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ON RINGS WHERE LEFT PRINCIPAL IDEALS ARE LEFT PRINCIPAL ANNIHILATORS

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Abstract. The rings in the title are studied and related to right principally injective rings. Many properties of these rings (called left pseudo-morphic by Yang) are derived, and conditions are given that an endomorphism ring is left pseudo-morphic. Some particular results: (1) Commutative pseudo-morphic rings are morphic; (2) Semiprime left pseudo-morphic rings are semisimple; and (3) A left and right pseudo-morphic ring satisfying (equivalent) mild finiteness conditions is a morphic, quasi-Frobenius ring in which every one-sided ideal is principal. Call a left ideal L a left principal annihilator if $L = \mathtt{l}(a) = \{r \in R \mid ra = 0\}$ for some $a \in R$. It is shown that if R is left pseudo-morphic, left mininjective ring with the ACC on left principal annihilators then R is a quasi-Frobenius ring in which every right ideal is principal and every left ideal is a left principal annihilator.

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1. Introduction

Unless otherwise noted, every ring R is associative with unity and all modules are unitary. We write the Jacobson radical as $J = J(R)$ and abbreviate to J when no confusion can result, with a similar convention for the unit group U, the left and right socles S_l and S_r, and the left and right singular ideals Z_l and Z_r of R. The ring of $n \times n$ matrices over R will be denoted by $M_n(R)$, and we write the left and right annihilators of a set X as $\mathtt{l}(X)$ and $\mathtt{r}(X)$ respectively. We denote the ring of integers by $\mathbb{Z}$ and write $\mathbb{Z}_n$ for the ring of integers modulo n. The term "regular ring" means von Neumann regular ring. We write $A \lhd R$ to indicate that A is a two-sided ideal of R, and the notations $N \subseteq^{ess} M$, $N \subseteq^{\oplus} M$ and $N \subseteq^{max} M$ signify that N is an essential submodule (respectively a direct summand, a maximal submodule) of a module M. We write module morphisms opposite the scalars, and we write $M^* = \hom(M, R)$ for the dual of the module M. Maps given by right or left multiplication by w will be written $\cdot w$ and $w\cdot$, respectively. Given a ring-theoretic condition $\mathfrak{c}$, a ring will be called a $\mathfrak{c}$-ring if it is both a left $\mathfrak{c}$-ring and a right $\mathfrak{c}$-ring, with a similar convention for elements.

A left ideal L of a ring R will be called a left *principal annihilator* if $L = \mathtt{l}(b)$ for some $b \in R$. In 2005, a ring R is called left *morphic* [9] if for all $a \in R$ there exists $b \in R$ such that $Ra = \mathtt{l}(b)$ and $\mathtt{l}(a) = Rb$. In 2007 R is called left *quasi-morphic* [2] if the sets of left principal ideals and left principal annihilators coincide: $\{Ra \mid a \in R\} = \{\mathtt{l}(b) \mid b \in R\}$. That same year, the rings R for which $\{Ra \mid a \in R\} \supseteq \{\mathtt{l}(b) \mid b \in R\}$ were called left *generalized morphic* rings by Zhu and Ding [17]. Our interest here is in the rings R satisfying the other inclusion: For all $a \in R$ there exists $b \in R$ such that $Ra = \mathtt{l}(b)$. These rings were called left *pseudo-morphic* by Yang [15] who investigated them in 2010.

An outline of the paper is as follows: The general properties of left pseudo-morphic rings are investigated in Section 2 (the commutative ones are morphic); Their relation with right principally injective rings is outlined in Section 3; Pseudo-morphic modules are considered in Section 4 (often with pseudo-morphic endomorphism rings); The semiprime pseudo-morphic rings are characterized in Section 5 (they are semisimple); and finally, in Section 6, it is proved that, in the presence if one of several (equivalent) mild finiteness conditions, the following are equivalent for a ring R : (1) R is a (left and right) pseudo-morphic ring, R is morphic and quasi-Frobenius, and (3) R is an artinian principal ideal ring, (extending an earlier characterization of these rings in [2, Theorem 19]). In fact we obtain a one-sided result: A left pseudo-morphic, left mininjective ring with the ACC on $\{\mathtt{l}(a) \mid a \in R\}$ is a quasi-Frobenius ring in which every right ideal is principal and every left ideal is a left principal annihilator.

2. Pseudo-morphic rings

We begin with a characterization of left pseudo-morphic elements.

Lemma 2.1. *The following are equivalent for an element a in a ring R :*

(1) $Ra = \mathtt{l}(b)$ *for some* $b \in R$.

(2) R/Ra *embeds in* ${}_RR$.

Proof. (1) $\Rightarrow$ (2) because $R/\mathtt{l}(b) \cong Rb$. Given (2), let $\sigma : R/Ra \to {}_RR$ be an R-monomorphism, and write $(1 + Ra)\sigma = b$. Then $Ra = \mathtt{l}(b)$ because σ is monic. Hence (2) $\Rightarrow$ (1). $\square$

Call an element $a \in R$ a left *pseudo-morphic element* if it satisfies these conditions. Hence a ring R is left *pseudo-morphic* if every element has this property.

Every regular element is left (and right) pseudo-morphic, so regular rings are pseudo-morphic. However, as we shall see, $\mathbb{Z}_n$ is pseudo-morphic for every $n \geq 2$. For another example, every classical artinian principal ideal ring is pseudo-morphic [2, Theorem 19].

