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ON CLASSES OF MODULES CLOSED UNDER INJECTIVE HULLS AND ARTINIAN PRINCIPAL IDEAL RINGS

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Abstract. In this work we consider some classes of modules closed under certain closure properties such as being closed under taking submodules, quotients, injective hulls and direct sums. We obtain some characterizations of artinian principal ideal rings using properties of big lattices of module classes.

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1. Introduction

In this work we consider some classes of modules closed under certain closure properties such as being closed under taking submodules, quotients, injective hulls and direct sums. We use the notation $\mathscr{L}_{\{\leq\}}$, $\mathscr{L}_{\{\twoheadrightarrow\}}$, $\mathscr{L}_{\{E\}}$ and $\mathscr{L}_{\{\oplus\}}$ describe as follows. We will denote $\mathscr{L}_{\{\leq\}}$ the class of hereditary classes, $\mathscr{L}_{\{\twoheadrightarrow\}}$ the class of module classes closed under taking quotients, $\mathscr{L}_{\{E\}}$ the class of module classes closed under taking injective hulls, $\mathscr{L}_{\{\oplus\}}$ the class of module classes closed under taking direct sums. $\mathscr{L}_{\{\leq,E\}}$ will denote the class of module classes closed under taking submodules and injective hulls. In general, if A is a set of closure properties, we denote by $\mathscr{L}_A$ the class of module classes closed with respect to the closure properties in A. If A denotes a subset of $\{\leq,\twoheadrightarrow,E,\oplus\}$ we should notice that $\mathscr{L}_A$ becomes a big lattice ordered by class inclusion with infima given by intersections.

There are many lattices of module classes of this type which are interesting to study for themselves. In this paper we will study lattices of module classes like $\mathscr{L}_{\{\leq,E\}}$, $\mathscr{L}_{\{\twoheadrightarrow,E\}}$.

We obtain some characterizations of artinian principal ideal rings using properties of big lattices of module classes. In the sequel, R denotes an associative ring with identity.

2. Generation in the lattices $\mathscr{L}_{\{\leq,E\}}$, $\mathscr{L}_{\{\twoheadrightarrow,E\}}$, $\mathscr{L}_{\{\leq,\oplus\}}$ and $\mathscr{L}_{\{\leq,\twoheadrightarrow\}}$

Definition 2.1. Let A be a set of closure properties and let $\mathscr{C}$ be a class of R-modules, we denote by $\xi_A(\mathscr{C})$ *the least class of modules* containing $\mathscr{C}$ and being closed under the properties in A.

We omit the easy verification of the following proposition.

Proposition 2.2. *If $\mathscr{C} \subseteq R$-mod, then*

1. $\xi_{\leq}(\mathscr{C}) = \{N|$ *there exists a monomorphism* $N \rightarrowtail C$ *with* $C \in \mathscr{C}\}$.
2. $\xi_E(\mathscr{C}) = \mathscr{C} \cup \{E(C)|C \in \mathscr{C}\}$.
3. $\xi_{\twoheadrightarrow}(\mathscr{C}) = \{M|$ *there exists a epimorphism* $N \twoheadrightarrow M$ *with* $N \in \mathscr{C}\}$.
4. $\xi_{\oplus}(\mathscr{C}) = \{M \mid M = \bigoplus_{i\in I} C_i$ *with I a set and* $C_i \in \mathscr{C}$ $\forall i \in I\}$.

Remark 2.3. *If $\mathscr{L}_P = \mathscr{L}_Q$, we have $\xi_P(\mathscr{C}) = \xi_Q(\mathscr{C})$ for each class $\mathscr{C}$ of R-modules.*

Proof. By definition, $\xi_P(\mathscr{C})$ is the least class in $\mathscr{L}_P$ containing $\mathscr{C}$, as $\mathscr{L}_P = \mathscr{L}_Q$, then $\mathscr{C} \subseteq \xi_P(\mathscr{C})$ and $\xi_P(\mathscr{C}) \in \mathscr{L}_Q$; thus $\xi_Q(\mathscr{C}) \subseteq \xi_P(\mathscr{C})$. By symmetry $\xi_P(\mathscr{C}) \subseteq \xi_Q(\mathscr{C})$. □

Now we describe generation in the lattices $\mathscr{L}_{\{\leq,E\}}$, $\mathscr{L}_{\{\leq,\oplus\}}$ and $\mathscr{L}_{\{\twoheadrightarrow,E\}}$.

Proposition 2.4. *If $\mathscr{C} \in \mathscr{L}_{\{E\}}$, then $\xi_{\leq}(\mathscr{C}) \in \mathscr{L}_{\{E\}}$.*

Proof. Let $N \in \xi_{\leq}(\mathscr{C})$ then there exists a monomorphism $t : N \rightarrowtail C$ with $C \in \mathscr{C}$. We obtain a commutative diagram:

$$\begin{array}{ccc} N & \hookrightarrow & E(N) \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle f} \\ C & \hookrightarrow & E(C). \end{array}$$

Inasmuch as N is essential in $E(N)$ we have that f is a monomorphism, $E(C) \in \mathscr{C}$, then $E(N) \in \xi_{\leq}(\mathscr{C})$. □

Remark 2.5. *If $\mathscr{C} \subseteq R$-mod, then $\xi_{\leq}\xi_E(\mathscr{C}) = \{M \in R$-mod$|$ there exists a monomorphism $M \rightarrowtail E(C)$ for some $C \in \mathscr{C}\}$.*

The next result is an immediate consequence of Proposition 2.4.

