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CHARACTERIZATIONS OF THE INTEGRAL DOMAINS WHOSE OVERRINGS ARE GOING-DOWN DOMAINS

Ahmed Ayache and David E. Dobbs

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Abstract. Let R be a (commutative integral) domain with quotient K; let R' be the integral closure of R (in K). Then each overring of R (inside K) is a going-down domain if and only if R' is a locally pseudo-valuation domain, $T \subseteq T'$ satisfies going-down for every overring T of R, and $tr.\deg[V_{R'}(M)/M(R')_M : R'/M] \leq 1$ for every maximal ideal M of R' (where $V_{R'}(M)$ denotes the valuation domain that is canonically associated to the pseudo-valuation domain $(R')_M$). Additional equivalences are given in case R is locally finite-dimensional. Applications include the case where R is integrally closed or R is not a Jaffard domain or $R[X]$ is catenarian.

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1. Introduction

All rings considered below are assumed to be commutative (integral) domains (with 1). Throughout, R denotes a domain with quotient field K, with R' denoting the integral closure of R (in K). As usual, $\operatorname{Spec}(R)$ (resp., $\operatorname{Max}(R)$) denotes the set of prime (resp., maximal) ideals of R; and, by an *overring* of R, we mean an R-subalgebra of K, that is, a ring T such that $R \subseteq T \subseteq K$. If an overring T of R is such that $T \neq R$ (resp., $T = R[u]$ for some $u \in K$), we say that T is a *proper overring* (resp., a *simple overring*) of R. We let $\dim(R)$ denote the (Krull) dimension of R, while $\dim_v(R)$ denotes the valuative dimension of R (that is, the supremum, which may be ∞, of $\dim(T)$ as T ranges over the set of overrings of R). Also, for a (ring) extension $R \subseteq S$ of domains with corresponding quotient fields $F \subseteq L$, we let $tr.\deg[S : R]$ denote the transcendence degree of S over R (that is, the transcendence degree of L over F).

Let us next recall some basic definitions and facts.

As in [13] and [21], R is said to be a *going-down domain* if the extension $R \subseteq T$ satisfies the going-down property for each domain T that contains R (as a subring); cf. also [32], [33], [15], [22], [17], [31]. By [21, Theorem 1], the rings T that need to be tested (to check that $R \subseteq T$ satisfies the going-down property) may be restricted to be either valuation overrings of R or simple overrings of R. The most natural examples of going-down domains are arbitrary Prüfer domains and domains of (Krull) dimension at most 1.

R is called a *treed domain* if $\mathrm{Spec}(R)$, when viewed as a poset via inclusion, is a tree; that is, if no prime ideal of R can contain incomparable prime ideals of R. It was shown in [13, Theorem 2.2] that each going-down domain must be treed; however, a construction of W. J. Lewis, reported in [22, Example 2.2], showed that the converse is false.

Following Hedstrom and Houston ([28], [29]), R is called a *pseudo-valuation domain* (or, in short, a PVD) if each prime ideal P of R is strongly prime; that is, if $xy \in P$, with $x \in K$ and $y \in K$, implies that either $x \in P$ or $y \in P$. Each PVD is a quasi-local going-down domain. (In fact, more was shown in [14, page 560], namely, that each PVD is a divided domain in the sense of [15].) Recall from [29] (cf. also [1]) that R is a PVD if and only if there is a (uniquely determined) valuation overring V of R such that $\mathrm{Spec}(R) = \mathrm{Spec}(V)$ (as sets); in this case, V is called the *canonically associated valuation overring* of R. It is shown in [1, Proposition 2.6] that PVDs can be characterized as the pullbacks $V \times_F k$ arising from a valuation domain (V, M) and a field $k \subseteq F := V/M$. In order to globalize the PVD concept, Dobbs and Fontana [18] introduced the following definition: R is called a *locally pseudo-valuation domain* (in short, an LPVD) if R_M is a PVD for each maximal ideal M of R. The class of LPVDs clearly contains all Prüfer domains, all PVDs (indeed, PVDs are the same as the quasi-local LPVDs), and an abundance of other semi-quasi-local domains arising from pullback constructions (see, for instance, [18, Example 2.5]).

Although each overring of a Prüfer domain must be a Prüfer domain, there exist one-dimensional domains with overrings of dimension greater than 1. By applying the classical $D+M$ construction (as in [26]) to such examples, it was shown in [13] that an overring of a going-down domain need not be a going-down domain. In fact, by an iterated pullback construction, it was shown in [19] that an integral overring of a going-down domain need not be a going-down domain. (Earlier, it had been shown that each integral overring of a going-down domain is a going-down domain if $\dim_v(R) \leq 2$ [14] or if R is both locally divided and locally finite-conductor [16, Theorem 3.2].) The *main goal of this paper* is to characterize the domains which are such that each overring is treed (or is itself a going-down domain).