A ring R is left semi-hereditary (a left PP-ring) if every finitely generated (principal) left ideal is projective. Lemma 2.1 gives:

Proposition 2.2. *The following are equivalent for a ring R :*

(1) *R is regular.*

(2) *R is left pseudo-morphic and left semihereditary.*

(3) *R is left pseudo-morphic and left PP.*

Proof. (1) $\Rightarrow$ (2) by the above remarks, and (2) $\Rightarrow$ (3) is clear. Given (3), let $a \in R$. Then R/Ra embeds in ${}_RR$ by Lemma 2.1, and so is projective by (3). Hence $Ra \subseteq^{\oplus} {}_RR$, proving (1). $\square$

As mentioned above, a ring R is called *left quasi-morphic* [2] if, for every $a \in R$, we have $Ra = \mathtt{l}(b)$ and $\mathtt{l}(a) = Rc$ for some b and c in R. If $b = c$ for each a, R is called *left morphic* [9]. These rings are clearly left pseudo-morphic. A ring R is called left *special* if it satisfies the following equivalent conditions [9, Theorem 9]:

(1) *${}_RR$ is uniserial of finite length.*

(2) *R is local and $J = Rc$ where $c \in R$ is nilpotent.*

(3) *R is left morphic, local and J is nilpotent.*

These rings are all left pseudo-morphic by (3). However, if we drop "local" or "$J = Rc$" in (2) then R need not be left pseudo-morphic, even if it is artinian (see Examples 2.3 and 2.4 below).

Question 1. *Let R be a local, left pseudo-morphic ring with J nilpotent. Is R left special? Equivalently is $J = Rc$ for some $c \in R$?*

Example 2.3. *Let $R = \begin{bmatrix} D & D \\ & D \end{bmatrix}$ where D is a division ring. Then R is artinian with $J^2 = 0$, and $J = R\gamma$ for some $\gamma \in R$, but R is neither left nor right pseudo-morphic by Proposition 2.6 below. However, R is not local.*

Example 2.4. *Let $R = \left\{ \begin{bmatrix} a & x & y \\ & a & z \\ & & a \end{bmatrix} \middle| a, x, y, z \in D \right\}$ where D is a division ring. Then R is a local, artinian ring that is neither left nor right pseudo-morphic. However, $J \neq R\gamma$ for any $\gamma \in R$.*

Proof. The ring R is clearly local and artinian with $J^3 = 0$. And $J \neq R\gamma$ for any $\gamma \in R$ because, as γ is not a unit, it is impossible that the matrix units e_{12} and e_{23} are both in $R\gamma$.

To see that R is not left pseudo-morphic, let $\alpha = \begin{bmatrix} 0 & 1 & 1 \\ & 0 & 0 \\ & & 0 \end{bmatrix}$ and assume that $R\alpha = \mathtt{l}(\beta)$ for some $\beta \in R$. Because $\alpha\beta = 0$ we have $\beta = \begin{bmatrix} 0 & p & q \\ & 0 & 0 \\ & & 0 \end{bmatrix}$, $p, q \in D$, so $\begin{bmatrix} 0 & D & D \\ & 0 & D \\ & & 0 \end{bmatrix} \subseteq \mathtt{l}(\beta) = R\alpha$. But $R\alpha = \left\{ \begin{bmatrix} 0 & a & a \\ & 0 & 0 \\ & & 0 \end{bmatrix} \middle| \, a \in D \right\}$, so R is not left pseudo-morphic. Using $\alpha' = \begin{bmatrix} 0 & 0 & 1 \\ & 0 & 1 \\ & & 0 \end{bmatrix}$ a similar argument shows that R is not right pseudo-morphic. $\square$

Example 2.5. *Let* $\Lambda = \begin{bmatrix} R & V \\ 0 & S \end{bmatrix}$ *be the split-null extension of rings R and S by a bimodule $V = {}_RV_S$.*

(1) *If V contains an element w such that $\mathtt{l}_R(w) = 0$, then R is not left pseudo-morphic.*
(2) *If V contains an element w such that $\mathtt{r}_S(w) = 0$, then R is not right pseudo-morphic.*

Proof. We prove (1); the proof of (2) is similar. As to (1), let $\alpha = \begin{bmatrix} 0 & w \\ 0 & 0 \end{bmatrix}$, and suppose $\alpha\Lambda = \mathtt{r}_\Lambda(\beta)$ for some $\beta \in R$. Then $\beta\alpha = 0$ so, by hypothesis, β has the form $\beta = \begin{bmatrix} 0 & v \\ 0 & s \end{bmatrix}$. But then we have $\begin{bmatrix} S & V \\ 0 & 0 \end{bmatrix} \subseteq \mathtt{r}_\Lambda(\beta) = \alpha\Lambda = \begin{bmatrix} 0 & wS \\ 0 & 0 \end{bmatrix}$. This contradiction proves (1). $\square$

Proposition 2.6. *No upper-triangular matrix ring is left or right pseudo morphic.*

Example 2.7. (Björk Example) [1, Page 70] *Let F be a field with an isomorphism $a \mapsto \bar{a}$ from F to a proper subfield $\bar{F} \subset F$, and let $S = \{a + bt \mid a, b \in F\}$ be the F-algebra on basis $\{1, t\}$ where $t^2 = 0$ and $ta = \bar{a}t$ for each $a \in F$. It is easy to see that S has a unique proper left ideal $Ft = St = J(S)$, so S is left special* (*and so local and left artinian*)*. Moreover, S may be taken to be right artinian by making the following choices*: *If $p \in \mathbb{Z}$ is a prime, take $F = \mathbb{Z}_p(x)$ to be the field of rational functions, and define $\bar{a} = a^p$ for all $a \in F$.*

Yang [15, Example 8] proves (1) of the next theorem; we include a proof for completeness.

