International Electronic Journal of Algebra
Volume 16 (2014) 16-31

COMPLETE HOMOMORPHISMS BETWEEN MODULE LATTICES

Patrick F. Smith

Received: 18 October 2013; Revised: 25 May 2014
Communicated by Christian Lomp

For my good friend John Clark on his 70th birthday

Abstract. We examine the properties of certain mappings between the lattice $\mathcal{L}(R)$ of ideals of a commutative ring R and the lattice $\mathcal{L}({}_RM)$ of submodules of an R-module M, in particular considering when these mappings are complete homomorphisms of the lattices. We prove that the mapping λ from $\mathcal{L}(R)$ to $\mathcal{L}({}_RM)$ defined by $\lambda(B) = BM$ for every ideal B of R is a complete homomorphism if M is a faithful multiplication module. A ring R is semiperfect (respectively, a finite direct sum of chain rings) if and only if this mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}({}_RM)$ is a complete homomorphism for every simple (respectively, cyclic) R-module M. A Noetherian ring R is an Artinian principal ideal ring if and only if, for every R-module M, the mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}({}_RM)$ is a complete homomorphism.

Mathematics Subject Classification 2010: 06B23, 06B10, 16D10, 16D80
Keywords: Lattice of ideals, lattice of submodules, multiplication modules, complete lattice, complete homomorphism

1. Introduction

In this paper we continue the discussion in [7] concerning mappings, in particular homomorphisms, between the lattice of ideals of a commutative ring and the lattice of submodules of a module over that ring.

A lattice L is called *complete* provided every non-empty subset S has a least upper bound $\vee S$ and a greatest lower bound $\wedge S$. Given complete lattices L and L' we say that a mapping $\varphi : L \to L'$ is a *complete homomorphism* provided

$$\varphi(\vee S) = \vee\{\varphi(x) : x \in S\} \text{ and } \varphi(\wedge S) = \wedge\{\varphi(x) : x \in S\},$$

for every non-empty subset S of L. A complete homomorphism which is a bijection (respectively, injection, surjection) will be called a *complete isomorphism* (respectively, *complete monomorphism*, *complete epimorphism*). The first result is standard and easy to prove.

Lemma 1.1. *The following statements are equivalent for a bijection φ from a complete lattice L to a complete lattice L'.*

(i) *φ is a complete isomorphism.*
(ii) $\varphi(\vee S) = \vee\{\varphi(x) : x \in S\}$ *for every non-empty subset S of L.*
(iii) $\varphi(\wedge S) = \wedge\{\varphi(x) : x \in S\}$ *for every non-empty subset S of L.*

Moreover, in this case the inverse mapping $\varphi^{-1} : L' \to L$ is also a complete isomorphism.

An element x of a complete lattice L is called *compact* in case whenever $x \leq \vee S$, for some non-empty subset S of L, there exists a finite subset F of S such that $x \leq \vee F$. The next result is also easy to prove.

Lemma 1.2. *Let $\varphi : L \to L'$ be a complete isomorphism from a complete lattice L to a complete lattice L' and let x be a compact element of L. Then $\varphi(x)$ is a compact element of L'.*

A lattice L is called *distributive* in case

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$$

for all elements x, y, z in L. The next result is also well known and easy to prove. It states that a lattice is distributive if and only if its dual lattice is distributive.

Lemma 1.3. *A lattice L is distributive if and only if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all x, y, z in L.*

Throughout this note all rings will be commutative with identity and all modules will be unital. Let R be a ring and M be any R-module. Let $\mathcal{L}(R)$ denote the lattice of all ideals of the ring R and let $\mathcal{L}({}_RM)$ denote the lattice of all submodules of the R-module M. In [7] we investigate the mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}({}_RM)$ defined by $\lambda(B) = BM$ for every ideal B of R and the mapping $\mu : \mathcal{L}({}_RM) \to \mathcal{L}(R)$ defined by $\mu(N) = (N :_R M)$ for every submodule N of M, where $(N :_R M)$ denotes the set of elements $r \in R$ such that $rM \subseteq N$. The module M is called a λ-*module* in [7] in case $\lambda : \mathcal{L}(R) \to \mathcal{L}({}_RM)$ is a homomorphism. Similarly, in [7] the module M is called a μ-*module* if the above mapping μ is a homomorphism. For any unexplained terminology and notation, please see [7].

Note that the lattice $\mathcal{L}({}_RM)$ is complete when we define

$$\wedge \mathcal{S} = \cap_{N \in \mathcal{S}} N \;\text{ and }\; \vee \mathcal{S} = \sum_{N \in \mathcal{S}} N,$$

for every non-empty collection $\mathcal{S}$ of submodules of M. In particular the lattice $\mathcal{L}(R)$ is complete. The module M will be called *λ-complete* in case the above mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}({}_RM)$ is a complete homomorphism. Similarly the module M will be called *μ-complete* if $\mu : \mathcal{L}({}_RM) \to \mathcal{L}(R)$ is a complete homomorphism. It is clear that every λ-complete module is a λ-module and every μ-complete module is a μ-module but, in each case, the converse is false in general, as we can easily show.