There have been several previous attempts at answering the above question or related questions, with success only under rather special additional hypotheses: cf. [13, Proposition 3.5], [4, Theorem 5.30], [12, Corollary 2.6], [6, Theorem 1], [7, Theorems 4.4 and 6.1] and [3, Theorem 10]. An underlying difficulty in such studies is that the nature of R has very subtle influences on its overrings. In addressing the general case, we have found inspiration from two sources: Ayache's recent result [3] that if an integrally closed domain R is such that each overring of R is treed, then R must be an LPVD; and the role, in the above-cited result from [12], of the transcendence degree of $R/M \subseteq V/M$, in case $(R.M)$ is a PVD with canonically associated valuation overring V. Our main result, Theorem 2.8, provides the following characterization for the general case: each overring of (a domain) R is a going-down domain if and only if R' is an LPVD, $T \subseteq T'$ satisfies going-down for every overring T of R, and $tr.\deg[V_{R'}(M)/M : R'/M] \leq 1$ for every maximal ideal M of R' (where $V_{R'}(M)$ denotes the valuation domain that is canonically associated to the pseudo-valuation domain $(R')_M$). Theorem 2.8 also shows that in case R is locally finite-dimensional, the above conditions are equivalent to: each overring T of R is treed and satisfies $\mathrm{ht}_{T'}(Q) = \mathrm{ht}_T(Q \cap R)$ for every $Q \in \mathrm{Max}(T')$. Some other interesting applications are also given along these lines, including the case where R is integrally closed (Proposition 2.2) or R is not a Jaffard domain (Proposition 2.5) or $R[X]$ is catenarian (Corollary 2.11).

If R is an LPVD and M is a maximal ideal of R, it will be convenient to let $V_R(M)$ denote the valuation domain which is canonically associated to the pseudo-valuation domain R_M. Also, $\subset$ will denote proper inclusion. Any unexplained material is standard, typically as in [26].

2. Results

For reference purposes, we begin by stating the above-mentioned recent result of the first-named author.

Theorem 2.1. ([3, Theorem 10]) *Let R be an integrally closed domain. If each overring of R is treed, then R is an LPVD.*

To appreciate the significance of the hypothesis in Proposition 2.2, note the following consequence of [17, Example 2.3]: the implication (ii) $\Rightarrow$ (i) in Proposition 2.2 would fail if the domain R is not integrally closed. Also, for the proof of

Proposition 2.2 (and later), we need to recall that a domain R is called a *Jaffard domain* if $\dim_v(R) = \dim(R) < \infty$.

Proposition 2.2. *Let R be an integrally closed domain. Then the following three conditions are equivalent:*

(i) *Each overring of R is a going-down domain;*

(ii) *Each overring of R is treed;*

(iii) *R is an* LPVD *and $tr.\deg[V_R(M)/MR_M : R/M] \leq 1$ for each maximal ideal M of R.*

If, in addition, R is locally finite-dimensional, then the above three conditions are also equivalent to the following three conditions:

(iv) *R is an* LPVD *and $\dim_v(R_M) - \dim(R_M) \leq 1$ for each maximal ideal M of R;*

(v) *For each maximal ideal M of R, either R_M is a valuation domain or R_M is a* PVD (*which is not a valuation domain*) *such that $\dim_v(R_M) = \dim(R_M) + 1$;*

(vi) *For each maximal ideal M of R, either R_M is a valuation domain or R_M is a* PVD (*which is not a valuation domain*) *such that $tr.\deg[V_R(M)/M : R/M] = 1$.*

Proof. By [13, Theorem 2.2], (i) $\Rightarrow$ (ii). According to [4, Proposition 5.1] (resp., [6, Proposition 2.2]), each overring of R is a going-down domain (resp., treed) if and only if R_M inherits the same property for every maximal ideal M of R. So, without loss of generality, we may assume that R is quasi-local with maximal ideal M. Therefore, (ii) $\Rightarrow$ (iii) comes from Theorem 2.1 and [7, Corollary 2.9], and (iii) $\Rightarrow$ (i) results from [7, Corollary 3.7] or [12, Corollary 2.6].

Assume henceforth that R is locally finite-dimensional.

(iii) $\Leftrightarrow$ (iv): Let M be a maximal ideal of R. Then R_M is a PVD. So, if $K(M) := V_R(M)/MR_M$, then R_M is isomorphic to the pullback ring $V_R(M) \times_{K(M)} R_M/MR_M$. Next, recall that each overring of the pseudo-valuation domain R_M is comparable with $V_R(M)$ (cf. the proof of [8, Theorem 3.1]); that valuative dimension can be found by taking the supremum of the dimensions of valuation overrings; and that the supremum of the dimensions of valuation domains that are intermediate between two given fields is the transcendence degree of the field extension (cf. [26, Theorem 20.9]). Therefore, it follows from [23, Proposition 2.1] that $\dim_v(R_M) = \dim(V_R(M)) + tr.\deg[V_R(M)/MR_M : R/M] = \dim(R_M) + tr.\deg[V_R(M)/MR_M : R/M]$. Thus, $tr.\deg[V_R(M)/MR_M : R/M] \leq 1$ if and only if $\dim_v(R_M) - \dim(R_M) \leq 1$.