Theorem 2.8. *Let S denote the Björk example. Then:*

(1) *S is left pseudo-morphic but not right pseudo-morphic.*
(2) *S is left pseudo-morphic but $M_2(S)$ is not left pseudo-morphic.*

Proof. We use the notation of Example 2.7.
(1) The ring S is left special (and so left pseudo-morphic) because 0, J and R are the *only* left ideals. To see that S is not right pseudo-morphic we show that tS is not a right principal annihilator. Observe first that $tS = \{ta \mid a \in F\} = \bar{F}t$. Suppose on the contrary that $tS = \mathtt{r}(x)$ for some $0 \neq x \in S$, say $x = a+bt$, $a, b \in F$. Then $0 = xt = at$ so $a = 0$. Thus $\bar{F}t = tS = \mathtt{r}(x) = \mathtt{r}(bt) = \{p + qt \mid b\bar{p} = 0\} = Ft$, a contradiction because $F \neq \bar{F}$.
(2) Write $R = M_2(S)$, choose $d \in F \setminus \bar{F}$, and let $\lambda = \begin{bmatrix} t & dt \\ 0 & 0 \end{bmatrix}$ in R.

Claim: $\mathtt{r}_R(\lambda) = \begin{bmatrix} Ft & Ft \\ Ft & Ft \end{bmatrix}$.

Proof. If $\mu = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in \mathtt{r}_R(\lambda)$ then $tx + dtz = 0$. Writing $x = a + bt$ and $z = a_1 + b_1t$ in S, we obtain $ta + dta_1 = 0$. This implies that $(\bar{a} + d\,\overline{a_1})t = 0$, so $\bar{a} + d\,\overline{a_1} = 0$. Because $d \notin \bar{F}$ this gives $a_1 = 0$. Hence $z \in Ft$ and $ta = 0$. Thus $a = 0$ so $x \in Ft$ too. Similarly $y, w \in Ft$, proving that $\begin{bmatrix} Ft & Ft \\ Ft & Ft \end{bmatrix}$. As the other inclusion is clear, the Claim follows.

We complete the proof by showing that $R\lambda = \mathtt{l}(\mu)$ with $\mu \in R$ is impossible. Indeed, it implies that $\mu \in \mathtt{r}_R(\lambda)$. If we write $\rho = \begin{bmatrix} t & 0 \\ 0 & 0 \end{bmatrix}$ then (by the Claim) $\rho \in \mathtt{l}(\mu) = R\lambda$, say $\rho = \beta\lambda$ where $\beta \in R$. If $\beta = \begin{bmatrix} u & v \\ w & z \end{bmatrix}$, we obtain $t = ut$ and $0 = udt$. Writing $u = m + nt$, $m, n \in F$, these become $t = mt$ and $0 = mdt$. But then $m = 1$, so $dt = 0$, a contradiction since $d \neq 0$. □

Theorem 2.8 leads to the question whether the other "half" of Morita invariance is true for the left pseudo-morphic rings.

Question 2. *If R is left pseudo-morphic and $e^2 = e \in R$, is eRe left pseudo-morphic? What if $ReR = R$?*

The answer is "yes" if R is left morphic [9, Theorem 15], and we also know that every left ideal of eRe is an annihilator [14, Lemma 8.10]. But the question remains open, even if R is left quasi-morphic.

Example 2.9. *If R is left pseudo-morphic and $U \subseteq R$ is a left denominator set, the ring of quotients*

$$Q = \{u^{-1}r \mid u \in U,\, r \in R\}$$

is also left pseudo-morphic. (*The converse is false as $\mathbb{Z} \subseteq \mathbb{Q}$ shows.*)

Proof. If $x = u^{-1}a \in Q$, let $Ra = \mathtt{l}_R(b)$, $b \in R$. We show that $Qx = \mathtt{l}_Q(b)$. We have $Qx \subseteq \mathtt{l}_Q(b)$ because $xb = u^{-1}(ab) = 0$. Conversely, let $y \in \mathtt{l}_Q(b)$, say $y = v^{-1}c$. Since $yb = 0$, it follows that $c \in \mathtt{l}_R(b) = Ra$, say $c = ra$, $r \in R$. Then $y = v^{-1}ra = (v^{-1}ru)x \in Qx$, as required. □

We are going show that all commutative pseudo-morphic rings are morphic. In fact all "reversible" left pseudo-morphic rings are left morphic, where a ring R is called *reversible* if $ab = 0$ implies $ba = 0$. The Björk example (Example 2.7) is reversible. Indeed, any local ring R with $J^2 = 0$ is reversible (if $xy = 0$ in S then either x or y is a unit or $x, y \in J$). In fact, every left special ring is reversible.

Theorem 2.10. *If R is a reversible ring, the following are equivalent:*

(1) *R is a left pseudo-morphic ring.*
(2) *R is left morphic.*
(3) *R is left quasi-morphic.*

In particular, a commutative pseudo-morphic ring is morphic.

