(1)}) \otimes R^{(2)} = \Sigma R^{(1)} \otimes r^{(1)} \otimes R^{(2)} r^{(2)}, \tag{7}$$

$$\Sigma R^{(1)} \otimes \Delta(R^{(2)}) = \Sigma R^{(1)} r^{(1)} \otimes r^{(2)} \otimes R^{(2)}, \tag{8}$$

for all $h \in H$, and such that there exists $R^{-1} \in \Delta(1)(H \otimes H)\Delta^{op}(1)$ with

$$RR^{-1} = \Delta^{op}(1), \ R^{-1}R = \Delta(1), \tag{9}$$

where $R = \Sigma R^{(1)} \otimes R^{(2)} = \Sigma r^{(1)} \otimes r^{(2)}$, $\Delta^{op}(h) = \Sigma h_2 \otimes h_1$ for all $h \in H$.

For any weak bialgebra H, it is well known that the maps $\Pi^L, \Pi^R, \overline{\Pi}^L$ and $\overline{\Pi}^R : H \to H$ are projections. They are given by the formulas: $\Pi^L(h) = \Sigma \varepsilon(1_1 h) 1_2$, $\overline{\Pi}^L(h) = \Sigma \varepsilon(1_2 h) 1_1$, $\Pi^R(h) = \Sigma \varepsilon(h 1_2) 1_1$ and $\overline{\Pi}^R(h) = \Sigma \varepsilon(h 1_1) 1_2$. Denote the image of the map Π^L by H^L and the image of the map Π^R by H^R. Then we also have $H^L = Im\overline{\Pi}^R$ and $H^R = Im\overline{\Pi}^L$. Let $t \in H$. Then t is said to be a left integral, if $ht = \Pi^L(h)t$ for all $h \in H$.

Definition 1.3. [11] Let H be a weak bialgebra. An algebra A is called a *weak left H-module algebra* if A is a left H-module via $h \otimes a \mapsto h \cdot a$ such that

$$h \cdot (ab) = \Sigma (h_1 \cdot a)(h_2 \cdot b), \ h \cdot 1_A = \Pi^L(h) \cdot 1_A, \tag{10}$$

for all $h \in H$, $a, b \in A$.

Definition 1.4. [11] Let H be a weak Hopf algebra with bijective antipode S and A a weak left H-module algebra. The *smash product algebra* $A\#H$ of A with H is defined on the vector space $A \otimes_{H^L} H$, where A is a right H^L-module via $a \cdot z = S^{-1}(z) \cdot a = a(z \cdot 1_A)$, its multiplication is given by

$$(a\#h)(b\#g) = \Sigma a(h_1 \cdot b)\#h_2 g \tag{11}$$

for all $a \in A$, $z \in H^L$, and the unit of $A\#H$ is $1_A\#1_H$.

By [8], we know that M is a left $A\#H$-module via $(a\#h) \cdot m = a \cdot (h \cdot m)$ if and only if M is a left A-module and a left H-module such that

$$h \cdot (a \cdot m) = \Sigma(h_1 \cdot a) \cdot (h_2 \cdot m). \tag{12}$$

for all $h \in H, a \in A, m \in M$. Hence, the weak left H-module algebra A is particularly a left $A\#H$-module via $(a\#h) \cdot b = a(h \cdot b)$ for all $h \in H, a, b \in A$.

Definition 1.5. [8] Let (H, R) be a quasitriangular weak Hopf algebra, and A a weak left H-module algebra. We say that A is *quantum commutative* with respect to (H, R) if

$$ab = \Sigma(R^{(2)} \cdot b)(R^{(1)} \cdot a), \tag{13}$$

for all $a, b \in A$.

If A is quantum commutative with respect to (H, R), then for all $a, b \in A$,

$$ab = \Sigma(R^{(1)} \cdot b)(S^{-1}(R^{(2)}) \cdot a). \tag{14}$$

In fact, since $(id \otimes S^{-1})(R) = R^{-1}$ by [12], we have

$$\Sigma(R^{(1)} \cdot b)(S^{-1}(R^{(2)}) \cdot a) = \Sigma(R^{(2)}S^{-1}(r^{(2)}) \cdot a)(R^{(1)}r^{(1)} \cdot b)$$
$$\overset{(9)}{=} \Sigma(1_1 \cdot a)(1_2 \cdot b) = ab.$$

Definition 1.6. [4] Let H be a weak bialgebra, and B a right H-comodule, which is also an algebra with a unit, such that

$$\rho(ab) = \rho(a)\rho(b), \tag{15}$$
$$\Sigma a_{(0)} \otimes \Pi^L(a_{(1)}) = \Sigma a1_{(0)} \otimes 1_{(1)}, \tag{16}$$

for all $a, b \in B$. Then B is called a *weak right H-comodule algebra*.

Let H be a weak Hopf algebra, and B a weak right H-comodule algebra. We define the subalgebra of coinvariants as $C := B^{coH} = \{x \in B|\ \Sigma x_{(0)} \otimes x_{(1)} = \Sigma x_{(0)} \otimes \Pi^L(x_{(1)})\}$. By [15], we know that $C = \{x \in B|\ \Sigma x_{(0)} \otimes x_{(1)} = \Sigma x1_{(0)} \otimes 1_{(1)}\} = \{x \in B|\ \Sigma x_{(0)} \otimes x_{(1)} = \Sigma 1_{(0)} x \otimes 1_{(1)}\}$.

If the following canonical map

$$can : B \otimes_C B \to B \otimes_{H^R} H,\ can(a \otimes b) = \Sigma ab_{(0)} \otimes b_{(1)}, \tag{17}$$

is bijective, then the extension B/C is called weak right H-Galois, where B is a right H^R-module via $a \cdot y = \Sigma \varepsilon(a_{(1)}y)a_{(0)}$, and H a left H^R-module via $y \cdot h = hS^{-1}(y)$, for any $a \in B, h \in H, y \in H^R$.

Define

$$can' : B \otimes_C B \to B \otimes_{H^R} H,\ can'(a \otimes b) = \Sigma a_{(0)}b \otimes a_{(1)},$$

and

$$\psi : B \otimes_{H^R} H \to B \otimes_{H^R} H, \ \psi(a \otimes h) = \Sigma a_{(0)} \otimes a_{(1)} S(h)$$

for all $a \in B, h \in H$. Then $can' = \psi \circ can$.

As a matter of fact, we have

$$\Sigma 1_{(0)} \otimes_{H^R} 1_{(1)} = 1_B \otimes_{H^R} 1_H, \tag{18}$$

since

$$
\begin{aligned}
\Sigma 1_{(0)} \otimes_{H^R} 1_{(1)} &= \Sigma 1_{(0)} \otimes S(1_{(1)}) \cdot 1_H = \Sigma 1_{(0)} \cdot S(1_{(1)}) \otimes 1_H \\
&= \Sigma \varepsilon(1_{(1)} S(1_{(2)})) 1_{(0)} \otimes 1_H = \Sigma \varepsilon(\Pi^L(1_{(1)})) 1_{(0)} \otimes 1_H \\
&= 1_B \otimes_{H^R} 1_H.
\end{aligned}
$$

Then for any $a, b \in B$, we have

$$
\begin{aligned}
\psi \circ can(a \otimes b) &= \psi(\Sigma a b_{(0)} \otimes b_{(1)}) = \Sigma a_{(0)} b_{(0)} \otimes a_{(1)} b_{(1)} S(b_{(2)}) \\
&= \Sigma a_{(0)} b_{(0)} \otimes a_{(1)} \Pi^L(b_{(1)}) \overset{(16)}{=} \Sigma a_{(0)} b 1_{(0)} \otimes a_{(1)} 1_{(1)} \\
&\overset{(18)}{=} \Sigma a_{(0)} b \otimes a_{(1)},
\end{aligned}
$$

as required.