For example, let $\mathbb{Z}$ denote the ring of rational integers and let p be any prime in $\mathbb{Z}$. Then the simple $\mathbb{Z}$-module $U = \mathbb{Z}/\mathbb{Z}p$ is a λ-module. Let q be any prime in $\mathbb{Z}$ other than p and let $\mathcal{S}$ denote the collection of ideals of $\mathbb{Z}$ of the form $\mathbb{Z}q^n$ for all positive integers n. Then

$$\lambda(\wedge \mathcal{S}) = \lambda(\cap_{n\geq 1} \mathbb{Z}q^n) = \lambda(0) = 0,$$

but

$$\wedge\{\lambda(B) \,:\, B \in \mathcal{S}\} = \cap_{n\geq 1} q^n U = U.$$

Thus U is not λ-complete.

Now let $\mathbb{Z}(p^\infty)$ denote the Prüfer p-group for any prime p in $\mathbb{Z}$. Let $V = \mathbb{Z}(p^\infty)$. Then the $\mathbb{Z}$-module V is a μ-module (see [7, Example 3.11]). However V contains an infinite collection $\mathcal{T}$ of proper submodules $V_i\,(i \in I)$ such that $V = \cup_{i\in I} V_i$. Thus

$$\mu(\vee\mathcal{T}) = \mu(V) = (V :_{\mathbb{Z}} V) = \mathbb{Z},$$

but

$$\vee\{\mu(W) \,:\, W \in \mathcal{T}\} = \sum_{i\in I} \mu(V_i) = \sum_{i\in I} (V_i :_{\mathbb{Z}} V) = 0.$$

Thus the $\mathbb{Z}$-module V is not μ-complete.

Proposition 1.4. *Given any ring R and R-module M the following statements are equivalent.*

(i) *The mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}({}_RM)$ is a complete isomorphism.*

(ii) *The mapping $\mu : \mathcal{L}({}_RM) \to \mathcal{L}(R)$ is a complete isomorphism.*

Moreover, in this case M is a faithful R-module.

Proof. (i) $\Leftrightarrow$ (ii) By Lemma 1.1 and [7, Corollary 1.5].

Now suppose that (i) holds. Let $A = \text{ann}_R(M)$. Then $\lambda(A) = AM = 0 = 0M = \lambda(0)$ so that $A = 0$ and M is faithful. □

Again let R be a ring and let M be an R-module. Let $A = \text{ann}_R(M)$. By defining

$$(r + A)m = rm \quad (r \in R,\, m \in M),$$

M becomes a faithful (R/A)-module with the property that a subset X of M is an R-submodule of M if and only if X is an (R/A)-submodule of M. Thus the lattice $\mathcal{L}({}_RM)$ is identical to the lattice $\mathcal{L}({}_{R/A}M)$. The mapping $\lambda : \mathcal{L}(R/A) \to \mathcal{L}({}_{R/A}M)$ will be denoted by $\overline{\lambda}$. Note that if $\overline{B}$ is any ideal of the ring R/A then $\overline{B} = B/A$ for a unique ideal B of R containing A and hence

$$\overline{\lambda}(\overline{B}) = \overline{\lambda}(B/A) = (B/A)M = BM.$$

In addition, the mapping $\mu : \mathcal{L}({}_{R/A}M) \to \mathcal{L}(R/A)$ is denoted by $\overline{\mu}$ so that

$$\overline{\mu}(N) = (N :_{R/A} M) = (N :_R M)/A,$$

for every submodule N of M, noting that, of course, $A \subseteq (N :_R M)$ for every submodule N of M.

Let R be any ring. An R-module M is called a *multiplication module* in case for each submodule N of M there exists an ideal B of R such that $N = BM$. Cyclic modules are multiplication modules as are projective ideals of R or ideals of R generated by idempotent elements (see [2]). We prove that for any ring R an R-module M is μ-complete if and only if M is a finitely generated multiplication module (Theorem 2.2). An easy consequence is that the mapping μ (respectively, λ) is a complete isomorphism if and only if M is a finitely generated faithful multiplication module (Corollary 2.4).

For any ring R, projective modules are λ-complete (Corollary 3.4) as are faithful multiplication modules (Theorem 3.6). We prove that a ring R is arithmetical if and only if every R-module is a λ-module (Theorem 4.6). The ring R is semiperfect if and only if every simple R-module is λ-complete (Theorem 4.2). On the other hand, R is a direct sum of chain rings if and only if every cyclic R-module M is λ-complete (Theorem 4.7). Note that we do not yet know which rings R have the property that every R-module is λ-complete. It is proved that a Noetherian ring R is an Artinian principal ideal ring if and only if every R-module is λ-complete (Theorem 4.12).