(v) $\Rightarrow$ (iv): Clear (as any finite-dimensional valuation domain is a Jaffard domain).

(iv) $\Rightarrow$ (v): Let M be a maximal ideal of R. Then R_M is a PVD such that either $\dim_v(R_M) = \dim(R_M)$ or $\dim_v(R_M) = \dim(R_M) + 1$. It suffices to show that if $\dim_v(R_M) = \dim(R_M)$, then R_M is a valuation domain. By the above reasoning, the assumed condition gives that $tr.\deg[V_R(M)/MR_M : R/M] = 0$, and so $R_M \subseteq V_R(M)$ is an integral extension. Then, since R_M is integrally closed, it follows that $R_M = V_R(M)$.

(v) $\Leftrightarrow$ (vi): As in the proof that (iii) $\Leftrightarrow$ (iv), $tr.\deg[V_R(M)/MR_M : R/M] = 1$ if and only if $\dim_v(R_M) - \dim(R_M) = 1$. $\square$

If $M \in \mathrm{Max}(R)$ and $\dim_v(R) \leq 2$, then

$$\dim_v(R_M) - \dim(R_M) \leq \dim_v(R) - \dim(R_M) \leq 1,$$

and we can then immediately deduce the following result from Proposition 2.2.

Corollary 2.3. *Let R be an integrally closed domain such that* $\dim_v(R) \leq 2$*. Then the following conditions are equivalent:*

(i) *Each overring of R is a going-down domain;*
(ii) *Each overring of R is treed;*
(iii) *R is an LPVD.*

Recall that R is *coequidimensional* if R is finite-dimensional and $\mathrm{ht}_R(M) = \dim(R)$ for every maximal ideal of R.

If each overring of R is treed, then $\dim_v(R) \leq \dim(R) + 1$ [7, Corollary 2.12]. If, in addition, R is coequidimensional and $M \in \mathrm{Max}(R)$, then $\dim_v(R_M) - \dim(R_M) \leq \dim_v(R) - \dim(R) \leq 1$. When combined with Proposition 2.2 and [26, Corollary 19.7 (2)] (noting that the latter can be used to show that $\dim_v(R) = \sup_{M \in \mathrm{Max}(R)} \dim_v(R_M)$ for all domains R), this easily leads to the following result.

Corollary 2.4. *Let R be an integrally closed coequidimensional domain. Then the following conditions are equivalent:*

(i) *Each overring of R is a going-down domain;*
(ii) *Each overring of R is treed;*
(iii) *R is an LPVD and* $\dim_v(R) \leq \dim(R) + 1$.

Let R be a finite-dimensional quasi-local integrally closed domain. It is known that if R is a Jaffard domain, then each overring of R is a going-down domain (or treed) if and only if R is a valuation domain [6, Lemma 1]. Our next proposition concerns the case where R is not a Jaffard domain (where it will turn out that, for a finite-dimensional quasi-local integrally closed domain which is not a Jaffard domain, having each overring being a going-down domain is equivalent to being a certain kind of PVD). This result points out that the domains that are quasi-local and integrally closed with all their overrings being going-down domains (or treed) are intimately connected to the domains for which each proper overring is (or satisfies) $\wp$, where $\wp$ is a ring-theoretic property, which means "(locally) Jaffard domain" or "(stably) strong S-domain" or "universally catenarian domain" or "satisfying the altitude formula". We will assume familiarity with results on all these concepts, as in [2], [5], [30], [10], but pause to collect the relevant definitions.

R is said to be *catenarian* in case, for each pair $P \subset Q$ of prime ideals of R, all saturated chains of prime ideals going from P to Q have a common finite length; R is called *universally catenarian* if the polynomial rings $R[X_1, X_2, \ldots, X_n]$ are catenarian for each positive integer n.

R is said to be a *strong* S-*domain* if, for each pair of consecutive prime ideals $P \subset Q$ of R, the extended primes $P[X] \subset Q[X]$ of $R[X]$ are consecutive; R is called a *stably strong* S-*domain* if the polynomial rings $R[X_1, X_2, \ldots, X_n]$ are strong S-domains for each positive integer n.

R is said to satisfy *the altitude formula* if, for each finite-type R-algebra S containing R and each prime ideal Q of S that lies over P, we have

$$\mathrm{ht}_S(Q) + tr.\deg[S/Q : R/P] = \mathrm{ht}_R(P) + tr.\deg[S : R].$$

With the above definitions recorded, we can give the following interesting characterizations.