2. Homological dimensions over quasitriangular weak Hopf algebra

In this section, we always assume that (H, R) is a finite dimension quasitriangular weak Hopf algebra. Then the antipode S of H is bijective by [3], and H has a dual pair of left integrals (t, λ) by Theorem 4.1 in [17]. Let A be a quantum commutative weak left H-module algebra, and $A\#H$ the weak smash product.

Lemma 2.1. *Let M is left $A\#H$-module. If there is a trace one element and M is projective as an A-module, then M is also projective as an $A\#H$-module.*

Proof. Let M, N be left $A\#H$-modules, and $\omega : M \to N$ be an A-module morphism. Denote the trace one element by c. Define the map $\widehat{\omega} : M \to N$ by

$$\widehat{\omega}(m) = \Sigma t_2 \cdot [(R^{(1)} \cdot c) \cdot \omega(S^{-1}(t_1 R^{(2)}) \cdot m)], \tag{19}$$

for all $a \in A$ and $m \in M$. Then, $\widehat{\omega}$ is an $A\#H$-module morphism.

For any $a \in A, h \in H$ and $m \in M$, on one hand, since $\Sigma S^{-1}(h_2) h_1 = \overline{\Pi}^R(h)$ and

$$\Sigma h_1 \otimes \Pi^L(h_2) = \Sigma 1_1 h \otimes 1_2, \ \Sigma \overline{\Pi}^R(h_1) \otimes h_2 = \Sigma 1_2 \otimes h S(1_1), \tag{20}$$

for $h \in H$ by [18], we have

$$
\begin{aligned}
h \cdot \widehat{\omega}(m) &= \Sigma h t_2 \cdot [(R^{(1)} \cdot c) \cdot \omega(S^{-1}(t_1 R^{(2)}) \cdot m)] \\
&= \Sigma h 1_2 t_2 \cdot [(R^{(1)} \cdot c) \cdot \omega(S^{-1}(1_1 t_1 R^{(2)}) \cdot m)] \\
&= \Sigma h 1_2 t_2 \cdot [(R^{(1)} \cdot c) \cdot \omega(S^{-1}(t_1 R^{(2)})S^{-1}(1_1) \cdot m)] \\
&= \Sigma h S(1_1) t_2 \cdot [(R^{(1)} \cdot c) \cdot \omega(S^{-1}(t_1 R^{(2)})1_2 \cdot m)] \\
&\overset{(20)}{=} \Sigma h_2 t_2 \cdot [(R^{(1)} \cdot c) \cdot \omega(S^{-1}(t_1 R^{(2)})\overline{\Pi}^R(h_1) \cdot m)] \\
&= \Sigma h_3 t_2 \cdot [(R^{(1)} \cdot c) \cdot \omega(S^{-1}(h_2 t_1 R^{(2)})h_1 \cdot m)] \\
&= \Sigma (h_2 t)_2 \cdot [(R^{(1)} \cdot c) \cdot \omega(S^{-1}((h_2 t)_1 R^{(2)})h_1 \cdot m)] \\
&= \Sigma \Pi^L(h_2)_2 t_2 \cdot [(R^{(1)} \cdot c) \cdot \omega(S^{-1}(\Pi^L(h_2)_1 t_1 R^{(2)})h_1 \cdot m)] \\
&= \Sigma 1_3 t_2 \cdot [(R^{(1)} \cdot c) \cdot \omega(S^{-1}(1_2 t_1 R^{(2)})1_1 h \cdot m)] \\
&= \Sigma t_2 \cdot [(R^{(1)} \cdot c) \cdot \omega(S^{-1}(1_2 t_1 R^{(2)})1_1 h \cdot m)] \\
&= \Sigma t_2 \cdot [(R^{(1)} \cdot c) \cdot \omega(S^{-1}(t_1 R^{(2)})h \cdot m)] = \widehat{\omega}(h \cdot m),
\end{aligned}
$$

on the other hand, since $\Sigma h_2 S^{-1}(h_1) = \overline{\Pi}^L(h)$ and

$$
\Sigma \overline{\Pi}^L(h_1) \otimes h_2 = \Sigma 1_1 \otimes 1_2 h, \tag{21}
$$

for $h \in H$ also by [18], we have

$$
\begin{aligned}
\widehat{\omega}(a \cdot m) &= \Sigma t_2 \cdot [(R^{(1)} \cdot c) \cdot \omega(S^{-1}(t_1 R^{(2)}) \cdot (a \cdot m))] \\
&\overset{(12)}{=} \Sigma t_3 \cdot [(R^{(1)} \cdot c) \cdot \omega((S^{-1}(t_2 R_2^{(2)}) \cdot a) \cdot (S^{-1}(t_1 R_1^{(2)}) \cdot m))] \\
&\overset{(8)}{=} \Sigma t_3 \cdot [(R^{(1)} r^{(1)} \cdot c)(S^{-1}(t_2 R^{(2)}) \cdot a) \cdot \omega(S^{-1}(t_1 r^{(2)}) \cdot m)] \\
&= \Sigma t_3 \cdot [(R^{(1)} r^{(1)} \cdot c)(S^{-1}(R^{(2)})S^{-1}(t_2) \cdot a) \cdot \omega(S^{-1}(t_1 r^{(2)}) \cdot m)] \\
&\overset{(14)}{=} \Sigma t_3 \cdot [(S^{-1}(t_2) \cdot a)(R^{(1)} \cdot c) \cdot \omega(S^{-1}(t_1 R^{(2)}) \cdot m)] \\
&= \Sigma [t_3 S^{-1}(t_2) \cdot a] \cdot [t_4 \cdot ((R^{(1)} \cdot c) \cdot \omega(S^{-1}(t_1 R^{(2)}) \cdot m))] \\
&= \Sigma [\overline{\Pi}^L(t_2) \cdot a] \cdot [t_3 \cdot ((R^{(1)} \cdot c) \cdot \omega(S^{-1}(t_1 R^{(2)}) \cdot m))] \\
&\overset{(21)}{=} \Sigma [1_1 \cdot a] \cdot [1_2 t_2 \cdot ((R^{(1)} \cdot c) \cdot \omega(S^{-1}(t_1 R^{(2)}) \cdot m))] \\
&= \Sigma a \cdot [t_2 \cdot ((R^{(1)} \cdot c) \cdot \omega(S^{-1}(t_1 R^{(2)}) \cdot m))] = a \cdot \widehat{\omega}(m).
\end{aligned}
$$

Hence, $\widehat{\omega}$ is an $A\#H$-module morphism.

Now, we assume that P is a left $A\#H$-module, which is projective as an A-module. What we will do next is to show that P is projective as an $A\#H$-module as well. Let $\alpha : M \to N$ be an $A\#H$-epimorphism and $\beta : P \to N$ be an $A\#H$-module morphism. Then, there exists an A-module morphism $\omega : P \to M$ such that $\alpha \circ \omega = \beta$ since P is a projective A-module. Take the map $\widehat{\omega} : P \to M$ as in (19). We claim that $\alpha \circ \widehat{\omega} = \beta$, which follows that P is a projective $A\#H$-module.