2. μ-complete modules

Let R be a ring and let M be an R-module. In this section we shall investigate μ-complete modules. We begin with the following basic result.

Lemma 2.1. *Given any ring R, an R-module M is μ-complete if and only if $(\sum_{N\in\mathcal{T}} N :_R M) = \sum_{N\in\mathcal{T}} (N :_R M)$ for any non-empty collection $\mathcal{T}$ of submodules of M.*

Proof. Let $\mathcal{T}$ be any non-empty collection of submodules of M. Then

$$\mu(\wedge\mathcal{T}) = \mu(\cap_{N\in\mathcal{T}} N) = (\cap_{N\in\mathcal{T}} N :_R M) = \cap_{N\in\mathcal{T}} (N :_R M)$$
$$= \wedge\{\mu(N) \,:\, N \in \mathcal{T}\}.$$

On the other hand

$$\mu(\vee\mathcal{T}) = \mu(\sum_{N\in\mathcal{T}} N) = (\sum_{N\in\mathcal{T}} N :_R M),$$

and

$$\vee\{\mu(N) \,:\, N \in \mathcal{T}\} = \sum_{N\in\mathcal{T}} (N :_R M).$$

The result follows. □

Note that, given any ring R and R-module M, the mapping μ is not a surjection in case M is not a faithful R-module because in this case no submodule N of M has the property that $(N :_R M) = 0$. The next result characterizes μ-complete modules.

Theorem 2.2. *Given any ring R, the following statements are equivalent for an R-module M with annihilator A in R.*

(i) *M is μ-complete.*
(ii) *M is a finitely generated multiplication module.*
(iii) *The mapping $\overline{\mu} : \mathcal{L}({}_{R/A}M) \to \mathcal{L}(R/A)$ is a complete isomorphism.*
(iv) *The mapping $\overline{\lambda} : \mathcal{L}(R/A) \to \mathcal{L}({}_{R/A}M)$ is a complete isomorphism.*

Moreover in this case the mapping $\mu : \mathcal{L}({}_RM) \to \mathcal{L}(R)$ is a monomorphism.

Proof. (i) ⇒ (ii) Let $\mathcal{T}$ denote the collection of all cyclic submodules of the μ-complete module M. Then $M = \sum_{N\in\mathcal{T}} N$. By Lemma 2.1,

$$R = (M :_R M) = (\sum_{N\in\mathcal{T}} N :_R M) = \sum_{N\in\mathcal{T}} (N :_R M),$$

and hence $R = (Rm_1 :_R M) + \cdots + (Rm_n :_R M)$ for some positive integer n and elements $m_i \in M\,(1 \leq i \leq n)$. It follows that

$$M = RM = (Rm_1 :_R M)M + \cdots + (Rm_n :_R M)M \subseteq Rm_1 + \cdots + Rm_n \subseteq M.$$

Therefore $M = Rm_1 + \cdots + Rm_n$. In other words, M is finitely generated. By [7, Theorem 3.8], M is also a multiplication module.

(ii) ⇒ (i) Suppose that M is a finitely generated multiplication module. By [7, Lemma 3.1 and Theorem 3.8] and induction,

$$(K_1 + \cdots + K_n :_R M) = (K_1 :_R M) + \cdots + (K_n :_R M),$$

for every positive integer n and submodules $K_i\,(1 \leq i \leq n)$. Let $L_i\,(i \in I)$ be any non-empty collection of submodules of M. Clearly,

$$\sum_{i\in I}(L_i :_R M) \subseteq (\sum_{i\in I} L_i :_R M).$$

Let $r \in (\sum_{i\in I} L_i :_R M)$. Then rM is a finitely generated submodule of $\sum_{i\in I} L_i$. There exists a finite subset I' of I such that $rM \subseteq \sum_{i\in I'} L_i$. Hence

$$r \in (\sum_{i\in I'} L_i :_R M) = \sum_{i\in I'}(L_i :_R M) \subseteq \sum_{i\in I}(L_i :_R M).$$

Thus $(\sum_{i\in I} L_i :_R M) \subseteq \sum_{i\in I}(L_i :_R M)$ and we have proved that $(\sum_{i\in I} L_i :_R M) = \sum_{i\in I}(L_i :_R M)$. By Lemma 2.1, M is μ-complete.

(ii) $\Rightarrow$ (iii) By [7, Lemma 2.9], the (R/A)-module M is a finitely generated faithful multiplication module and hence the mapping $\overline{\mu}$ is a bijection by [7, Theorem 4.3]. By the proof of (ii) $\Rightarrow$ (i), the mapping $\overline{\mu}$ is a complete isomorphism.

(iii) $\Leftrightarrow$ (iv) By Proposition 1.4.

(iii) $\Rightarrow$ (ii) By the proof of (i) $\Rightarrow$ (ii), the (R/A)-module M is a finitely generated multiplication module and hence the R-module M is a finitely generated multiplication module by [7, Lemma 2.9].