Proposition 2.5. *Let (R, M) be a finite-dimensional quasi-local integrally closed domain. If R is not a Jaffard domain, then the following conditions are equivalent:*

(i) *Each overring of R is a going-down domain;*

(ii) *Each overring of R is treed;*

(iii) *Each proper overring of R is a Jaffard domain;*

(iv) *Each proper overring of R is a locally Jaffard domain;*

(v) *Each proper overring of R is a strong* S-*domain;*

(vi) *Each proper overring of R is a stably strong* S-*domain;*

(vii) *Each proper overring of R is a universally catenarian domain;*

(viii) *Each proper overring of R satisfies the altitude formula;*

(ix) *R is a* PVD *and* $\dim_v(R) = \dim(R) + 1$.

(x) *R is a* PVD *and* $tr.\deg[V_R(M)/M : R/M] = 1$.

Proof. The assertions (iii), (iv), (v), (vi), (vii), (viii), (ix), and (x) are equivalent by combining [9, Theorem 1.4, Theorem 1.8 and Proposition 2.2].

(x) $\Rightarrow$ (i) $\Leftrightarrow$ (ii): These assertions follow directly from Proposition 2.2.

(ii) $\Rightarrow$ (ix): Assume (ii). Then R is a PVD by Theorem 2.1. As R is not a Jaffard domain, then R is not a valuation domain. Thus, $\dim_v(R) = \dim(R) + 1$, by Proposition 2.2. □

We turn next to the case where R is not necessarily integrally closed. First, it will be convenient to provide a characterization of the integral extensions $R \subseteq S$ such that each intermediate ring between R and S is treed. Notice that the next result improves [7, Theorem 6.4] since the extension $R \subset S$ in it is not assumed to satisfy the going-down property.

Lemma 2.6. *Let $R \subset S$ be an integral extension of domains. Then the following conditions are equivalent:*

(i) *Each intermediate ring T between R and S (that is, each ring T such that $R \subseteq T \subseteq S$) is treed;*

(ii) *R is treed and every non-maximal prime ideal of R is unibranched in S.*

Proof. [7, Lemma 6.3] handles the implication (i) $\Rightarrow$ (ii). For the converse, assume, by way of contradiction, that there exists a non-treed intermediate ring T between R and S. It is easy to see, by the lying-over property and (ii), that each non-maximal prime ideal of R is unibranched in T. Choose q and q' to be incomparable prime ideals of T that are contained in some maximal ideal m' of T, and let $p := q \cap R$ and $p' := q' \cap R$ be their contractions to R. Then p and p' are each contained in the maximal ideal $m := m' \cap R$ of R. As R is treed, then p and p' must be comparable, say $p \subseteq p' \subset m$. If $p = p'$, then $q = q'$ (since p is unibranched in T), a contradiction. Now, suppose that $p \subset p'$. As $R \subseteq T$ enjoys the going-up property, there is a prime ideal q'' of T such that $q \subset q''$ and $q'' \cap R = p'$. Again, since p' is unibranched in T, we conclude that $q' = q''$, and so $q \subset q'$, another contradiction. Hence, T is treed, as desired. □

Lemma 2.7. *Let R be a domain such that each overring of R is treed. Then the following conditions are equivalent:*

(i) *R is a going-down domain;*

(ii) *$R \subseteq R'$ satisfies going-down;*

If, in addition, R is locally finite-dimensional, then the above two conditions are equivalent to the following condition:

(iii) *For each $M \in \mathrm{Max}(R')$, $\mathrm{ht}_{R'}(M) = \mathrm{ht}_R(M \cap R)$.*

Proof. It is clear that (i) $\Rightarrow$ (ii). Conversely, suppose (ii). Then, by [3, Theorem 10], R' is an LPVD, hence a going-down domain. A standard argument (which, for the sake of completeness, is given in the following paragraph) can be used to show that $R \subseteq T$ satisfies going-down for each overring T of R. Thus, (ii) $\Rightarrow$ (i).

In this paragraph, we provide the above-promised "standard argument." Consider prime ideals $P_2 \subset P_1$ of R and Q_1 of T such that $Q_1 \cap R = P_1$. Our task is to find $Q_2 \in \operatorname{Spec}(T)$ such that $Q_2 \cap R = P_2$ and $Q_2 \subseteq Q_1$. Put $A := R'T$. Note that A is integral over T. By the lying-over property of integrality, there exists $W_1 \in \operatorname{Spec}(A)$ such that $W_1 \cap T = Q_1$. Then $q_1 := W_1 \cap R' \in \operatorname{Spec}(R')$ satisfies $q_1 \cap R = P_1$. As $R \subseteq R'$ satisfies going-down, there exists $q_2 \in \operatorname{Spec}(R')$ such that $q_2 \cap R = P_2$ and $q_2 \subseteq q_1$. Next, since $R' \subseteq A$ satisfies going-down, there exists $W_2 \in \operatorname{Spec}(A)$ such that $W_2 \cap R' = q_2$ and $W_2 \subseteq W_1$. Then $Q_2 := W_2 \cap T \in \operatorname{Spec}(T)$ has the desired properties, thus completing the proof of the "standard argument."