Proof. The last statement and (2) ⇒ (3) ⇒ (1) are clear. Assume (1). Given $a \in R$ and $Ra = \mathtt{l}(b)$ for some $b \in R$, we prove (2) by showing that $\mathtt{l}(a) = Rb$. Since R is reversible we have

$$\mathtt{l}(a) = \mathtt{r}(a) = \mathtt{r}(Ra) = \mathtt{rl}(b) = \mathtt{lr}(b)$$

where the reversible hypothesis is used at the first and last steps. But $\mathtt{lr}(b) = Rb$ by Lemma 3.1 and Theorem 3.2 below, so $\mathtt{l}(a) = Rb$ as required. This proves that (1) ⇒ (2). □

The following example shows that the "reversible" hypothesis is essential in Theorem 2.10.

Example 2.11. *Write $M_\omega(D) = end(_DV)$ where V is a left vector space on a basis $\{v_0, v_1, \ldots\}$ over a division ring D. Then $M_\omega(D)$ is pseudo-morphic* (*in fact regular*) *but it is not left morphic.*

Proof. The ring $R = M_\omega(D)$ is clearly regular. Assume that R is left morphic. Then R is unit regular by a result of Ehrlich [5] (see [9, Proposition 5]). Define the shift operator $\sigma : V \to V$ by $v_i\sigma = v_{i+1}$ for each i. If $\tau : V \to V$ is defined by $v_0\tau = 0$ and $v_{i+1}\tau = v_i$ for each $i \geq 0$, then $\mathtt{r}_R(\tau) = 0$ but $R\tau \neq R$ because τ is not a unit in R. But $R\tau = \mathtt{l}(\gamma)$ for some $\gamma \in R$ (because R is left morphic). Hence $\gamma \in \mathtt{r}(\tau) = 0$ and so $R\tau = \mathtt{l}(\gamma) = R$, a contradiction. □

3. Principally injective rings

The next result, part of [11, Lemma 1.1], identifies a class of rings that are important for this paper.

Lemma 3.1. *The following conditions are equivalent for an element a in a ring R:*

(1) *R-linear maps $aR \to R_R$ all extend to $R_R \to R_R$.*
(2) $\mathtt{lr}(a) = Ra$.
(3) *If $\mathtt{r}(a) \subseteq \mathtt{r}(b)$, $b \in R$, then $Rb \subseteq Ra$.*

A ring R is called right *principally injective* (right *P-injective* for short) if every element $a \in R$ satisfies these conditions. Clearly right self-injective rings are right P-injective, as are all regular rings. The following results of Yang (Theorems 6(2) and 11(2) in [15]) will be needed, and we include short proofs for completeness.

Theorem 3.2. *Let R be a left pseudo-morphic ring. Then*:

(1) *R is right P-injective.*
(2) *R is pseudo-morphic if and only if R is quasi-morphic.*

Proof. (1) If $a \in R$ and $Ra = \mathtt{l}(b)$ then $\mathtt{lr}(a) = \mathtt{lr}(Ra) = \mathtt{lrl}(b) = \mathtt{l}(b) = Ra$. Use Lemma 3.1.
(2) Every quasi-morphic ring is pseudo-morphic. Conversely, if R is pseudo-morphic and $a \in R$, let $aR = \mathtt{r}(c)$ where $c \in R$. Then $\mathtt{l}(a) = \mathtt{lr}(c) = Rc$ by (1), so R is left quasi-morphic. Similarly R is right quasi-morphic because it is also left P-injective—by the analogue of (1). □

Note. The converse to (1) of Theorem 3.2 is false: Example 3.3 below is a finite commutative P-injective ring (in fact quasi-Frobenius) that is not pseudo-morphic.

If both R and $M_2(R)$ are left pseudo-morphic then Theorem 3.2 and [14, Proposition 5.36] show that R is right 2-injective (maps $aR + bR \to R_R$ extend to R).

Question 3. *If R is left pseudo-morphic and right 2-injective, is $M_2(R)$ right pseudo-morphic*?

A ring is called *quasi-Frobenius* if it is right or left self-injective and right or left artinian (all four combinations are equivalent). These rings grew out of the theory of representations of a finite group as a group of matrices over a field.

Example 3.3. *The group ring $R = \mathbb{Z}_4 C_2$ is a finite, commutative, local, quasi-Frobenius ring that is not pseudo-morphic.*

Proof. The ring R is local by [3, Example 20]. And R is self-injective by a theorem of Connell [4, Theorem 4.1], so R is quasi-Frobenius (being artinian). In particular R is P-injective, but R is not pseudo-morphic because $R(2+2g) \neq \mathtt{l}(x)$ for all $x \in R$. This is clear if either $x = 0$ or x is a unit. But the set of nonzero, nonunits of R is $\{1+g,\ 1-g,\ -1+g,\ -1-g,\ 0,\ 2,\ 2g,\ 2+2g\}$. Observe:

$\mathtt{l}(1+g) = \mathtt{l}(-1-g)$ contains $1+g$,

$\mathtt{l}(1-g) = \mathtt{l}(-1+g)$ contains $1-g$,

$\mathtt{l}(2) = \mathtt{l}(2g)$ and $\mathtt{l}(2+2g)$ both contain 2.

Since $R(2+2g) = \{0, 2+2g\}$, it follows that $\mathtt{l}(x) \neq R(2+2g)$ for any x. □

The following implications hold for any ring:

Left quasi-morphic $\Rightarrow$ Left pseudo-morphic $\Rightarrow$ Right P-injective

Example 3.3 shows that the converse to the second implication is not true, but (surprisingly) the converse to the first implication is still open:

Question 4. [15, Question 10] *Does there exist a left pseudo-morphic ring that is not left quasi-morphic*?