Finally, suppose that there exist submodules N and L of M such that $\mu(N) = \mu(L)$. By [2, p. 756],

$$N = (N :_R M)M = \mu(N)M = \mu(L)M = (L :_R M)M = L.$$

Thus μ is a monomorphism. □

Given a ring R and an R-module M, note that Theorem 2.2 shows that whenever the mapping $\mu : \mathcal{L}(_RM) \to \mathcal{L}(R)$ is a complete homomorphism then it is a monomorphism. This is not true if μ is merely a homomorphism (see, for example, [7, Example 3.11 and Proposition 3.12]).

Corollary 2.3. *Every homomorphic image of a μ-complete module M is μ-complete.*

Proof. By Theorem 2.2. □

In contrast to Corollary 2.3 homomorphic images of λ-complete modules need not be λ-complete. For example, the $\mathbb{Z}$-module $\mathbb{Z}$ is λ-complete but we have already noted that the simple $\mathbb{Z}$-module $\mathbb{Z}/\mathbb{Z}p$ is not λ-complete for every prime p in $\mathbb{Z}$. (Note that every homomorphic image of a λ-module over the ring $\mathbb{Z}$ is also a λ-module by [7, Theorem 2.3].)

Corollary 2.4. *Given a ring R, the following statements are equivalent for an R-module M.*

(i) *The mapping* $\lambda : \mathcal{L}(R) \to \mathcal{L}({}_RM)$ *is a complete isomorphism.*
(ii) *The mapping* $\mu : \mathcal{L}({}_RM) \to \mathcal{L}(R)$ *is a complete isomorphism.*
(iii) *The* R*-module* M *is a finitely generated faithful multiplication module.*

Proof. By Proposition 1.4 and Theorem 2.2. □

Corollary 2.5. *Let* R *be a ring and let* M *be any* μ*-complete* R*-module with* $A = ann_R(M)$*. Then the* (R/A)*-module* M *is a* λ*-complete module.*

Proof. By [7, Lemma 2.9], Theorem 2.2 and Corollary 2.4. □

Note that in general μ-complete modules are not λ-complete. For, let R be a domain that is not Prüfer. By [7, Theorem 2.3], there exists a cyclic R-module M which is not a λ-module and hence is not λ-complete. However, every cyclic module over any ring is a finitely generated multiplication module.

3. λ-complete modules

In contrast to the case of μ-complete modules, the situation for (non-faithful) λ-complete modules is more complex. We already know that simple modules over $\mathbb{Z}$ are not λ-complete although they are clearly finitely generated multiplication modules. First we prove an elementary result characterizing λ-complete modules.

Lemma 3.1. *Let* R *be a ring. Then an* R*-module* M *is* λ*-complete if and only if* $(\cap_{B\in\mathcal{S}} B)M = \cap_{B\in\mathcal{S}} (BM)$ *for every non-empty collection* $\mathcal{S}$ *of ideals of* R*.*

Proof. Let $\mathcal{S}$ be any non-empty collection of ideals of R. Then

$$\lambda(\vee\mathcal{S}) = (\sum_{B\in\mathcal{S}} B)M = \sum_{B\in\mathcal{S}} (BM) = \vee\{\lambda(B) : B \in \mathcal{S}\}.$$

In addition, $\lambda(\wedge\mathcal{S}) = (\cap_{B\in\mathcal{S}} B)M$ and $\wedge\{\lambda(B) : B \in \mathcal{S}\} = \cap_{B\in\mathcal{S}} (BM)$. The result follows. □

Corollary 3.2. *Let* A *be any ideal of a ring* R*. Then the* R*-module* R/A *is* λ*-complete if and only if* $\cap_{B\in\mathcal{S}}(A+B) = A+(\cap_{B\in\mathcal{S}}B)$ *for every non-empty collection* $\mathcal{S}$ *of ideals of* R*.*

Proof. Apply Lemma 3.1 to the module $M = R/A$. □

Lemma 3.3. *Let* R *be any ring. Then*

(a) *Every direct summand of a* λ*-complete module is* λ*-complete.*
(b) *Every direct sum of* λ*-complete modules is also* λ*-complete.*

Proof. (a) Let K be a direct summand of a λ-complete module M. Let $\mathcal{S}$ be any non-empty collection of ideals of R. Then

$$(\cap_{B\in\mathcal{S}} B)K = K\cap(\cap_{B\in\mathcal{S}} B)M = K\cap(\cap_{B\in\mathcal{S}}(BM))$$
$$= \cap_{B\in\mathcal{S}}(K\cap BM) = \cap_{B\in\mathcal{S}}(BK).$$

By Lemma 3.1 K is a λ-complete module.