Indeed, by Proposition 5.6 in [12], we have

$$
\Sigma \Pi^L(R^{(1)}) \otimes R^{(2)} = \Sigma 1_2 \otimes 1_1, \tag{22}
$$

then for any $p \in P$, since α is $A\#H$-linear and c is a trace one element, we obtain

$$
\begin{aligned}
\alpha \circ \widehat{\omega}(p) \quad &= \quad \alpha(\Sigma t_2 \cdot ((R^{(1)} \cdot c) \cdot (\omega(S^{-1}(t_1 R^{(2)}) \cdot p)))) = \Sigma t_2 \cdot [(R^{(1)} \cdot c) \cdot \alpha \circ \omega(S^{-1}(t_1 R^{(2)}) \cdot p)] \\
&= \quad \Sigma[t_2 R^{(1)} \cdot c] \cdot [t_3 \cdot \beta(S^{-1}(t_1 R^{(2)}) \cdot p)] = \Sigma[t_2 R^{(1)} \cdot c] \cdot \beta(t_3 S^{-1}(t_1 R^{(2)}) \cdot p) \\
&\overset{(6)}{=} \quad \Sigma[R^{(1)} t_1 \cdot c] \cdot \beta(S^{-1}(R^{(2)} t_2 S(t_3)) \cdot p) = \Sigma[R^{(1)} t_1 \cdot c] \cdot \beta(S^{-1}(R^{(2)} \Pi^L(t_2)) \cdot p) \\
&\overset{(20)}{=} \quad \Sigma[R^{(1)} 1_1 t \cdot c] \cdot \beta(S^{-1}(R^{(2)} 1_2) \cdot p) = \Sigma[R^{(1)} \cdot 1_A] \cdot \beta(S^{-1}(R^{(2)}) \cdot p) \\
&= \quad \Sigma[\Pi^L(R^{(1)}) \cdot 1_A] \cdot \beta(S^{-1}(R^{(2)}) \cdot p) \overset{(22)}{=} \Sigma[1_2 \cdot 1_A] \cdot \beta(S^{-1}(1_1) \cdot p) \\
&= \quad \Sigma[\Pi^L(1_1) \cdot 1_A] \cdot [1_2 \cdot \beta(p)] = \Sigma[1_1 \cdot 1_A] \cdot [1_2 \cdot \beta(p)] = \beta(p).
\end{aligned}
$$

Hence, we complete the proof of Lemma 2.1. $\square$

Corollary 2.2. *If H is semisimple and a left $A\#H$-module M is projective as an A-module, then M is also projective as an $A\#H$-module.*

Proof. Since H is semisimple, there is a left integral $t \in H$ such that $\Pi^L(t) = 1_H$ by [3]. Then $t \cdot 1_A = \Pi^L(t) \cdot 1_A = 1_A$. If choose $c = 1_A$ as the trace one element, then we have that M is a projective $A\#H$-module by Lemma 2.1. $\square$

Let B/C be a weak H-Galois extension. We use following formal notation for the inverse of *can*: for any $h \in H$,

$$
can^{-1}(1_B \otimes_{H^R} h) = \Sigma l_i(h) \otimes r_i(h) \in B \otimes_C B,
$$

such that

$$
\Sigma l_i(h) r_i(h)_{(0)} \otimes_{H^R} r_i(h)_{(1)} = 1_B \otimes_{H^R} h,
$$

by definition.

Lemma 2.3. *Let H be a finite dimensional weak Hopf algebra with a dual pair of left integrals (T, t), and B/C a weak right H-Galois extension. Define $tr(a) = a_{(0)} T(a_{(1)})$ for all $a \in B$. Then, tr is a C-bimodule map from B onto C, and the following formula holds:*

$$
\Sigma l_i(S^{-1}(t)) tr(r_i(S^{-1}(t))) = 1_B = \Sigma tr(l_i(S^{-1}(t))) r_i(S^{-1}(t)). \tag{23}
$$

Proof. Since $T \in H^*$ is a left integral, by [13] we have $\Sigma g_1 \langle T, h g_2 \rangle = \Sigma S(h_1) \langle T, h_2 g \rangle$ for all $h, g \in H$. Then, for any $a \in B$,

$$
\begin{aligned}
\Sigma \rho(a_{(0)} T(a_{(1)})) \quad &= \quad \Sigma a_{(0)} \otimes a_{(1)} T(a_{(2)}) \\
&= \quad \Sigma a_{(0)} \otimes S(1_1) \langle T, 1_2 a_{(1)} \rangle \in B \otimes H^L,
\end{aligned}
$$

so $tr(B) \subseteq C$.

For all $a \in B, x \in C$, since

$$
\begin{aligned}
tr(ax) &= \Sigma a_{(0)}x_{(0)}T(a_{(1)}x_{(1)}) = \Sigma a_{(0)}1_{(0)}xT(a_{(1)}1_{(1)}) \\
&= \Sigma a_{(0)}xT(a_{(1)}) = tr(a)x, \\
tr(xa) &= \Sigma x_{(0)}a_{(0)}T(x_{(1)}a_{(1)}) = \Sigma x1_{(0)}a_{(0)}T(1_{(1)}a_{(1)}) \\
&= \Sigma xa_{(0)}T(a_{(1)}) = xtr(a),
\end{aligned}
$$

tr is a C-bimodule map.

Furthermore, since the following diagrams

$$
\begin{array}{ccc}
B \otimes_C B & \xrightarrow{\;\;can\;\;} & B \otimes_{H^R} H \\
{\scriptstyle id_B \otimes tr} \searrow & & \swarrow {\scriptstyle id_B \otimes T} \\
& B &
\end{array}
$$

$$
\begin{array}{ccc}
B \otimes_C B & \xrightarrow{\;\;can'\;\;} & B \otimes_{H^R} H \\
{\scriptstyle tr \otimes id_B} \searrow & & \swarrow {\scriptstyle id_B \otimes T} \\
& B &
\end{array}
$$

commute, and $T(S^{-1}(t)) = 1_k$ by the proof of Theorem 6.4 in [17], we have

$$
\begin{aligned}
& (id_B \otimes T) \circ can = id_B \otimes_C tr \\
\implies\; & (id_B \otimes T) \circ can(\Sigma l_i(S^{-1}(t)) \otimes_C r_i(S^{-1}(t))) = (id_B \otimes_C tr)(\Sigma l_i(S^{-1}(t)) \otimes_C r_i(S^{-1}(t))) \\
\implies\; & 1_B T(S^{-1}(t)) = \Sigma l_i(S^{-1}(t))tr(r_i(S^{-1}(t))) \\
\implies\; & 1_B = \Sigma l_i(S^{-1}(t))tr(r_i(S^{-1}(t))),
\end{aligned}
$$

and on the other hand,

$$
\begin{aligned}
& (id_B \otimes T) \circ can' = tr \otimes_C id_B \\
\implies\; & (id_B \otimes T) \circ \psi \circ can(\Sigma l_i(S^{-1}(t)) \otimes_C r_i(S^{-1}(t))) = (tr \otimes_C id_B)(\Sigma l_i(S^{-1}(t)) \otimes_C r_i(S^{-1}(t))) \\
\implies\; & (id_B \otimes T) \circ \psi(1_B \otimes_{H^R} S^{-1}(t)) = \Sigma tr(l_i(S^{-1}(t)))r_i(S^{-1}(t)) \\
\implies\; & (id_B \otimes T)(\Sigma 1_{(0)} \otimes_{H^R} 1_{(1)}t) = \Sigma tr(l_i(S^{-1}(t)))r_i(S^{-1}(t)) \\
\overset{(18)}{\implies}\; & (id_B \otimes T)(1_B \otimes_{H^R} t) = \Sigma tr(l_i(S^{-1}(t)))r_i(S^{-1}(t)) \\
\implies\; & 1_B T(t) = \Sigma tr(l_i(S^{-1}(t)))r_i(S^{-1}(t)) \\
\implies\; & 1_B = \Sigma tr(l_i(S^{-1}(t)))r_i(S^{-1}(t)).
\end{aligned}
$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

The following proposition extends Theorem 5 in [5].

Proposition 2.4. *Let H be a finite dimensional weak Hopf algebra with a dual pair of left integrals (T, t), and B/C a weak right H-Galois extension. Let V be a left B-module and W a left C-module. Then there exists a space isomorphism*

$$
\gamma : {}_C Hom(V, W) \to {}_B Hom(V, B \otimes_C W), \; \gamma(f)(v) = \Sigma l_i(S^{-1}(t)) \otimes_C f(r_i(S^{-1}(t))v).
$$

In particular, the map $\gamma : {}_CHom(B,W) \to {}_BHom(B, B \otimes_C W) \cong B \otimes_C W$, is left B-linear isomorphic, where the B-module structures of ${}_CHom(B,W)$ and ${}_BHom(B, B \otimes_C W)$ are induced by the right multiplication of B, respectively.