4. Pseudo-morphic modules

It is always instructive to view a ring-theoretic property in an endomorphism ring.

Lemma 4.1. *Let ${}_RM$ be a module and write $E = end(M)$. The following are equivalent for $\alpha \in E$:*

(1) *$M\alpha = ker(\beta)$ for some $\beta \in E$.*
(2) *$M/M\alpha \cong M\beta$ for some $\beta \in E$.*
(3) *$M/M\alpha \cong N$ for some submodule $N \subseteq M$.*

Proof. (1) $\Rightarrow$ (2) because $M/M\alpha = M/ker(\beta) \cong M\beta$, and (2) $\Rightarrow$ (3) is obvious. (3) $\Rightarrow$ (1) Let $\sigma : M/M\alpha \to N$ be an R-isomorphism, $N \subseteq M$. Define $\beta : M \to M$ by $m\beta = (m + M\alpha)\sigma$ for each $m \in M$. Then $\beta \in E$ and $ker(\beta) = M\alpha$. □

Definition 4.2. A module ${}_RM$ is called *pseudo-morphic* if the conditions in Lemma 4.1 are satisfied for every $\alpha \in end(M)$.

Thus R is a left pseudo-morphic ring if and only if ${}_RR$ is a pseudo-morphic module.

A module ${}_RM$ is called *image direct* if $M\alpha \subseteq^{\oplus} M$ for all $\alpha \in E = end({}_RM)$. Such a module M is pseudo-morphic because, if $M\alpha \oplus K = M$, $\alpha \in E$, then $M\alpha = ker(1-\pi)$ where π is the projection onto $M\alpha$ with kernel K. In particular M is pseudo-morphic if $end({}_RM)$ is a regular ring.

A module ${}_RM$ is called *image projective* if $M\gamma \subseteq M\alpha$, $\gamma, \alpha \in E = end(M)$, implies $\gamma \in E\alpha$:

$$\begin{array}{ccccc} & & M & & \\ & \overset{\lambda}{\swarrow} & \downarrow \gamma & & \\ M & \overset{\alpha}{\to} & M\alpha & \to 0 & \end{array}$$

Clearly every quasi-projective module is image projective.

Part (b) $\Rightarrow$ (a) in the following theorem gives a condition that an endomorphism ring is left pseudo-morphic, and extends parts of [10, Lemma 31]. The proof of (a) $\Rightarrow$ (b) involves the following notion: We say that a module ${}_RM$ *generates* its submodule K if $K = \Sigma\{M\lambda \mid \lambda \in E,\ M\lambda \subseteq K\}$, and we say that M *generates its kernels* if it generates $ker(\beta)$ for all $\beta \in end(M)$.

Theorem 4.3. *Let ${}_RM$ be a module and write $E = end(M)$. Consider the following conditions:*

(a) *E is a left pseudo-morphic ring.*

(b) *M is both pseudo-morphic and image projective.*

Always (b) $\Rightarrow$ (a); *if M generates its kernels then* (a) $\Rightarrow$ (b).

Proof. (b) $\Rightarrow$ (a) Let $\alpha \in E$. As M is pseudo-morphic we have $M\alpha = ker(\beta)$, $\beta \in E$. Then $M\alpha\beta = 0$ so $\alpha\beta = 0$ and we have $E\alpha \subseteq \mathtt{l}_E(\beta)$. On the other hand, if $\gamma \in \mathtt{l}_E(\beta)$ then $\gamma\beta = 0$ so $M\gamma \subseteq ker(\beta) = M\alpha$. As M is image projective this implies $\gamma \in E\alpha$, so $\mathtt{l}_E(\beta) \subseteq E\alpha$. This proves (a).

(a) $\Rightarrow$ (b) Given (a), we show first that M is image projective. Let $M\gamma \subseteq M\alpha$, $\gamma, \alpha \in E$. If $\alpha\theta = 0$, $\theta \in E$, then $M\gamma\theta \subseteq M\alpha\theta = 0$, so $\gamma\theta = 0$. This means that $\mathtt{r}_E(\alpha) \subseteq \mathtt{r}_E(\gamma)$, so $E\gamma \subseteq E\alpha$ by Lemma 3.1 because E is right P-injective (Theorem 3.2). Hence M is image projective.

To see that M is left pseudo-morphic, let $\alpha \in E$. By (a) we have $E\alpha = \mathtt{l}_E(\beta)$ for some $\beta \in E$. Then $\alpha\beta = 0$ so $M\alpha \subseteq ker(\beta)$. For the other inclusion, our hypothesis gives $ker(\beta) = \Sigma\{M\lambda \mid \lambda \in E,\ M\lambda \subseteq ker(\beta)\}$. But $M\lambda \subseteq ker(\beta)$ means $\lambda \in \mathtt{l}_E(\beta) = E\alpha$, so $M\lambda \subseteq ME\alpha \subseteq M\alpha$. It follows that $ker(\beta) \subseteq M\alpha$. □

Remark 4.4. *The proof shows that if $end({}_RM)$ is a left pseudo-morphic ring then M is always an image projective module.*

Question 5. *If $end({}_RM)$ is a left pseudo-morphic ring, when is M pseudo-morphic*?

If R is any ring then R^n is image projective and generates its submodules. Hence we have:

Theorem 4.5. *If R is a ring, $M_n(R)$ is left pseudo-morphic if and only if R^n is pseudo-morphic as a left R-module.*

We conclude this section with the following remarkable result that will play an important role in the ring case (Theorem 5.3).