(b) Let $L_i\,(i\in I)$ be any collection of λ-complete modules and let $L=\oplus_{i\in I}L_i$. Given any non-empty collection $\mathcal{S}$ of ideals of R we have:

$$(\cap_{B\in\mathcal{S}} B)L = \oplus_{i\in I}(\cap_{B\in\mathcal{S}} B)L_i = \oplus_{i\in I}(\cap_{B\in\mathcal{S}}(BL_i)) = \cap_{B\in\mathcal{S}}(BL).$$

By Lemma 3.1 L is λ-complete. □

Corollary 3.4. *Given any ring R, every projective R-module is λ-complete.*

Proof. Clearly the R-module R is λ-complete. Apply Lemma 3.3. □

Recall the following result (see [2, Theorem 1.2] or [7, Lemma 2.10]).

Lemma 3.5. *Let R be any ring. Then an R-module M is a multiplication module if and only if for each maximal ideal P of R either*

(a) *for each m in M there exists p in P such that $(1-p)m=0$, or*
(b) *there exist $x\in M$ and $q\in P$ such that $(1-q)M\subseteq Rx$.*

We now strengthen [7, Theorem 2.12].

Theorem 3.6. *Let R be any ring. Then every faithful multiplication R-module is a λ-complete module.*

Proof. Let M be a faithful multiplication R-module. Let $\mathcal{S}$ be any non-empty collection of ideals of R. Then $(\cap_{B\in\mathcal{S}} B)M\subseteq\cap_{B\in\mathcal{S}}(BM)$. Suppose that there exists $m\in\cap_{B\in\mathcal{S}}(BM)$ with $m\notin(\cap_{B\in\mathcal{S}} B)M$. Let $I=\{r\in R : rm\in(\cap_{B\in\mathcal{S}} B)M\}$. Then I is a proper ideal of R. Let P be a maximal ideal of R such that $I\subseteq P$. Clearly $(1-p)m=0$ for some $p\in P$ implies that $1-p\in I$, a contradiction. By Lemma 3.5 there exist $x\in M$ and $q\in P$ such that $(1-q)M\subseteq Rx$. Note that for each ideal B in $\mathcal{S}$ $(1-q)m\in(1-q)BM=B(1-q)M\subseteq Bx$. Thus $(1-q)m=r_Bx$ for some $r_B\in B$ for each ideal B in $\mathcal{S}$. If B and C are ideals in $\mathcal{S}$ then $(r_B-r_C)x=0$ and hence $(1-q)(r_B-r_C)M=(r_B-r_C)(1-q)M\subseteq(r_B-r_C)Rx=0$. Because M is faithful we have $(1-q)(r_B-r_C)=0$ and $(1-q)r_B=(1-q)r_C$. It follows that $(1-q)r_C\in\cap_{B\in\mathcal{S}}B$. Thus $(1-q)^2m=(1-q)r_Cx\in(\cap_{B\in\mathcal{S}}B)M$. This implies that $(1-q)^2\in I\subseteq P$, a contradiction. Thus $\cap_{B\in\mathcal{S}}(BM)=(\cap_{B\in\mathcal{S}} B)M$ for every non-empty subset $\mathcal{S}$ of ideals of R. By Lemma 3.1 M is λ-complete. □

We have already noted that for any prime p in $\mathbb{Z}$, the simple $\mathbb{Z}$-module $\mathbb{Z}/\mathbb{Z}p$ is a multiplication module which is not λ-complete. Thus Theorem 3.6 requires that the module be faithful as well as a multiplication module.

If R is any ring and M the free R-module $R \oplus R$, then it is not hard to check that the mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}({}_RM)$ is a complete monomorphism which is not an epimorphism. On the other hand, compare the following result with Theorem 2.2.

Proposition 3.7. *Let R be a ring and let I be a proper ideal of R which is generated by idempotent elements such that $ann_R(I) = 0$. Then the R-module I is a faithful multiplication module and the mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}({}_RI)$ is a complete epimorphism but not a monomorphism.*

Proof. By [7, Proposition 2.15] and Theorem 3.6. □

4. Special rings

Let R be any ring. Then every cyclic R-module is μ-complete by Theorem 2.2. However, the same theorem shows that the 2-generated R-module $M = R \oplus R$ is not μ-complete because M is not a multiplication module. Thus no non-zero ring R has the property that every finitely generated R-module is μ-complete. We saw in Corollary 3.4 that for every ring R every projective R-module is λ-complete. In addition for every ring R, every faithful multiplication module is λ-complete by Theorem 3.6. In this section we investigate rings R with the property that every module in a certain class of R-modules is λ-complete. The classes that we shall look at are the classes of simple R-modules, semisimple R-modules, cyclic R-modules, finitely generated R-modules and all R-modules.

First we investigate when simple modules are λ-complete. Following [1, p. 303] we call a ring R with Jacobson radical J a *semiperfect ring* in case R/J is semiprime Artinian and idempotents lift modulo J. For properties of semiperfect rings see [1, Theorem 27.6] or [10, Theorem 42.6]. By a *local* ring we mean any (commutative) ring which contains only one maximal ideal. It is well known that a (commutative) ring R is semiperfect if and only if R is the (finite) direct sum of local rings (see, for example, [1, Theorem 27.6]). Given any ring R, a submodule N of an R-module M has a *supplement* K in case K is a submodule of M minimal with respect to the property that $M = N + K$.