Proof. For any $b \in B$, we have

$$\Sigma bl_i(S^{-1}(t)) \otimes_C r_i(S^{-1}(t)) = \Sigma l_i(S^{-1}(t)) \otimes_C r_i(S^{-1}(t))b. \tag{24}$$

Indeed, since $S^{-1}(t)$ is the right integral of H and $\Sigma b_{(0)} \otimes \Pi^R(b_{(1)}) = \Sigma b1_{(0)} \otimes \Pi^R(1_{(1)})$ for all $b \in B$ (cf. [19]), applying *can* to the both sides of above equation, we obtain

$$
\begin{aligned}
can(\Sigma bl_i(S^{-1}(t)) \otimes_C r_i(S^{-1}(t))) &= \Sigma bl_i(S^{-1}(t))r_i(S^{-1}(t))_{(0)} \otimes_{H^R} r_i(S^{-1}(t))_{(1)} \\
&= b \otimes_{H^R} S^{-1}(t) \\
can(\Sigma l_i(S^{-1}(t)) \otimes_C r_i(S^{-1}(t))b) &= \Sigma l_i(S^{-1}(t))r_i(S^{-1}(t))_{(0)}b_{(0)} \otimes_{H^R} r_i(S^{-1}(t))_{(1)}b_{(1)} \\
&= \Sigma b_{(0)} \otimes_{H^R} S^{-1}(t)b_{(1)} = \Sigma b_{(0)} \otimes_{H^R} S^{-1}(t)\Pi^R(b_{(1)}) \\
&= \Sigma b1_{(0)} \otimes_{H^R} S^{-1}(t)\Pi^R(1_{(1)}) = \Sigma b1_{(0)} \otimes_{H^R} S^{-1}(t)1_{(1)} \\
&= (b \otimes_{H^R} S^{-1}(t))(\Sigma 1_{(0)} \otimes_{H^R} 1_{(1)}) \\
&= b \otimes_{H^R} S^{-1}(t).
\end{aligned}
$$

Next, we show that the map γ is well-defined. For any $b \in B, v \in V$ and $f \in {}_CHom(V,W)$,

$$
\begin{aligned}
\gamma(f)(bv) &= \Sigma l_i(S^{-1}(t)) \otimes_C f(r_i(S^{-1}(t))bv) \\
&\overset{(24)}{=} \Sigma bl_i(S^{-1}(t)) \otimes_C f(r_i(S^{-1}(t))v) \\
&= b\gamma(f)(v).
\end{aligned}
$$

Hence, the map $\gamma(f)$ is left B-linear.

For any $g \in {}_BHom(V, B \otimes_C W)$, define

$$\eta : {}_BHom(V, B \otimes_C W) \to {}_CHom(V,W), \quad \eta(g) = (tr \otimes_C id_W) \circ g.$$

It is obvious that $\eta(g)$ are left C-linear because of the C-linearity of tr and g.

Now by Lemma 2.3, we have for all $v \in V$,

$$
\begin{aligned}
\eta(\gamma(f))(v) &= (tr \otimes_C id_W)(\Sigma l_i(S^{-1}(t)) \otimes_C f(r_i(S^{-1}(t))v)) \\
&= \Sigma tr(l_i(S^{-1}(t)))f(r_i(S^{-1}(t))v) \\
&= f(\Sigma tr(l_i(S^{-1}(t)))r_i(S^{-1}(t))v) \\
&= f(v).
\end{aligned}
$$

On the other hand, let $v \in V$ and denote $g(v) = \Sigma b_i \otimes_C w_j$, then for all $b \in B$ we have $g(bv) = \Sigma b b_j \otimes_C w_j$. It follows that

$$
\begin{aligned}
\gamma(\eta(g))(v) &= \Sigma l_i(S^{-1}(t)) \otimes_C \eta(g)(r_i(S^{-1}(t))v) \\
&= \Sigma l_i(S^{-1}(t)) \otimes_C (tr \otimes_C id_W) \circ g(r_i(S^{-1}(t))v) \\
&= \Sigma l_i(S^{-1}(t)) \otimes_C tr(r_i(S^{-1}(t))b_j)w_j \\
&= \Sigma l_i(S^{-1}(t)) tr(r_i(S^{-1}(t))b_j) \otimes_C w_j \\
&\overset{(24)}{=} \Sigma b_j l_i(S^{-1}(t)) tr(r_i(S^{-1}(t))) \otimes_C w_j \\
&= \Sigma b_j \otimes_C w_j = g(v).
\end{aligned}
$$

This shows that $\eta \circ \gamma$ and $\gamma \circ \eta$ are identities, so that we have our desired isomorphism. $\qquad\square$

Lemma 2.5. *For any left $A\#H$-module M, define a right A-action on M by*

$$
\varphi : M \otimes A \to M, \ m \otimes a \mapsto m \leftharpoonup a := \Sigma(R^{(1)} \cdot a) \cdot (S^{-1}(R^{(2)}) \cdot m). \qquad (25)
$$

for all $a \in A$ and $m \in M$. Then M becomes an (A, A)-bimodule and an (H^L, A)-bimodule.

Proof. M is a right A-module. For any $a, b \in A$ and $m \in M$, we have

$$
\begin{aligned}
m \leftharpoonup 1_A &= \Sigma(R^{(1)} \cdot 1_A) \cdot (S^{-1}(R^{(2)}) \cdot m) = \Sigma(\Pi^L(R^{(1)}) \cdot 1_A) \cdot (S^{-1}(R^{(2)}) \cdot m) \\
&\overset{(22)}{=} \Sigma(1_2 \cdot 1_A) \cdot (S^{-1}(1_1) \cdot m) = \Sigma(S(1_1) \cdot 1_A) \cdot (1_2 \cdot m) \\
&= \Sigma(1_1 \cdot 1_A) \cdot (1_2 \cdot m) = m, \\
(m \leftharpoonup a) \leftharpoonup b &= \Sigma(R^{(1)} \cdot b) \cdot (S^{-1}(R^{(2)}) \cdot (m \leftharpoonup a)) \\
&= \Sigma(R^{(1)} \cdot b) \cdot (S^{-1}(R^{(2)}) \cdot ((r^{(1)} \cdot a) \cdot (S^{-1}(r^{(2)}) \cdot m))) \\
&= \Sigma(R^{(1)} \cdot b)(S^{-1}(R_2^{(2)})r^{(1)} \cdot a) \cdot (S^{-1}(r^{(2)} R_1^{(2)}) \cdot m) \\
&\overset{(8)}{=} \Sigma(R^{(1)} \breve{R}^{(1)} \cdot b)(S^{-1}(R^{(2)})r^{(1)} \cdot a) \cdot (S^{-1}(r^{(2)} \breve{R}^{(2)}) \cdot m) \\
&\overset{(14)}{=} \Sigma(r^{(1)} \cdot a)(\breve{R}^{(1)} \cdot b) \cdot (S^{-1}(r^{(2)} \breve{R}^{(2)}) \cdot m) \\
&\overset{(7)}{=} \Sigma(R_1^{(1)} \cdot a)(R_2^{(1)} \cdot b) \cdot (S^{-1}(R^{(2)}) \cdot m) \\
&= \Sigma(R^{(1)} \cdot (ab)) \cdot (S^{-1}(R^{(2)}) \cdot m) = m \leftharpoonup (ab).
\end{aligned}
$$

What we next do is to prove that φ indeed makes M into an (A, A)-bimodule.

$$
\begin{aligned}
(a \cdot m) \leftharpoonup b &= \Sigma(R^{(1)} \cdot b) \cdot (S^{-1}(R^{(2)}) \cdot (a \cdot m)) \\
&= \Sigma(R^{(1)} \cdot b)(S^{-1}(R_2^{(2)}) \cdot a) \cdot (S^{-1}(R_1^{(2)}) \cdot m) \\
&\overset{(8)}{=} \Sigma(R^{(1)} r^{(1)} \cdot b)(S^{-1}(R^{(2)}) \cdot a) \cdot (S^{-1}(r^{(2)}) \cdot m) \\
&\overset{(14)}{=} \Sigma a(r^{(1)} \cdot b) \cdot (S^{-1}(r^{(2)}) \cdot m) \\
&= a \cdot (m \leftharpoonup b).
\end{aligned}
$$