Theorem 4.6. *Let ${}_RM$ be a pseudo-morphic module and write $E = end(M)$. If $N \subseteq M$ is finitely generated as an E-submodule then $N = ker(\beta)$ for some $\beta \in E$.*

Proof. If N_E is principal there is nothing to prove. In general, let $N = \Sigma_{i=1}^{n} M\alpha_i = N_0 + M\alpha_n$ where, inductively, $N_0 = \Sigma_{i=1}^{n-1} M\alpha_i = ker(\beta_1)$ for some $\beta_1 \in E$. Let $M\alpha_n = ker(\beta_2)$, and then let $M\alpha_n\beta_1 = ker(\beta_3)$. It follows that $N \subseteq ker(\beta_1\beta_3)$, and we complete the proof by showing that this is equality. Given $m \in ker(\beta_1\beta_3)$ we have $m\beta_1 \in ker(\beta_3) = M\alpha_n\beta_1$, say $m\beta_1 = m_1\alpha_n\beta_1$, $m_1 \in M$. It follows that $(m - m_1\alpha_n) \in ker(\beta_1) = N_0$, whence $m \in N_0 + M\alpha_2 = N$, as required. □

5. Semiprime left pseudo-morphic rings

A ring R is called *right Kasch* if every simple right R-module embeds in R_R. The following Lemma is well known; the cyclic proof is left to the reader.

Lemma 5.1. *The following are equivalent for a ring R :*

(1) *R is right Kasch.*
(2) *If M is a maximal right ideal of R then $M = \mathtt{r}(a)$ for some $a \in R$.*
(3) *If $T \subset R$ is a right ideal then $\mathtt{l}(T) \neq 0$.*

Example 5.2. *The ring $R = M_w(D)$ in Example 2.11 is pseudo-morphic (indeed regular), but it is neither right nor left Kasch.*

Proof. Suppose R is right Kasch, and let $M \subseteq^{max} R_R$. Then $M = \mathtt{r}(a)$ for some $0 \neq a \in R$. As $Z_r = 0$ (R is regular), $a \notin Z_r$ so $\mathtt{r}(a) = M$ is not essential in R_R. This implies that $M \subseteq^{\oplus} R_R$ so R is semisimple, a contradiction. Hence R is not right Kasch; similarly R is not left Kasch. □

The left pseudo-morphic rings are just the rings where every *principal* left ideal is a left principal annihilator; surprisingly this extends to finitely generated left ideals [15, Theorem 5(2)].

Theorem 5.3. *Let R be any left pseudo-morphic ring.*

(1) *If $L \subseteq R$ is any finitely generated left ideal then $L = \mathtt{l}(b)$ for some $b \in R$.*
(2) *In this case $\mathtt{lr}(L) = L$.*

Proof. For (1) the proof of Theorem 4.6 goes through if we take $M = {}_RR$. As to (2), if $L = \mathtt{l}(b)$ then $\mathtt{lr}(L) = \mathtt{lrl}(b) = \mathtt{l}(b) = L$. $\square$

Call a ring R left *finitely Kasch* if it satisfies the following condition:

If L is a finitely generated left ideal of R and $\mathtt{r}(L) = 0$, then $L = R$.

The name comes from the fact that R is left Kasch if and only if *every* left ideal L satisfies this condition (see Lemma 5.1). Thus left Kasch rings are left finitely Kasch, and the converse holds if maximal left ideals are finitely generated.

Theorem 5.4. *Every left pseudo-morphic ring R is left finitely Kasch. The converse is false.*

Proof. Let $L \subseteq R$ be a finitely generated left ideal with $\mathtt{r}(L) = 0$. By Theorem 5.3 we have $L = \mathtt{l}(b)$ where $b \in R$. Then $Lb = 0$ so $b \in \mathtt{r}(L) = 0$. This means that $L = \mathtt{l}(b) = \mathtt{l}(0) = R$, as required. The converse fails (even if R is left Kasch) by the following example. $\square$

Example 5.5. *If F is a field and V is an F-space of dimension 2, write $R = \left\{ \begin{bmatrix} a & v \\ & a \end{bmatrix} \middle| a \in F,\ v \in V \right\}$. Then R is a commutative, local, artinian, Kasch ring, but R is not pseudo-morphic.*

Proof. R is clearly commutative, local and artinian, and it is Kasch because $J = J(R) = \begin{bmatrix} 0 & V \\ & 0 \end{bmatrix}$ is the only maximal ideal and $\mathtt{l}(J) = J \neq 0$. Let $0 \neq v \in V$ and consider $\alpha = \begin{bmatrix} 0 & v \\ & 0 \end{bmatrix}$. Then $R\alpha = \begin{bmatrix} 0 & Rv \\ & 0 \end{bmatrix}$, and we claim $R\alpha = \mathtt{l}(\beta)$ is impossible for $\beta \in R$. Indeed, such a β is not a unit and $\beta \neq 0$, so $\beta = \begin{bmatrix} 0 & v \\ & 0 \end{bmatrix}$, $0 \neq v \in V$. But then $\mathtt{l}(\beta) = J \neq R\alpha$. $\square$