Lemma 4.1. *Let R be a ring and let U be a simple R-module with annihilator P. Then the R-module U is λ-complete if and only if P has a supplement in ${}_RR$.*

Proof. Suppose first that U is λ-complete. Let $\mathcal{S}$ denote the collection of ideals B of R such that $R = P+B$. By Corollary 3.2 $R = P+C$ where $C = \cap_{B\in\mathcal{S}} B$. Clearly C is a supplement of P in ${}_RR$. Conversely, suppose that P has a supplement G in ${}_RR$. Let $\mathcal{T}$ be any non-empty collection of ideals of R. Then

$$P + (\cap_{D\in\mathcal{T}} D) = P = \cap_{D\in\mathcal{T}} (P + D),$$

unless $D \nsubseteq P$ for all $D \in \mathcal{T}$. Now suppose that $D \nsubseteq P$ for all $D \in \mathcal{T}$. Let $D \in \mathcal{T}$. Then $R = P+G = P+D$ implies that $R = P+(D\cap G)$ and hence $G = D\cap G \subseteq D$. It follows that

$$R = P + G \subseteq P + (\cap_{D\in\mathcal{T}} D) \subseteq \cap_{D\in\mathcal{T}} (P + D) \subseteq R.$$

Thus in any case $P + (\cap_{D\in\mathcal{T}} D) = \cap_{D\in\mathcal{T}} (P + D)$. By Corollary 3.2, the R-module U is λ-complete. □

Theorem 4.2. *The following statements are equivalent for a ring R.*

(i) *Every semisimple R-module is λ-complete.*
(ii) *Every simple R-module is λ-complete.*
(iii) *The ring R is semiperfect.*

Proof. (i) ⇒ (ii) Clear.

(ii) ⇒ (iii) By Lemma 4.1 and [10, Theorem 42.6].

(iii) ⇒ (i) By Lemma 4.1 and [10, Theorem 42.6] every simple R-module is λ-complete and by Lemma 3.3 every semisimple R-module is λ-complete. □

Next we investigate rings R with the property that every cyclic R-module is λ-complete. First we recall a result of Stephenson (see [9, Theorem 1.6]).

Lemma 4.3. *The following statements are equivalent for a module M over a ring R.*

(i) *The lattice $\mathcal{L}({}_RM)$ is distributive (i.e. $L \cap (K + N) = (L \cap K) + (L \cap N)$ for all submodules K, L, N of M).*
(ii) *$K + (L \cap N) = (K + L) \cap (K + N)$ for all submodules K, L, N of M.*
(iii) *$R = (Rx :_R Ry) + (Ry :_R Rx)$ for all $x, y \in M$.*

Corollary 4.4. *The following statements are equivalent for a module M over a ring R.*

(i) *The lattice $\mathcal{L}({}_RM)$ is distributive.*
(ii) *Every finitely generated submodule of M is a μ-module.*
(iii) *Every 2-generated submodule of M is a μ-module.*

(iv) $R = (N :_R L) + (L :_R N)$ *for all finitely generated submodules* N *and* L *of* M.

(v) *Every finitely generated submodule of* M *is a multiplication module.*

Proof. By Lemma 4.3 and [7, Corollary 3.9]. □

The next result is [7, Lemma 2.1].

Lemma 4.5. *An* R*-module* M *is a* λ*-module if and only if* $(B \cap C)M = BM \cap CM$ *for all (finitely generated) ideals* B *and* C *of* R.

We can now generalize [7, Theorems 2.3 and 3.13]. Recall that a ring R is called a *chain ring* in case the ideals of R form a chain, that is, for any ideals B and C of R either $B \subseteq C$ or $C \subseteq B$. For any ring R and prime ideal P of R the localization of R at P will be denoted by R_P as usual. (See [6, Chapter 5] for a good account of localization.) In 1949 Fuchs [3] called a ring R *arithmetical* provided the lattice $\mathcal{L}(R)$ is distributive and Jensen [4, Lemma 1] showed that a ring R is arithmetical if and only if the local ring R_P is a chain ring for every prime ideal P of R.

Theorem 4.6. *The following statements are equivalent for a ring* R.

(i) R *is an arithmetical ring.*

(ii) *Every* R*-module is a* λ*-module.*

(iii) *Every homomorphic image of a* λ*-module is a* λ*-module.*

(iv) *Every cyclic* R*-module is a* λ*-module.*

(v) *Every finitely generated ideal of* R *is a multiplication* R*-module.*

(vi) *Every finitely generated ideal of* R *is a* μ*-module over the ring* R.