For any $a \in A, h \in H$ and $m \in M$, we have

$$
h \cdot (m \leftharpoonup a) = \Sigma(h_1 \cdot m) \leftharpoonup (h_2 \cdot a). \qquad (26)
$$

In fact, by (6) we have $R^{-1}\Delta^{op}(h) = \Delta(h)R^{-1}$, that is,

$$\Sigma S(R^{(1)})h_2 \otimes R^{(2)}h_1 = \Sigma h_1 S(R^{(1)}) \otimes h_2 R^{(2)},$$
$$\Sigma R^{(1)}h_2 \otimes S^{-1}(R^{(2)})h_1 = \Sigma h_1 R^{(1)} \otimes h_2 S^{-1}(R^{(2)}).$$

Hence

$$\begin{aligned} h \cdot (m \leftharpoonup a) &= \Sigma h \cdot [(R^{(1)} \cdot a) \cdot (S^{-1}(R^{(2)}) \cdot m)] \\ &= \Sigma(h_1 R^{(1)} \cdot a) \cdot (h_2 S^{-1}(R^{(2)}) \cdot m) \\ &= \Sigma(R^{(1)}h_2 \cdot a) \cdot (S^{-1}(R^{(2)})h_1 \cdot m) \\ &= \Sigma(h_1 \cdot m) \leftharpoonup (h_2 \cdot a). \end{aligned}$$

Now, for any $x \in H^L$, since $\Delta(x) = \Sigma 1_1 x \otimes 1_2$, we compute

$$\begin{aligned} x \cdot (m \leftharpoonup a) &= \Sigma(x_1 \cdot m) \leftharpoonup (x_2 \cdot a) \\ &= \Sigma(1_1 x \cdot m) \leftharpoonup (1_2 \cdot a) \\ &\overset{(29)}{=} (x \cdot m) \leftharpoonup a. \end{aligned}$$

It follows that M is an (H^L, A)-bimodule. □

Now, we are ready to give a necessary and sufficient condition of the global dimension of weak smash product $A\#H$ equals to that of its subalgebra A, which is quantum commutative as a weak H-module algebra, as the main result of this section. In the following, we denote $gl.dim(A)$ ($proj.dim(A)$) by the global dimension (projective dimension) of an algebra A, respectively.

Theorem 2.6. *Let (H, R) be a finite dimension quasitriangular weak Hopf algebra, and A a quantum commutative weak left H-module algebra. Then $gl.dim(A\#H) = n$ if and only if $gl.dim(A) = n$ and there exists a trace one element $c \in A$.*

Proof. " $\implies$ " Suppose that $gl.dim(A\#H) = n < \infty$. Let M be an arbitrary left $A\#H$-module. There is an $A\#H$-epimorphism $(A\#H) \otimes_A M \to M$ given by $(a\#h)\otimes_A m \mapsto (a\#h)\cdot m$. In view of the fact that $_A Hom(A\#H, M) \cong (A\#H)\otimes_A M$ as $A\#H$-modules by Proposition 2.4 since the extension $A\#H/A$ is weak right H-Galois (see [14]), we have an epimorphism

$$_A Hom(A\#H, M) \to M \to 0. \tag{27}$$

Let $m = p.dim_{A\#H}A$, then $m \leq n$. Applying the functor $_{A\#H}Hom(A, -)$ to (27), we get a long exact sequence

$$\cdots \to {}_{A\#H}Ext^m(A, {}_A Hom(A\#H, M)) \to {}_{A\#H}Ext^m(A, M) \to 0.$$

Since

$$_{A\#H}Ext^t(A, {}_A Hom(A\#H, M)) \cong {}_{A\#H}Ext^t(A, (A\#H) \otimes_A M) \cong {}_A Ext^t(A, M),$$

$_AExt^t(A, M) = 0$ for all $t > 0$, and so $m = 0$. Hence A is projective as an $A\#H$-module. Then there exists some $c \in A$ such that $t \cdot c = 1_A$ by Proposition 3 in [21], and we claim that the map $\xi : A\#H \to A$, $a\#h \mapsto a \cdot \Pi^L(h)$ is a split $A\#H$-epimorphism.

It is obvious that ξ is both well-defined from the fact that Π^L is a left H^L-linear map by [3], and a split epimorphism. Hence it suffices to show that ξ is an $A\#H$-linearity.

In fact, since $\Delta(x) = \Sigma x 1_1 \otimes 1_2$ for $x \in H^L$, and

$$\Sigma\varepsilon(h_2 g)h_1 = hS^{-1}\Pi^L(g) \tag{28}$$

for $h, g \in H$, we obtain

$$\begin{aligned}
\xi[(a\#h)(b\#g)] &= \xi[\Sigma a(h_1 \cdot b)\#h_2 g] = \Sigma[a(h_1 \cdot b)] \cdot \Pi^L(h_2 g) \\
&= \Sigma S^{-1}\Pi^L(h_2 g) \cdot [a(h_1 \cdot b)] = \Sigma[S^{-1}(\Pi^L(h_2 g)_2) \cdot a][S^{-1}(\Pi^L(h_2 g)_1)h_1 \cdot b] \\
&= \Sigma[S^{-1}(1_2) \cdot a][S^{-1}(\Pi^L(h_2 g)1_1)h_1 \cdot b] = \Sigma[1_1 \cdot a][1_2 S^{-1}(\Pi^L(h_2 g))h_1 \cdot b] \\
&= a[\Sigma\epsilon(1_1 h_2 g)S^{-1}(1_2)h_1 \cdot b] = a[\Sigma\epsilon(1_1 1_2' h_2 g)S^{-1}(1_2)1_1' h_1 \cdot b] \\
&= a[\Sigma\epsilon(1_2 h_2 g)S^{-1}(1_3)1_1 h_1 \cdot b] = a[\Sigma S^{-1}(1_2)1_1 hS^{-1}(\Pi^L(g) \cdot b] \\
&= a[hS^{-1}(\Pi^L(g) \cdot b] = (a\#h) \cdot [S^{-1}(\Pi^L(g) \cdot b] \\
&= (a\#h) \cdot [b \cdot \Pi^L(g)] = (a\#h) \cdot \xi(b\#g).
\end{aligned}$$

We claim that A is an (A, A)-bimodule direct summand of $A\#H$. Indeed, A and $A\#H$ are (A, A)-bimodules via left multiplication and the action φ defined in (25). Since A is H-quantum commutative, $a \leftharpoonup b = ab$ for all $a, b \in A$. Furthermore, ξ is an (A, A)-bimodule homomorphism. For any $a, b \in A$ and $h \in H$, by

$$\begin{aligned}
\xi((a\#h) \leftharpoonup b) &= \xi[\Sigma(R^{(1)} \cdot b)(S^{-1}(R^{(2)}) \cdot (a\#h))] \\
&= \Sigma\xi[(R^{(1)} \cdot b)(S^{-1}(R_2^{(2)}) \cdot a)\#S^{-1}(R_1^{(2)})h)] \\
&= \Sigma[(R^{(1)} \cdot b)(S^{-1}(R_2^{(2)}) \cdot a)] \cdot \Pi^L(S^{-1}(R_1^{(2)})h) \\
&= \Sigma\varepsilon(1_1 S^{-1}(R_1^{(2)})h)[(R^{(1)} \cdot b)(S^{-1}(R_2^{(2)}) \cdot a)] \cdot 1_2 \\
&= \Sigma\varepsilon(\overline{\Pi}^R(1_1)S^{-1}(R_1^{(2)})h)S^{-1}(1_2) \cdot [(R^{(1)} \cdot b)(S^{-1}(R_2^{(2)}) \cdot a)] \\
&= \Sigma\varepsilon(S^{-1}(1_1)S^{-1}(R_1^{(2)})h)(S^{-1}(1_3)R^{(1)} \cdot b)(S^{-1}(R_2^{(2)}1_2) \cdot a) \\
&= \Sigma\varepsilon(S^{-1}(R_1^{(2)}1_1)h)(S^{-1}(1_2')R^{(1)} \cdot b)(S^{-1}(R_2^{(2)}1_2 1_1') \cdot a) \\
&= \Sigma\varepsilon(S^{-1}(R_1^{(2)})h)(S^{-1}(1_2)R^{(1)} \cdot b)(S^{-1}(R_2^{(2)}1_1) \cdot a) \\
&= \Sigma\varepsilon(S^{-1}(R_1^{(2)})h)(R^{(1)} \cdot b)(S^{-1}(R_2^{(2)}) \cdot a) \\
&\stackrel{(27)}{=} \Sigma(R^{(1)} \cdot b)(S^{-1}(R^{(2)})S^{-1}\Pi^L(h) \cdot a) \\
&= (S^{-1}\Pi^L(h) \cdot a)b = (a \cdot \Pi^L(h))b = \xi(a\#h) \leftharpoonup b.
\end{aligned}$$

Therefore, for any left A-module N, the map $\xi \otimes id_N : (A\#H) \otimes_A N \to N$ is a split A-epimorphism (the left A-action of $(A\#H) \otimes_A N$ via the left component).