Lemma 5.6. *Every left nonsingular, left finitely Kasch ring R is semisimple.*

Proof. Assume that $Z_l = 0$. If L is any left ideal of R we show that $L \subseteq^{\oplus} {}_RR$. By Zorn's Lemma choose a left ideal M such that $L \oplus M \subseteq^{ess} {}_RR$; we show that $\mathtt{r}(L \oplus M) \subseteq Z_l$. If $a \in \mathtt{r}(L \oplus M)$ then $L \oplus M \subseteq \mathtt{l}(a)$. It follows that $\mathtt{l}(a) \subseteq^{ess} {}_RR$; that is $a \in Z_l$, as required. $\square$

In 1968 Yohe [16, Theorem II] proved that a semiprime ring is in which every one-sided ideal is principal is semisimple. The following theorem extends this.[1]

Theorem 5.7. *Every semiprime, left pseudo-morphic ring is semisimple.*

Proof. R is left finitely Kasch by Theorem 5.4 so, by Lemma 5.6, it suffices to show that $Z_l = 0$. Suppose that $0 \neq a \in Z_l$. Then $\mathtt{l}(a) \subseteq^{ess} {}_R R$ so $Ra \cap \mathtt{l}(a) \neq 0$. But $[Ra \cap \mathtt{l}(a)]^2 \subseteq (Ra)\mathtt{l}(a) = 0$, a contradiction because R is semiprime. $\square$

6. Finiteness conditions

We first consider some finiteness conditions on pseudo-morphic rings. We need:

Lemma 6.1. *Let R be right P-injective and define*

$$\Theta : \{Ra \mid a \in R\} \to \{\mathtt{r}(a) \mid a \in R\} \quad by \quad \Theta(Ra) = \mathtt{r}(a).$$

Then Θ is a order-reversing bijection.

Proof. Θ is well defined, onto and order-reversing for any ring. It is one-to-one here because R is right P-injective: $\mathtt{r}(a) = \mathtt{r}(b)$ implies $Ra = \mathtt{lr}(a) = \mathtt{lr}(b) = Rb$. $\square$

The following eight conditions on a ring R will be referred to as *P-conditions*:

$$\left.\begin{array}{lll} \text{ACC or DCC} & \text{on} & \{\mathtt{l}(b) \mid b \in R\} \text{ or } \{\mathtt{r}(b) \mid b \in R\}, \\ \text{ACC or DCC} & \text{on} & \{Ra \mid a \in R\} \text{ or } \{aR \mid a \in R\}. \end{array}\right\}$$

Any quasi-morphic ring satisfies the following four conditions by [2, Lemma 18]:

$$\text{ACC or DCC on } \{Ra \mid a \in R\} \quad \Leftrightarrow \quad \text{ACC or DCC on } \{aR \mid a \in R\}. \qquad (*)$$

The next result extends this.

Proposition 6.2. *If R is a pseudo-morphic ring the eight P-conditions are all equivalent.*

Proof. If R is pseudo-morphic, it is quasi-morphic by Theorem 3.2. Hence R satisfies (*), and the proposition follows because Lemma 6.1 shows that

$$\text{ACC or DCC on } \{Ra \mid a \in R\} \quad \Leftrightarrow \quad \text{DCC or ACC on } \{\mathtt{r}(a) \mid a \in R\},$$

and

$$\text{ACC or DCC on } \{aR \mid a \in R\} \quad \Leftrightarrow \quad \text{DCC or ACC on } \{\mathtt{l}(a) \mid a \in R\}. \quad \square$$

A regular ring R becomes semisimple if it satisfies any of the eight P-conditions. If we impose any of these conditions on a pseudo-morphic ring it becomes an *artinian principal ideal ring* (artinian and every one-sided ideal is principal). This is

[1] In 1972 Jaegermann and Krempa [7, Theorem 3.1] characterized the semiprime, *general rings* (possibly with no unity) in which every one-sided ideal is a principal annihilator, and showed that they have a unity and are semisimple.

part of the following theorem which extends (and reformulates) another characterization of the artinian principal ideal rings in [2, Theorem 19]. It also extends [15, Theorem 11(3)] which assumes that R is left or right noetherian.

Theorem 6.3. *The following conditions are equivalent for a ring R :*

(1) *R is pseudo-morphic and satisfies any of the eight P-conditions.*
(2) *R is quasi-morphic and satisfies any of the eight P-conditions.*
(3) *R is an artinian principal ideal ring.*
(4) *R is morphic and quasi-Frobenius.*
(5) *$R \cong M_{n_1}(R_1) \times M_{n_2}(R_2) \times \cdots \times M_{n_k}(R_k)$ where each R_i is special.*

Proof. (4) $\Rightarrow$ (1) is clear, (1) $\Rightarrow$ (2) by Theorem 3.2, (2) $\Rightarrow$ (3) by Proposition 6.2 and [2, Theorem 19], and (3) $\Rightarrow$ (5) $\Rightarrow$ (4) by [2, Theorem 19]. □

We now turn to a *one-sided* version of Theorem 6.3. We need a preliminary observation involving an injectivity condition weaker than P-injectivity. A ring R is called right *mininjective* if, for every simple right ideal K, all R-linear maps $K \to R_R$ extend to $R_R \to R_R$. The analogue of Lemma 3.1 is:

Lemma 6.4. [12, Lemma 1.1] *The following are equivalent for a ring R :*

(1) *R is right mininjective.*
(2) *If kR is simple, $k \in R$, then $\mathtt{lr}(k) = Rk$.*
(3) *If kR is simple and $\mathtt{r}(k) \subseteq \mathtt{r}(a)$, $k, a \in R$, then $Ra \subseteq Rk$.*

Right P-injective rings are right mininjective; the converse fails as semiprime rings are mininjective.