Proof. (i) ⇒ (ii) Let B and C be any finitely generated ideals of R. By Corollary 4.4, $R = (B :_R C) + (C :_R B)$. Then

$$BM \cap CM = [(B :_R C) + (C :_R B)](BM \cap CM)$$

$$\subseteq (B :_R C)CM + (C :_R B)BM \subseteq (B \cap C)M.$$

It follows that $BM \cap CM = (B \cap C)M$. By Lemma 4.5 the R-module M is a λ-module.

(ii) ⇒ (iii) Clear.

(iii) ⇒ (iv) Because ${}_RR$ is a λ-module.

(iv) ⇒ (i) Let A, B and C be any ideals of R. Then the cyclic R-module R/A being a λ-module implies, by Lemma 4.5, $(B\cap C)(R/A) = (B(R/A))\cap(C(R/A))$ and hence $((B\cap C)+A)/A = ((B+A)/A)\cap((C+A/A)$. It follows that $(A+B)\cap(A+C) = A + (B \cap C)$. By Lemma 4.3, R is an arithmetical ring.

(i) ⇔ (v) ⇔ (vi) By Corollary 4.4. □

Theorem 4.6 applies to Prüfer domains because every finitely generated ideal is invertible and hence a multiplication module. More generally, if R is a semihereditary ring (that is, every finitely generated ideal of R is a projective R-module), then every finitely generated ideal of R is a multiplication module by [8, Theorem 1] and hence Theorem 4.6 applies to R. It also applies to von Neumann regular rings because every ideal of such a ring is generated by idempotent elements and hence is a multiplication module (see [2, Corollary 1.3]).

Corollary 4.7. *The following statements are equivalent for a ring R.*

(i) *Every cyclic R-module is λ-complete.*

(ii) *The ring $R = R_1 \oplus \cdots \oplus R_n$ is the direct sum of chain rings $R_i\,(1 \le i \le n)$ for some positive integer n.*

Proof. (i) ⇒ (ii) By Theorem 4.2 and [1, Theorem 27.6], the ring $R = R_1 \oplus \cdots \oplus R_n$ is the direct sum of local rings $R_i\,(1 \le i \le n)$ for some positive integer n. By Theorem 4.6 and [4, Lemma 1], R_i is a chain ring for all $1 \le i \le n$.

(ii) ⇒ (i) Without loss of generality we can suppose that R is a chain ring. Let A be any ideal of the chain ring R and let $\mathcal{S}$ be any non-empty collection of ideals of R. Then $A \subseteq \cap_{B\in\mathcal{S}} B$ or $\cap_{B\in\mathcal{S}} B \subseteq A$. Suppose first that $A \subseteq \cap_{B\in\mathcal{S}} B$. Then

$$A + (\cap_{B\in\mathcal{S}} B) = \cap_{B\in\mathcal{S}} B = \cap_{B\in\mathcal{S}} (A + B).$$

Now suppose that $\cap_{B\in\mathcal{S}} B \subset A$. Then there exists an ideal C in $\mathcal{S}$ such that $A \nsubseteq C$ and hence $C \subseteq A$ because R is a chain ring. In this case, it is easy to see that

$$A + (\cap_{B\in\mathcal{S}} B) = A = \cap_{B\in\mathcal{S}} (A + B).$$

In any case, we have proved that $A + (\cap_{B\in\mathcal{S}} B) = \cap_{B\in\mathcal{S}} (A + B)$. By Corollary 3.2 every cyclic R-module is λ-complete, as required. □

Now we consider finitely generated modules and ask the question: Which rings R have the property that every finitely generated module is λ-complete? Are these precisely the rings for which every cyclic module is λ-complete? This amounts to asking whether chain rings R have the property that every finitely generated R-module is λ-complete. Some chain rings do have this property. Contrast the following result with Theorem 4.6.

Theorem 4.8. *Let R be a local principal ideal domain. Then R is a chain ring such that every finitely generated R-module is λ-complete but no non-zero injective R-module is λ-complete.*

Proof. It is well known that if P is the unique maximal ideal of R then the only ideals of R are the ideals R, P^n $(n \geq 1)$ and $0 = \cap_{n\geq 1} P^n$. Thus R is a chain ring. Let M be any finitely generated R-module. Then M is a finite direct sum of cyclic R-modules (see, for example, [6, Theorem 10.30]) and hence M is λ-complete by Theorem 4.7 and Lemma 3.3. Now let X be any non-zero injective R-module. By [5, Proposition 2.6] and [6, Corollary 8.27],

$$\cap_{n\geq 1}(P^n X) = X \neq 0 = (\cap_{n\geq 1} P^n)X.$$

Thus X is not λ-complete by Lemma 3.1. □

Finally in this section we consider rings R with the property that every R-module is λ-complete. Note first the following simple fact which can be contrasted with Corollary 2.3.

Proposition 4.9. *The following statements are equivalent for a ring R.*

(i) *Every R-module is λ-complete.*
(ii) *Every homomorphic image of every λ-complete module is λ-complete.*

Proof. (i) ⇒ (ii) Clear.