Hence, N is a direct summand of $(A\#H) \otimes_A N$. It follows that

$$proj.dim_A(N) \leq proj.dim_A((A\#H) \otimes_A N).$$

We now consider the left $A\#H$-module $(A\#H) \otimes_A N$. Since its any projective $A\#H$-module resolution is also a projective resolution over A, we have

$$proj.dim_A((A\#H) \otimes_A N) \leq proj.dim_{A\#H}((A\#H) \otimes_A N).$$

Thus $gl.dim(A) \leq gl.dim(A\#H) < \infty$.

" $\Longleftarrow$ " We assume that $gl.dim(A) = n < \infty$ and $t \cdot c = 1_A$ for some $c \in A$. Let M be any left $A\#H$-module. If M is projective as an A-module, then M is also projective as an $A\#H$ by Lemma 2.1. Let N be an $A\#H$-module. Consider an exact sequence of $A\#H$-modules

$$0 \to X \to P_{n-1} \to \cdots \to P_0 \to N \to 0,$$

where $_{A\#H}P_i(i = 0, \ldots, n-1)$ is projective, we have $_AX$ is projective since $gl.dim(A) = n$. Hence $_{A\#H}X$ is projective and $proj.dim_{A\#H}(N) \leq n = proj.dim_A(N)$. It follows that $gl.dim(A\#H) \leq gl.dim(A)$. The proof is completed. $\square$

As a consequent of Theorem 2.6 and Proposition 3 in [21], we have

Corollary 2.7. *Let* (H, R) *be a finite dimension quasitriangular weak Hopf algebra, and* A *a quantum commutative weak left* H-module algebra. *Then*

(1) $gl.dim(A\#H) = n$ *if and only if* $gl.dim(A) = n$ *and* A *is a projective left $A\#H$-module.*

(2) *If* H *is furthermore semisimple, then* $gl.dim(A\#H) = gl.dim(A)$. *Hence, A is semisimple if and only if $A\#H$ is semisimple, and A is hereditary if and only if $A\#H$ is hereditary.*

Let us recall from [2] that the finitistic dimension of an algebra A is defined by the formula

$$fin.dim(A) = sup\{proj.dim(A) < \infty | \ M \text{ is a left A-module and } proj.dim(M) < \infty\}.$$

In the following, we shall investigate the relation between the finitistic dimension of a quantum commutative weak H-module algebra A and that of weak smash product $A\#H$.

Proposition 2.8. *Suppose that (H, R) is a finite dimension quasitriangular weak Hopf algebra, its dual H^* is semisimple, and let A be a quantum commutative weak left H-module algebra. If there exists a trace one element $c \in A$, then $fin.dim(A\#H) = fin.dim(A)$.*

Proof. By Theorem 2.6, for any left $A\#H$-module M, we have $proj.dim_{A\#H} M = proj.dim_A M$. This implies that $fin.dim(A\#H) \leq fin.dim(A)$.

Since $A\#H$ is a left H^*-module algebra in a natural way by [11], we can form a weak smash product $(A\#H)\#H^*$. Since H^* is semisimple, $fin.dim((A\#H)\#H^*) \leq fin.dim(A\#H)$ by Lemma 6.2 in [9]. By Proposition 6.3 in [9], we know $(A\#H)\#H^*$ is Morita equivalent to A, then $fin.dim((A\#H)\#H^*) = fin.dim(A)$. Hence,

$$fin.dim(A) = fin.dim((A\#H)\#H^*) \leq fin.dim(A\#H) \leq fin.dim(A).$$

Therefore $fin.dim(A\#H) = fin.dim(A)$.   □

Corollary 2.9. *Suppose that (H, R) is a finite dimension semisimple quasitriangular weak Hopf algebra, its dual H^* is semisimple, and let A be a quantum commutative weak left H-module algebra. Then $fin.dim(A\#H) = fin.dim(A)$.*

3. In the case of twisted weak Hopf algebra

In this section, we always assume that the weak Hopf algebra H is finite dimensional.

Definition 3.1. [13] A *twist* for H is a pair (Θ, Θ^{-1}), with

$$\Theta \in \Delta(1_H)(H \otimes H), \ \Theta^{-1} \in (H \otimes H)\Delta(1_H), \ and \ \Theta\Theta^{-1} = \Delta(1_H), \qquad (29)$$

satisfying the following axioms:

$$(\varepsilon \otimes id)(\Theta) = (id \otimes \varepsilon)(\Theta) = (\varepsilon \otimes id)(\Theta^{-1}) = (id \otimes \varepsilon)(\Theta^{-1}) = 1_H, \quad (30)$$

$$(\Delta \otimes id)(\Theta)(\Theta \otimes 1_H) = (id \otimes \Delta)(\Theta)(\Theta \otimes 1_H), \qquad (31)$$

$$(\Theta^{-1} \otimes 1_H)(\Delta \otimes id)(\Theta^{-1}) = (1_H \otimes \Theta^{-1})(id \otimes \Delta)(\Theta^{-1}), \qquad (32)$$

$$(\Delta \otimes id)(\Theta^{-1})(id \otimes \Delta)(\Theta) = (\Theta \otimes 1_H)(1_H \otimes \Theta^{-1}), \qquad (33)$$

$$(id \otimes \Delta)(\Theta^{-1})(\Delta \otimes id)(\Theta) = (1_H \otimes \Theta)(\Theta^{-1} \otimes 1_H). \qquad (34)$$

Lemma 3.2. *Let (Θ, Θ^{-1}) be a twist for H. Then there is a weak Hopf algebra H_Θ having the same algebra structure and counit as H with a comultiplication and antipode given by*

$$\Delta_\Theta(h) := \Sigma h_1^\Theta \otimes h_2^\Theta = \Theta^{-1}\Delta(h)\Theta, \ S_\Theta(h) = v^{-1}S(h)v, \qquad (35)$$

for all $h \in H_\Theta$, where $v = \Sigma S(\Theta^{(1)})\Theta^{(2)}$ is invertible with inverse $v^{-1} = \Sigma\Theta^{-(1)}$ $S(\Theta^{-(2)})$ in H_Θ. The counital maps of H_Θ are

$$\Pi_\Theta^L(h) = \Sigma\varepsilon(\Theta^{(1)}h)\Theta^{(2)}, \ \Pi_\Theta^R(h) = \Sigma\varepsilon(h\Theta^{-(2)})\Theta^{-(1)},$$

for all $h \in H_\Theta$.