It follows from Lemma 2.6 that if R is a locally finite-dimensional domain and each overring of R is treed, then $\operatorname{ht}_{R'}(P) = \operatorname{ht}_R(P \cap R)$ for every non-maximal prime ideal P of R'. Therefore, by combining Lemma 2.6 with [32, Propositions 1 and 3], we obtain the equivalence of (iii) with (i) and (ii). $\square$

Contrary to the case where R is integrally closed, if each overring of R is treed, it need not be the case that each overring of R is a going-down domain. Indeed, in [17, Example 2.3], the second-named author built a two-dimensional domain R such that R is not a going-down domain and each overring of R is treed. This ring R was not integrally closed and did not satisfy the conditions in Lemma 2.7.

We can now present our main result. To motivate condition (ii) in Theorem 2.8, we mention the following consequence of [12, Theorem 2.5]. Let R be a PVD whose canonically associated valuation overring is R'. (So, R' is an LPVD.) Then $tr.\deg[V_{R'}(M)/M(R')_M : R'/M] \leq 1$ for each (that is, for the) maximal ideal M of R' if and only if $T \subseteq T'$ satisfies the going-down property for every overring T of R.

Theorem 2.8. *Let R be a domain. Then the following two conditions are equivalent:*

(i) *Each overring of R is a going-down domain;*

(ii) *R' is an* LPVD*, $tr.\deg[V_{R'}(M)/M(R')_M : R'/M] \leq 1$ for each maximal ideal M of R', and $T \subseteq T'$ satisfies the going-down property for every overring T of R.*

If, in addition, R is locally finite-dimensional, then the above conditions are equivalent to the following condition:

(iii) *Each overring* T *of* R *is treed and satisfies* $\mathrm{ht}_{T'}(Q) = \mathrm{ht}_T(Q \cap T)$ *for every* $Q \in \mathrm{Max}(T')$.

Proof. (i) $\Rightarrow$ (ii) is immediate from Proposition 2.2.

(ii) $\Rightarrow$ (i): Assume (ii). Then, in view of the hypotheses, Proposition 2.2 gives that each overring of R' is a going-down domain. Now, let T be any overring of R. Then T' is an overring of R', and so T' is a going-down domain. As $T \subseteq T'$ satisfies going-down, [32, Lemma B] gives that T must be a going-down domain.

Assume henceforth that R is locally finite-dimensional.

(i) $\Rightarrow$ (iii): Assume (i). Then each overring of R is treed, by [13, Theorem 2.2]. Hence, by Lemma 2.7, $\mathrm{ht}_{T'}(Q) = \mathrm{ht}_T(Q \cap T)$ for every $Q \in \mathrm{Max}(T')$.

(iii) $\Rightarrow$ (ii): By Proposition 2.2, $tr.\deg[V_{R'}(M)/M(R')_M : R'/M] \leq 1$ for each maximal ideal M of R' and R' is an LPVD. Let T be an overring of R. It remains to prove that $T \subseteq T'$ satisfies going-down. Note that each overring of T is treed. We claim that T is locally finite-dimensional. Indeed, if Q is a maximal ideal of T', $P := Q \cap R'$ is its contraction to R', and M is a maximal ideal of R' that contains P, we have overring inclusions $(R')_M \subseteq (R')_P \subseteq (T')_Q$. But $(T')_Q$ is comparable to $V_{R'}(M)$ under containment by [24, Proposition 1.3(a)]. So, either $V_{R'}(M) \subseteq (T')_Q$ (in which case, $\dim((T')_Q) \leq \dim(V_{R'}(M)) = \dim((R')_M) < \infty$ since R' is locally finite-dimensional), or $(R')_M \subseteq (T')_Q \subseteq V_{R'}(M)$ (in which case, $\dim((T')_Q) \leq$

$\dim_v((T')_Q) \leq \dim_v((R')_M) = \dim(V_{R'}(M)) + tr.\deg[V_{R'}(M)/M(R')_M : R'/M] =$

$\dim((R')_M) + tr.\deg[V_{R'}(M)/M(R')_M : R'/M] \leq \dim((R')_M) + 1 < \infty$. Thus, T' is locally finite-dimensional. Hence, T is locally finite-dimensional, since $T \subseteq T'$ satisfies $\mathrm{ht}_{T'}(Q) = \mathrm{ht}_T(Q \cap T)$ for each $Q \in \mathrm{Max}(T')$. This proves the above claim. Finally, Lemma 2.7 shows that $T \subseteq T'$ satisfies going-down. $\square$

It is known that if the integral closure of R is a valuation domain, then each overring of R is a going-down domain [14, Corollary 2.5]. However, as [17, Example 2.3] illustrates, this conclusion need not follow if the integral closure of R is a Prüfer domain. We next characterize when each overring of R is a going-down domain, provided that R' is a Prüfer domain. As mentioned in the introduction, condition (vi) in Corollary 2.9 has been much-studied in the literature and need not hold for an arbitrary going-down domain R.