(ii) ⇒ (i) Let M be any R-module. There exist a free R-module F and a submodule K of F such that $M \cong F/K$. By Corollary 3.4 the module F is λ-complete and hence so too is M. □

In the case of Noetherian rings we can give a complete classification. We shall require the following two lemmas.

Lemma 4.10. *Let R be a ring such that every R-module is λ-complete and let A be any ideal of R. Then every (R/A)-module is λ-complete.*

Proof. Let $\mathcal{S}$ be any non-empty collection of ideals of the ring R/A. Then every ideal of $\mathcal{S}$ has the form B/A for some ideal B of R. Let $\mathcal{S}'$ denote the collection of ideals B of R such that B/A belongs to $\mathcal{S}$. Let M be any (R/A)-module. Then M is an R-module in the usual way and we have

$$(\cap_{C\in\mathcal{S}}C)M = (\cap_{B\in\mathcal{S}'}(B/A))M = ((\cap_{B\in\mathcal{S}'}B)/A)M = (\cap_{B\in\mathcal{S}'}B)M$$

$$= \cap_{B\in\mathcal{S}'}(BM) = \cap_{B\in\mathcal{S}'}((B/A)M) = \cap_{C\in\mathcal{S}}(CM).$$

By Lemma 3.1, the (R/A)-module M is λ-complete. □

Lemma 4.11. *The following statements are equivalent for a domain R with field of fractions F.*

(i) *R is a field.*
(ii) *Every R-module is λ-complete.*
(iii) *The R-module F is λ-complete.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) Clear by Lemma 3.1.

(iii) $\Rightarrow$ (i) Let $B_i (i \in I)$ denote the collection of all non-zero ideals of R. Then Lemma 3.1 gives that

$$F = \cap_{i\in I} (B_i F) = (\cap_{i\in I} B_i)F.$$

Thus $\cap_{i\in I} B_i \neq 0$. It follows that R has non-zero socle and hence $R = F$. □

Contrast the following result with Theorem 4.8.

Theorem 4.12. *A Noetherian ring R has the property that every R-module is λ-complete if and only if R is an Artinian principal ideal ring.*

Proof. Suppose first that every R-module is λ-complete. Let P be any prime ideal of R. By Lemma 4.10, every (R/P)-module is λ-complete and hence the domain R/P is a field by Lemma 4.11. Thus every prime ideal of R is maximal. By [5, Theorem 4.6], the ring R is Artinian. Next, by Theorem 4.6 every ideal of R is a multiplication module and hence, by [2, Corollary 2.9] every ideal of R is principal. Thus R is a principal ideal ring.

Conversely, suppose that R is an Artinian principal ideal ring. Let M be any R-module. Let $\mathcal{S}$ be any non-empty collection of ideals of R. Because R is Artinian, there exists a finite subset $\mathcal{S}'$ of $\mathcal{S}$ such that $\cap_{B\in\mathcal{S}} B = \cap_{B\in\mathcal{S}'} B$. Noting that R is a principal ideal ring and so every ideal of R is a multiplication module, Theorem 4.6 and [7, Lemma 2.1] together give that $(\cap_{B\in\mathcal{S}'} B)M = \cap_{B\in\mathcal{S}'} (BM)$. Thus,

$$\cap_{B\in\mathcal{S}} (BM) \subseteq \cap_{B\in\mathcal{S}'} (BM) = (\cap_{B\in\mathcal{S}'} B)M = (\cap_{B\in\mathcal{S}} B)M,$$

and hence $(\cap_{B\in\mathcal{S}} B)M = \cap_{B\in\mathcal{S}} (BM)$. By Lemma 3.1 the R-module M is λ-complete. □

5. Other homomorphisms

In general there will be many complete homomorphisms $\nu : \mathcal{L}(R) \to \mathcal{L}(_R M)$ for a given ring R and R-module M (see [7, Section 5]). Note the following result.

Proposition 5.1. *Let R be a ring and let M be an R-module such that there exists a complete isomorphism $\nu : \mathcal{L}(R) \to \mathcal{L}(_R M)$. Then M is a finitely generated R-module.*

Proof. By Lemma 1.2 because M is a finitely generated R-module if and only if M is a compact element of $\mathcal{L}({}_RM)$. □

Recall that a ring R is called *semilocal* provided it contains only a finite number of maximal ideals.

Corollary 5.2. *Let R be a ring and let M be an R-module such that there exists a complete isomorphism $\nu : \mathcal{L}(R) \to \mathcal{L}({}_RM)$. Suppose further that either*

(a) *R is a local ring, or*

(b) *R is a semilocal ring and M is a faithful R-module.*

Then M is a cyclic R-module.

Proof. By Proposition 5.1 and [7, Theorem 5.3]. □

Acknowledgment. The author would like to thank the referee for various helpful suggestions and in particular for bringing references [3] and [4] to his attention.

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Patrick F. Smith
Department of Mathematics
University of Glasgow
Glasgow G12 8QW
Scotland UK
e-mail: Patrick.Smith@glasgow.ac.uk