Proof. See Proposition 3.1.2 in [6] and Proposition 6.1.2 in [13]. $\qquad\square$

Lemma 3.3. *Let (Θ, Θ^{-1}) be a twist for H, and A a weak left H-module algebra. Define the operation $*$ on A by $a * b = \Sigma(\Theta^{(1)} \cdot a)(\Theta^{(2)} \cdot b)$, Then A becomes a weak left H_Θ-module algebra, which we denote by A_Θ.*

Proof. First, A_Θ is an associative algebra with unit 1_A. Since $\Sigma S(1_1) \otimes 1_2$ is a separability element of H^L by [3], for any $x \in H^L$,

$$\Sigma x S(1_1) \otimes 1_2 = \Sigma S(1_1) \otimes 1_2 x. \tag{36}$$

Meanwhile, by [6]

$$\Sigma\Pi^L(\Theta^{(1)})\Theta^{(2)} = 1_H, \tag{37}$$

$$\Sigma S^{-1}\Pi^L(\Theta^{(2)})\Theta^{(1)} = 1_H. \tag{38}$$

Then for any $a \in A_\Theta$, we have

$$
\begin{aligned}
1_A * a &= \Sigma(1_1\Theta^{(1)} \cdot 1_A)(1_2\Theta^{(2)} \cdot a) = \Sigma(1_1\Pi^L(\Theta^{(1)}) \cdot 1_A)(1_2\Theta^{(2)} \cdot a) \\
&= \Sigma(\Pi^L(\Theta^{(1)})1_1 \cdot 1_A)(1_2\Theta^{(2)} \cdot a) = \Sigma(\Pi^L(\Theta^{(1)})S(1_1) \cdot 1_A)(1_2\Theta^{(2)} \cdot a) \\
&\overset{(36)}{=} \Sigma(S(1_1) \cdot 1_A)(1_2\Pi^L(\Theta^{(1)})\Theta^{(2)} \cdot a) = \Sigma(1_1 \cdot 1_A)(1_2 \cdot a) \overset{(37)}{=} a, \\
a * 1_A &= \Sigma(\Theta^{(1)} \cdot a)(\Theta^{(2)} \cdot 1_A) = \Sigma(1_1\Theta^{(1)} \cdot a)(1_2\Pi^L(\Theta^{(2)}) \cdot 1_A) \\
&\overset{(36)}{=} \Sigma(1_1S^{-1}\Pi^L(\Theta^{(2)})\Theta^{(1)} \cdot a)(1_2 \cdot 1_A) = \Sigma(1_1 \cdot a)(1_2 \cdot 1_A) \overset{(38)}{=} a.
\end{aligned}
$$

Hence, 1_A is the unit of A_Θ. Moreover, for any $a, b, c \in A_\Theta$,

$$
\begin{aligned}
(a * b) * c &= \Sigma[(\Theta^{(1)} \cdot a)(\Theta^{(2)} \cdot b)] * c \\
&= \Sigma\bar{\Theta}^{(1)} \cdot [(\Theta^{(1)} \cdot a)(\Theta^{(2)} \cdot b)](\bar{\Theta}^{(2)} \cdot c) \\
&= \Sigma(\bar{\Theta}_1^{(1)}\Theta^{(1)} \cdot a)(\bar{\Theta}_2^{(1)}\Theta^{(2)} \cdot b)(\bar{\Theta}^{(2)} \cdot c) \\
&\overset{(31)}{=} \Sigma(\bar{\Theta}^{(1)} \cdot a)(\bar{\Theta}_1^{(2)}\Theta^{(1)} \cdot b)(\bar{\Theta}_2^{(2)}\Theta^{(2)} \cdot c) \\
&= \Sigma(\bar{\Theta}^{(1)} \cdot a)\bar{\Theta}^{(2)} \cdot [(\Theta^{(1)} \cdot b)(\Theta^{(2)} \cdot c)] = a * (b * c).
\end{aligned}
$$

That is A_Θ is associative, where $\Theta = \Sigma\Theta^{(1)} \otimes \Theta^{(2)} = \Sigma\bar{\Theta}^{(1)} \otimes \bar{\Theta}^{(2)}$. Hence, A_Θ is an associative algebra with unit 1_A. Furthermore, it is obvious that A_Θ is an H_Θ-module since H_Θ has the same algebra structure as H by Lemma 3.2.

Now, for any $a, b \in A_\Theta$ and $h \in H_\Theta$,

$$
\begin{aligned}
h \cdot (a * b) &= h \cdot [\Sigma(\Theta^{(1)} \cdot a)(\Theta^{(2)} \cdot b)] = \Sigma(h_1 \Theta^{(1)} \cdot a)(h_2 \Theta^{(2)} \cdot b) \\
&= \Sigma(1_1 h_1 \Theta^{(1)} \cdot a)(1_2 h_2 \Theta^{(2)} \cdot b) = \Sigma(\bar{\Theta}^{(1)} \Theta^{-(1)} h_1 \Theta^{(1)} \cdot a)(\bar{\Theta}^{(2)} \Theta^{-(2)} h_2 \Theta^{(2)} \cdot b) \\
&= \Sigma(\Theta^{-(1)} h_1 \Theta^{(1)} \cdot a) * (\Theta^{-(2)} h_2 \Theta^{(2)} \cdot b) \\
&= \Sigma(h_1^\Theta \cdot a) * (h_2^\Theta \cdot b).
\end{aligned}
$$

It follows that A_Θ is a weak left H_Θ-module algebra. $\qquad\square$

We have the following which extends Theorem 1.7 in [7].

Proposition 3.4. *Let (Θ, Θ^{-1}) be a twist for H, and A a weak left H-module algebra. Then left $A\#H$-module category $_{A\#H}\mathfrak{M}$ and left $A_\Theta\#H_\Theta$-module category $_{A_\Theta\#H_\Theta}\mathfrak{M}$ are equivalent.*

Proof. Let M be an $A\#H$-module. Define an $A_\Theta\#H_\Theta$-action by for all $a \in A_\Theta, h \in H_\Theta$

$$
(a\#h) \rightharpoonup m = [\Sigma(\Theta^{(1)} \cdot a)\#\Theta^{(2)}h] \cdot m. \tag{39}
$$

Then, M is an $A_\Theta\#H_\Theta$-module.

The above action (39) is well defined, since for any $a \in A_\Theta, h \in H_\Theta$ and $x \in H_\Theta^L$, and followed by the fact that $\Sigma S_\Theta(\Theta^{(1)}) \otimes \Theta^{(2)}$ is a separability element of H_Θ^L, we have

$$
\begin{aligned}
(a \cdot x\#h) \rightharpoonup m &= \Sigma[(\Theta^{(1)} S_\Theta^{-1}(x) \cdot a)\#\Theta^{(2)}h] \cdot m \\
&= \Sigma[(\Theta^{(1)} \cdot a)\#\Theta^{(2)}xh] \cdot m \\
&= (a\#xh) \rightharpoonup m.
\end{aligned}
$$

Moreover, $(1_A\#1_H) \rightharpoonup m = m$ for all $m \in M$.

$$
\begin{aligned}
(1_A\#1_H) \rightharpoonup m &= [\Sigma(\Theta^{(1)} \cdot 1_A)\#\Theta^{(2)}] \cdot m \\
&= [\Sigma(\Pi^L(\Theta^{(1)}) \cdot 1_A)\#\Theta^{(2)}] \cdot m \\
&= [\Sigma(1_A \cdot \Pi^L(\Theta^{(1)}))\#\Theta^{(2)}] \cdot m \\
&= [\Sigma 1_A\#\Pi^L(\Theta^{(1)})\Theta^{(2)}] \cdot m \\
&\stackrel{(37)}{=} (1_A\#1_H) \cdot m = m.
\end{aligned}
$$