Corollary 2.9. *If the integral closure* R' *of* R *is a Prüfer domain, then the following two conditions are equivalent:*

(i) *Each overring of* R *is a going-down domain;*

(ii) *$T \subseteq T'$ satisfies going-down for each overring T of R.*

If, in addition, R is locally finite-dimensional, then the above conditions are equivalent to the following four conditions:

(iii) *Each integral overring T of R (that is, each ring T such that $R \subseteq T \subseteq R'$) is treed and $\mathrm{ht}_{R'}(Q) = \mathrm{ht}_R(Q \cap R)$ for every $Q \in \mathrm{Max}(R')$;*

(iv) *R is treed, and for each prime ideal Q of R', one has either $Q \notin \mathrm{Max}(R')$ and Q is the unique prime ideal of R' lying over $Q \cap R$ or $Q \in \mathrm{Max}(R')$ and $\mathrm{ht}_{R'}(Q) = \mathrm{ht}_R(Q \cap R)$.*

(v) *R is a going-down domain and every non-maximal prime ideal of R is unibranched in R';*

(vi) *Each integral overring of R is a going-down domain.*

Proof. (i) $\Rightarrow$ (ii): Trivial.

(ii) $\Rightarrow$ (i): Since R' is a Prüfer domain, R' is locally finite-dimensional and $V_{R'}(M) = (R')_M$ for all $M \in \mathrm{Max}(R')$. Thus, (ii) $\Rightarrow$ (i) by Theorem 2.8.

Assume henceforth that R is locally finite-dimensional.

(ii) $\Rightarrow$ (iii): Assume (ii). By combining (i) with Theorem 2.8, we get (iii).

(iii) $\Rightarrow$ (iv): Apply Lemma 2.6.

(iv) $\Rightarrow$ (v): Assume (iv). By [32, Lemma D], it suffices to prove that if $Q \in \mathrm{Spec}(R')$, with $n := \mathrm{ht}_{R'}(Q)$ $(< \infty)$, and $q := Q \cap R$ $(\in \mathrm{Max}(R))$, with $m := \mathrm{ht}_R(q)$ $(< \infty)$, then $n = m$. In view of the conditions in (iv), we can assume, without loss of generality, that Q is a non-maximal prime ideal of R' and that q is unibranched in R'. As the assertion is trivial if Q (or q) is (0), we can also assume that $n \neq 0$ and $m \neq 0$. Note, by the incomparable property of integrality, that $m \geq n$. Thus, it suffices to prove that $m \leq n$.

Since R is treed, note that the set of prime ideals of R which are contained in q forms a (finite maximal) chain, $(0) = q_0 \subset \ldots \subset q_m = q$. By using the lying-over, going-up and incomparable properties of integrality, we can find a chain $(0) = Q_0 \subset \ldots \subset Q_m$ of prime ideals of R' such that $Q_i \cap R = q_i$ for all $i = 0, \ldots, m$. Necessarily, $Q_m = Q$ since q is unibranched in R'. Thus, $n = \mathrm{ht}_{R'}(Q) = \mathrm{ht}_{R'}(Q_m) \geq m$.

(v) $\Rightarrow$ (vi): Let T be an integral overring of R. Since (v) gives that R is treed, Lemma 2.6 gives that T is treed. Then it follows by reasoning as in the proof of [33, Proposition 2.12] that T is a going-down domain.

(vi) $\Rightarrow$ (i): As R' is a Prüfer domain, it follows easily from [33, Proposition 2.12] that $A \subseteq B$ satisfies the incomparable property, for all overrings $A \subseteq B$ of R. In addition, each integral overring of R is treed, and so we get that each overring of

R is treed, by [7, Theorem 5.5]. Let T be an overring of R. Set $T_o := R' \cap T$. By hypothesis, T_o is a going-down domain; and the extension $T_o \subseteq T$ satisfies the incomparable property, by the above comments. Since, in addition, T is treed, a standard argument (which, for the sake of completeness, is included in the next paragraph) shows that T is a going-down domain, as desired.