Our one-sided version of Theorem 6.3 entails assuming that R is merely left pseudo-morphic and applying one of the P-conditions; the one we choose is the ACC on $\{\mathtt{l}(b) \mid b \in R\}$. First:

Lemma 6.5. *Let R be a left pseudo-morphic ring with the ACC on $\{\mathtt{l}(b) \mid b \in R\}$. Then*:

(1) *R is left noetherian.*
(2) *$L = \mathtt{lr}(L)$ for every left ideal L of R.*
(3) *Every left ideal is a left principal annihilator.*

Proof. (1) $\Rightarrow$ (3) by Theorem 5.3 and (3) $\Rightarrow$ (2) is routine. So we prove (1). Suppose a left ideal $L \subseteq R$ is not finitely generated. Choose $0 \neq a_1 \in L$, so $Ra_1 \subset L$. Then let $a_2 \in L \setminus Ra_1$ so $Ra_1 \subset Ra_1 + Ra_2 \subset L$. Continuing we get $Ra_1 \subset Ra_1 + Ra_2 \subset Ra_1 + Ra_2 + Ra_3 \subset \cdots$. By Theorem 5.3, this takes the form $\mathtt{l}(b_1) \subset \mathtt{l}(b_2) \subset \mathtt{l}(b_3) \subset \cdots$, $b_i \in R$, contradicting the ACC. □

A ring satisfying conditions (1) and (2) in Theorem 6.5 is called a left *Johns* ring after B. Johns [8].

Finally, we can prove our one-sided version of Theorem 6.3. With an eye on Theorem 6.3 one might hope that if R is left pseudo-morphic with the ACC on $\{\mathtt{l}(b) \mid b \in R\}$, then R would be quasi-Frobenius. However the Björk example (Example 2.7) has both these properties but is not quasi-Frobenius. In fact it is not left mininjective by [14, Example 2.5]. So *some* other condition is needed to guarantee that the ring is quasi-Frobenius. Surprisingly left mininjectivity is enough.

Theorem 6.6. *The following are equivalent for a ring R :*

(1) *R is left pseudo-morphic, left mininjective, with the ACC on $\{\mathtt{l}(b) \mid b \in R\}$.*
(2) *R is quasi-Frobenius and every right ideal is principal.*
(3) *R is quasi-Frobenius and every left ideal is a left principal annihilator.*

Proof. (1) $\Rightarrow$ (2) Given (1), R is left Johns by Lemma 6.5. Since R is left mininjective by hypothesis, it is quasi-Frobenius by [13, Theorem 4.6]. If T is a right ideal of R write $\mathtt{l}(T) = \mathtt{l}(a)$ for some $a \in R$, again by Lemma 6.5. Hence $T = \mathtt{rl}(T) = \mathtt{rl}(a) = aR$, proving (2).
(2) $\Rightarrow$ (3) Given a left ideal $L \subseteq R$, use (2) to write $\mathtt{r}(L) = bR$, $b \in R$. Since R is quasi-Frobenius, $L = \mathtt{lr}(L) = \mathtt{l}(b)$, proving (3).
(3) $\Rightarrow$ (1) Since R is quasi-Frobenius, it is clearly left mininjective with the ACC on $\{\mathtt{l}(b) \mid b \in R\}$. If $L \subseteq R$ is a left ideal then $L = \mathtt{l}(b)$ by (3). So R is certainly left pseudo-morphic, proving (1). □

A ring satisfying the conditions in Theorem 6.6 will be called a *QF-PRI* ring.

Note that the ring $R = \mathbb{Z}_4 C_2$ in Example 3.3 is a commutative, finite, quasi-Frobenius ring but R is not QF-PRI. Indeed, $J(R) = R(1+g) + R(1-g)$ is not principal (in fact it is the *only* non principal ideal by [2, Example 20]).

Remark 6.7. *Ghorbani* [6, Proposition 2.1] *has a version of (2) $\Leftrightarrow$ (3) in Theorem 6.6 for reflexive modules.*

Question 6. *Which of the P-conditions can replace the ACC on $\{\mathtt{l}(b) \mid b \in R\}$ in* Theorem 6.6? *One possibility is the DCC on $\{aR \mid a \in R\}$.*

Corollary 6.8. *Let R be a left perfect, left minjective, left pseudo-morphic ring. Then R is QF-PRI.*

Proof. Since R is left perfect it has the DCC on $\{aR \mid a \in R\}$. Because R is right P-injective, it has the ACC on $\{\mathtt{l}(a) \mid a \in R\}$ by Lemma 6.1. As R is left mininjective, Theorem 6.6 applies. □

In view of condition (4) in Theorem 6.3 we ask:

Question 7. *If a ring R is QF-PRI, must R be left morphic? The Björk example does not rule this out because it is not left mininjective.*

Question 8. *If R is a QF-PRI ring, is R right pseudo-morphic?*

Remark 6.9. *$\mathbb{Z}_4C_2$ and $\mathbb{Z}_2C_4$ are both local, commutative, quasi-Frobenius rings and so are mininjective with the ACC on $\{\mathtt{l}(b) \mid b \in R\}$. However $\mathbb{Z}_4C_2$ is not pseudo-morphic, not a PRI ring, and J is not principal; while $\mathbb{Z}_2C_4$ is special (and so is morphic, PRI and J is principal).*

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