At the same time, for any $a, b \in A_\Theta, h, g \in H_\Theta$, we have

$$
\begin{aligned}
[(a\#h)(b\#g)] \rightharpoonup m &= [\Sigma a * (h_1^\Theta \cdot b)\#h_2^\Theta g] \rightharpoonup m \\
&= [\Sigma a * (\Theta^{-(1)} h_1 \Theta^{(1)} \cdot b)\#\Theta^{-(2)} h_2 \Theta^{(2)} g] \rightharpoonup m \\
&= [\Sigma(\bar{\Theta}^{(1)} \cdot a)(\bar{\Theta}^{(2)} \Theta^{-(1)} h_1 \Theta^{(1)} \cdot b)\#\Theta^{-(2)} h_2 \Theta^{(2)} g] \rightharpoonup m \\
&\stackrel{(39)}{=} [\Sigma(\breve{\Theta}_1^{(1)} \bar{\Theta}^{(1)} \cdot a)(\breve{\Theta}_2^{(1)} \bar{\Theta}^{(2)} \Theta^{-(1)} h_1 \Theta^{(1)} \cdot b)\#\breve{\Theta}^{(2)} \Theta^{-(2)} h_2 \Theta^{(2)} g] \cdot m
\end{aligned}
$$

$$\overset{(33)}{=} [\Sigma(\breve{\Theta}_1^{(1)}\Theta_1^{-(1)}\bar{\Theta}^{(1)}\cdot a)(\breve{\Theta}_2^{(1)}\Theta_2^{-(1)}\bar{\Theta}_1^{(2)}h_1\Theta^{(1)}\cdot b)\#\breve{\Theta}^{(2)}\Theta^{-(2)}\bar{\Theta}_2^{(2)}h_2\Theta^{(2)}g]\cdot m$$

$$\overset{(29)}{=} [\Sigma(1_{11}\bar{\Theta}^{(1)}\cdot a)(1_{12}\bar{\Theta}_1^{(2)}h_1\Theta^{(1)}\cdot b)\#1_2\bar{\Theta}_2^{(2)}h_2\Theta^{(2)}g]\cdot m$$

$$= [\Sigma(1_1\bar{\Theta}^{(1)}\cdot a)(1_{21}\bar{\Theta}_1^{(2)}h_1\Theta^{(1)}\cdot b)\#1_{22}\bar{\Theta}_2^{(2)}h_2\Theta^{(2)}g]\cdot m$$

$$= [\Sigma(\bar{\Theta}^{(1)}\cdot a)(\bar{\Theta}_1^{(2)}h_1\Theta^{(1)}\cdot b)\#\bar{\Theta}_2^{(2)}h_2\Theta^{(2)}g]\cdot m$$

$$= [\Sigma(\bar{\Theta}^{(1)}\cdot a\#\bar{\Theta}^{(2)}h)(\Theta^{(1)}\cdot b\#\Theta^{(2)}g)]\cdot m$$

$$= \Sigma(\bar{\Theta}^{(1)}\cdot a\#\bar{\Theta}^{(2)}h)\cdot[(\Theta^{(1)}\cdot b\#\Theta^{(2)}g)\cdot m]$$

$$= (a\#h)\rightharpoonup[(b\#g)\rightharpoonup m].$$

Hence, M is a left $A_\Theta\#H_\Theta$-module.

Conversely, let N be an $A_\Theta\#H_\Theta$-module. Define an $A\#H$-action by for all $a\in A, h\in H$

$$(a\#h)\rightharpoonup n = \Sigma[(\Theta^{-(1)}\cdot a)\#\Theta^{-(2)}h]\cdot n. \tag{40}$$

Then, N is an $A\#H$-module.

The well-definition of the above action (40) can be similarly checked as before. For any $a,b\in A, h,g\in H$ and $n\in N$, it follows $(1_A\#1_H)\rightharpoonup n = n$ from the equality $\Sigma\Pi^L(\Theta^{-(1)})\Theta^{-(2)} = 1_H$, and we have

$$(a\#h)\rightharpoonup[(b\#g)\rightharpoonup n] = (a\#h)\rightharpoonup[\Sigma(\Theta^{-(1)}\cdot b\#\Theta^{-(2)}g)\cdot n]$$

$$= \Sigma(\Theta^{-(1)}\cdot a\#\Theta^{-(2)}h)\cdot(\breve{\Theta}^{-(1)}\cdot b\#\breve{\Theta}^{-(2)}g)\cdot n$$

$$= [\Sigma(\Theta^{-(1)}\cdot a)*(\hat{\Theta}^{-(1)}\Theta_1^{-(2)}h_1\Theta^{(1)}\breve{\Theta}^{-(1)}\cdot b)\#\hat{\Theta}^{-(2)}\Theta_2^{-(2)}h_2\Theta^{(2)}\breve{\Theta}^{-(2)}g]\cdot n$$

$$= [\Sigma(\Theta^{-(1)}\cdot a)*(\breve{\Theta}^{-(1)}\Theta_1^{-(2)}h_1\cdot b)\#\breve{\Theta}^{-(2)}\Theta_2^{-(2)}h_2g]\cdot n$$

$$= [\Sigma(\Theta^{(1)}\Theta^{-(1)}\cdot a)(\Theta^{(2)}\breve{\Theta}^{-(1)}\Theta_1^{-(2)}h_1\cdot b)\#\breve{\Theta}^{-(2)}\Theta_2^{-(2)}h_2g]\cdot n$$

$$\overset{(32)}{=} [\Sigma(\Theta^{(1)}\Theta^{-(1)}\breve{\Theta}_1^{-(1)}\cdot a)(\Theta^{(2)}\Theta^{-(2)}\breve{\Theta}_2^{-(1)}h_1\cdot b)\#\breve{\Theta}^{-(2)}h_2g]\cdot n$$

$$= [\Sigma(\breve{\Theta}_1^{-(1)}\cdot a)(\breve{\Theta}_2^{-(1)}h_1\cdot b)\#\breve{\Theta}^{-(2)}h_2g]\cdot n$$

$$= [\Sigma\breve{\Theta}^{-(1)}\cdot(a(h_1\cdot b))\#\breve{\Theta}^{-(2)}h_2g]\cdot n$$

$$= [\Sigma a(h_1\cdot b)\#h_2g]\rightharpoonup n$$

$$= [(a\#h)(b\#g)]\rightharpoonup n.$$

Hence, N is a left $A\#H$-module.

Define the functor $\mathcal{F}: {}_{A\#H}\mathfrak{M}\to {}_{A_\Theta\#H_\Theta}\mathfrak{M}$ by $\mathcal{F}(M):=M$ as a k-space with the $A_\Theta\#H_\Theta$-module structure defined in (39), and the functor $\mathcal{G}: {}_{A_\Theta\#H_\Theta}\mathfrak{M}\to {}_{A\#H}\mathfrak{M}$ by $\mathcal{G}(N):=N$ as a k-space with the $A\#H$-module structure defined in (40). Then $(\mathcal{F},\mathcal{G})$ are equivalent functors. Hence, the categories ${}_{A\#H}\mathfrak{M}$ and ${}_{A_\Theta\#H_\Theta}\mathfrak{M}$ are equivalent. $\square$

Since a twisting of the weak quasitriangular Hopf algebra (H,R) is again quasitriangular with the structure given by $(H_\Theta,\Theta_{21}^{-1}R\Theta)$ (see Remark 6.1.3 in [13]), we have the following.

Theorem 3.5. *Let (H,R) be a finite dimension quasitriangular weak Hopf algebra with a twist (Θ,Θ^{-1}), and A a quantum commutative weak left H-module algebra. Then the following assertions are equivalent:*

(1) $gl.dim(A\#H) = n.$

(2) $gl.dim(A_\Theta\#H_\Theta) = n.$

(3) $gl.dim(A) = n$ and $t \cdot c = 1_A$ for some $c \in A.$

(4) $gl.dim(A_\Theta) = n$ and $t \cdot c = 1_A$ for some $c \in A.$

Proof. (1) and (3) are equivalent followed by Theorem 2.6. Since $A\#H$ and $A_\Theta\#H_\Theta$ are Morita equivalent by Proposition 3.4, $gl.dim(A\#H) = gl.dim(A_\Theta\#H_\Theta).$ Now to show (2) and (4) are equivalent, it suffices to show that if A is quantum H-commutative, then A_Θ is quantum H_Θ-commutative. That is to show that

$$a * b = \Sigma(\Theta^{-(1)}R^{(2)}\Theta^{(2)} \cdot b) * (\Theta^{-(2)}R^{(1)}\Theta^{(1)} \cdot a)$$

for all $a, b \in A_\Theta$, which can be easily checked. $\square$

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