By [21, Theorem], it is enough to prove that $T \subseteq V$ satisfies going-down for each valuation overring $(V, \mathfrak{P})$ of T. Put $J := \mathfrak{P} \cap R'$. As R' is a Prüfer domain, $V = (R')_J$. Consider prime ideals $P_1 \subset P_2$ of T and Q_2 of V such that $Q_2 \cap T = P_2$. Our task is to find $Q_1 \in \operatorname{Spec}(V)$ such that $Q_1 \cap T = P_1$ and $Q_1 \subseteq Q_2$. By replacing V with V_{Q_2}, we may suppose that $Q_2 = \mathfrak{P}$. Put $p_i := P_i \cap T_o$, for $i = 1, 2$. Note that $p_1 \subset p_2$ by the incomparable property of the extension $T_o \subseteq T$. Also note that $J \cap T_o = p_2$. Since (vi) ensures that $T_o \subseteq R'$ satisfies going-down, we therefore get $I \in \operatorname{Spec}(R')$ such that $I \cap T_o = p_1$ and $I \subset J$. Consider $P_3 := IV \cap T \in \operatorname{Spec}(T)$. We have that $P_3 \cap T_o = I \cap T_o = p_1$ and $P_3 \subseteq P_2$. Then, since the extension $T_o \subseteq T$ satisfies the incomparable property, either $P_1 = P_3$ or P_1 and P_3 are incomparable. But P_1 and P_3 are not incomparable, since T is treed. Hence $P_1 = P_3$. Then $Q_1 := I(R')_J = IV$ has the desired properties, which completes the proof. $\square$

Remark 2.10. *(a) We point out how the quasi-local QQR-domain (D, M) of* [27, Example 4.3] *relates to conditions (i) and (ii) in Corollary 2.9. Note that D is not a pseudo-valuation domain, by* [28, Theorem 1.7]*, since its integral closure D' is not quasi-local. In fact, D' is a Prüfer domain with exactly two maximal ideals, and D' is the only integral proper overring of D. Moreover, $D \subset D'$ is the special kind of minimal (over)ring extension that was studied in* [27]*, namely, the type of ring extension where every proper overring of D contains D'. Consequently, each proper overring of D is a Prüfer domain (hence a going-down domain, hence a treed domain). However, D fails to be a treed domain (and so D is the only overring of D which is not a going-down domain, thus showing that D "narrowly" fails to satisfy condition (i) in Corollary 2.9). Indeed,* [25, Remark 3.3] *established, i.a., the following more general fact: if (A, N) is a quasi-local QQR-domain and $A \subset A'$ is (an integral minimal ring extension which is) decomposed (in the sense that $A'/N \cong A/N \times A/N$ as algebras over A/N), then A is not a treed domain. (In contrast, note that each of the base rings denoted by R in* [17, Examples 2.1 and 2.3] *is a treed domain.) Similarly, D "narrowly" fails to satisfy condition (ii) in Corollary 2.9, since D is the only overring T of D such that $T \subseteq T'$ does not satisfy going-down.*

(b) With respect to condition (ii) in Corollary 2.9, we can use the fact that (filtered) direct limits preserve going-down [20] *to show that an arbitrary domain R (whose integral closure need not be a Prüfer domain) is such that $T \subseteq T'$ satisfies going-down for each overring T of R if and only if $A = R[u_1, \dots, u_n] \subset A[u]$ satisfies going-down for all finite lists $u_1, \dots, u_n, u$ in (the quotient field) K such that u is integral over A.*

We close with the following interesting case, which generalizes [6, Corollary 2] to the context of an arbitrary domain (with arbitrary valuative dimension). Recall from [10, Lemma 2.3] that if $R[X]$ is catenarian, then R is a strong S-domain, so $\mathrm{ht}_{R[X]}(P[X]) = \mathrm{ht}_R(P)$ for each prime ideal P of R (cf. [30]).

Corollary 2.11. *Assume that at least one of the following two conditions holds:*

(1) *$R[X]$ is a catenarian domain,*

(2) *R' is coequidimensional and $\mathrm{ht}_{R[X]}(M[X]) = \mathrm{ht}_R(M)$ for each maximal ideal M of R.*

Then the following three conditions are equivalent:

(i) *Each overring of R is a going-down domain;*

(ii) *Each overring of R is treed;*

(iii) *R is a going-down domain and every non-maximal prime ideal of R is unibranched in R'.*

Proof. (1) Note that each catenarian domain is locally finite-dimensional, by definition.

(i) ⇒ (ii): Apply [13, Theorem 2.2].

(ii) ⇒ (iii): Assume (ii). Then, by Theorem 2.1, R' is a going-down domain. We claim that $R \subseteq R'$ satisfies going-down.

By a result of McAdam, the claim will follow if we show that $R \subseteq R[u]$ satisfies going-down for all $u \in R'$. Identify $R[u] = R[X]/Q_o$, where Q_o is some nonzero prime ideal of the polynomial ring $R[X]$ such that $Q_o \cap R = (0)$. Since R is treed and locally finite-dimensional and $R \subseteq R[u]$ has the incomparable property, it follows from [32, Lemma C] that the claim will follow if we prove that $\mathrm{ht}_{R[u]}(Q') = \mathrm{ht}_R(Q' \cap R)$ for each prime ideal Q' of $R[u]$. In fact, since $R[u]$ is catenarian (as a homomorphic image of the catenarian domain $R[X]$), it is enough to test ideals Q' which are maximal ideals of $R[u]$. (To see this, it follows from the proof of [32, Lemma C] that it is enough to know that, if Q' is a prime ideal of $R[u]$ which is contained in a maximal ideal M' of $R[u]$ such that $n := \mathrm{ht}_{R[u]}(M')$, then there is a chain $\mathcal{C}$ of length n (i.e., with $n + 1$ elements) consisting of prime ideals of

$R[u]$ such that both M' and Q' are members of $\mathcal{C}$. This, in turn, follows from the catenariaty of $R[u]$, as one need only take $\mathcal{C}$ to be the union of a maximal chain of primes descending from M' to Q' with a maximal chain of primes descending from Q'.) For convenience, we will denote Q', now that it can be assumed to be a maximal ideal of $R[u]$, by M'. We can write $M' = M/Q_o$, for some maximal ideal M of $R[X]$ that contains Q_o. Then, since $R[X]$ is catenarian,

$$\mathrm{ht}_{R[u]}(M') = \mathrm{ht}_{R[X]/Q_o}(M/Q_o) = \mathrm{ht}_{R[X]}(M) - \mathrm{ht}_{R[X]}(Q_o) = \mathrm{ht}_{R[X]}(M) - 1. \quad (\lambda)$$

Set $m := M \cap R$. Then m is a maximal ideal of R and $m[X] \subset M$, and so

$$\mathrm{ht}_{R[X]/m[X]}(M/m[X]) = 1. \quad (\mu)$$

Furthermore, the strong S-domain property of R gives that $\mathrm{ht}_{R[X]}(M) = \mathrm{ht}_R(m)$. Thus, since $R[X]$ is catenarian, $\mathrm{ht}_{R[X]}(M) =$

$$\mathrm{ht}_{R[X]}(m[X]) + \mathrm{ht}_{R[X]/m[X]}(M/m[X]) = \mathrm{ht}_R(m) + \mathrm{ht}_{R[X]/m[X]}(M/m[X]). \quad (\nu)$$

By combining (λ), (μ) and (ν), we obtain $\mathrm{ht}_{R[u]}(M') = \mathrm{ht}_R(m)$, thus proving the above claim that $R \subseteq R'$ satisfies going-down.

Now, it follows from Lemma 2.7 that R is a going-down domain and that $\mathrm{ht}_{R'}(M') = \mathrm{ht}_R(M' \cap R)$ for all $M' \in \mathrm{Max}(R')$. As $\mathrm{ht}_{R[X]}(M[X]) = \mathrm{ht}_R(M)$ for each maximal ideal M of R, it follows from [4, Corollary 5.28] that R' is a Prüfer domain. (Note that R satisfies the tacit hypothesis in [4, Corollary 5.28] of being locally finite-dimensional.) Also, by [32, Lemma C], $\mathrm{ht}_{R'}(Q) = \mathrm{ht}_R(Q \cap R)$ for all $Q \in \mathrm{Spec}(R')$. The proof of Corollary 2.9 now permits us to infer (iii).

(iii) $\Rightarrow$ (i): Assume (iii). Once again, [4, Corollary 5.28] shows that R' is a Prüfer domain. Then Corollary 2.9 yields (i).

(2) Assume that R' is coequidimensional and $\mathrm{ht}_{R[X]}(M[X]) = \mathrm{ht}_R(M)$ for every maximal ideal M of R. We need only prove that (ii) $\Rightarrow$ (iii), as one can prove that (i) $\Rightarrow$ (ii) and that (iii) $\Rightarrow$ (i) by reasoning as above with (1). Assume (ii). If M' is any maximal ideal of R', then

$$\dim(R') = \mathrm{ht}_{R'}(M') \leq \mathrm{ht}_R(M' \cap R) \leq \dim(R) = \dim(R'),$$

and so $\mathrm{ht}_{R'}(M') = \mathrm{ht}_R(M \cap R)$. Therefore, an application of Corollary 2.9 yields (iii), thus completing the proof. For an alternate way to end the proof without appealing to Corollary 2.9, apply Lemma 2.7 to get that R is a going-down domain, and then apply Lemma 2.6 to get that every non-maximal prime ideal of R is unibranched in R'. $\square$

Lastly, we note a motivation for condition (2) in Corollary 2.11. Recall the result [11] that a going-down domain R is universally catenarian if (and only if) R is a locally finite-dimensional strong S-domain. It follows that, if we assume slightly more than condition (2) in Corollary 2.11, specifically that R' is coequidimensional and R is a strong S-domain, then whenever the equivalent conditions (i)-(iii) in Corollary 2.11 hold, then condition (1) in Corollary 2.11 also holds.

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Ahmed Ayache
Faculty of Science
Sana'a University
P.O. Box 12460, Sana'a, Yemen
e-mail: aaayache@yahoo.com

David E. Dobbs
Department of Mathematics
University of Tennessee
Knoxville, TN 37996-1320, U.S.A.
e-mail: ddobbs1@utk.edu