Proposition 2.6. *If $\mathscr{C}$ is a class of R-modules, then $\xi_{\leq,E}(\mathscr{C}) = \xi_{\leq}\xi_E(\mathscr{C})$.*

Example 2.7. *Notice that in general $\xi_{\leq}\xi_E(\mathscr{C}) \neq \xi_E\xi_{\leq}(\mathscr{C})$. For example $\mathscr{C} = \{_{\mathbb{Z}}\mathbb{Z}\}$, then $\xi_{\leq}\xi_E(_{\mathbb{Z}}\mathbb{Z}) = \{M|$there exists a monomorphism $M \rightarrowtail {_{\mathbb{Z}}\mathbb{Q}}\ \}$ and $\xi_E\xi_{\leq}(_{\mathbb{Z}}\mathbb{Z}) = \{M|$there exists a monomorphism $M \rightarrowtail {_{\mathbb{Z}}\mathbb{Z}}\ \}\bigcup\{E(M)|$ there exists a monomorphism $M \rightarrowtail {_{\mathbb{Z}}\mathbb{Z}}\ \}$. Just notice that $\{\frac{a}{2^n}|a \in \mathbb{Z}, n \in \mathbb{Q}\}$ belongs to the class $\xi_{\leq}\xi_E(\mathscr{C})$ but not belongs to the second class $\xi_E\xi_{\leq}(\mathscr{C})$.*

Remark 2.8. *Notice that $\mathscr{L}_{\{\leq,E\}}$ is a complete and distributive big lattice, where infima and suprema are given by intersection and union of classes respectively. Furthermore R-mod and $\{0\}$ are the greatest and least elements of the lattice.*

Remark 2.9. *If $\mathscr{C}$ is a class of R-modules, then $\xi_{\leq}\xi_{\oplus}(\mathscr{C}) = \{M|$ there exists a monomorphism $M \rightarrowtail \bigoplus_{i\in I} C_i$, with I a set, and $C_i \in \mathscr{C}$, for each $i \in I\}$.*

Proposition 2.10. *If $\mathscr{C}$ is a class of R-modules, then $\xi_{\leq,\oplus}(\mathscr{C}) = \xi_{\leq}\xi_{\oplus}(\mathscr{C})$.*

Proof. Clearly $\mathscr{C} \subseteq \xi_{\leq}\xi_{\oplus}(\mathscr{C})$. Now we will prove that $\xi_{\leq}\xi_{\oplus}(\mathscr{C})$ is a class closed under taking submodules and direct sums. As $\mathscr{C}$ clearly is a hereditary class, it suffices to show that it is closed under taking direct sums. Let $\{M_i\}_{i\in I} \subseteq \xi_{\leq}\xi_{\oplus}(\mathscr{C})$ be a family, then for every M_i there exists a family $\{C_{ij}\}_{j\in J_i}$ of modules in $\mathscr{C}$ such that there is a monomorphism $M_i \rightarrowtail \bigoplus_{j\in J_i} C_{ij}$, thus there is a monomorphism $\bigoplus_{i\in I} M_i \rightarrowtail \bigoplus_{i\in I}\{\bigoplus_{j\in J_i} C_{ij}\}$. Hence $\bigoplus_{i\in I} M_i \in \xi_{\leq}\xi_{\oplus}(\mathscr{C})$. Let $\mathscr{D}$ is a class of modules closed under taking submodules and direct sums containing $\mathscr{C}$. If $M \in \xi_{\leq}\xi_{\oplus}(\mathscr{C})$, then there exists a monomorphism $M \rightarrowtail \bigoplus_{i\in I} C_i$ with $\{C_i\}_{i\in I} \subseteq \mathscr{C}$. Hence $\{C_i\}_{i\in I} \subseteq \mathscr{D}$, then by hypothesis $\bigoplus_{i\in I} C_i \in \mathscr{D}$, thus $M \in \mathscr{D}$. Therefore $\xi_{\leq}\xi_{\oplus}(\mathscr{C}) \subseteq \mathscr{D}$. □

Definition 2.11. Let $\eta = \xi_E\xi_{\twoheadrightarrow}$ we define $\eta^0 = Id$, $\eta^{n+1} = \eta\eta^n$ for all $n \in \mathbb{N}$ and for a class of R-modules $\mathscr{C}$ we define $\eta^{\infty}(\mathscr{C}) = \underset{n\in\mathbb{N}}{\cup}\eta^n(\mathscr{C})$.

Remark 2.12. *Observe that for each $n \in \mathbb{N}$ and for each class $\mathscr{C}$ of R-modules, we have that $\eta^{n+1}(\mathscr{C}) = \xi_{\twoheadrightarrow}(\eta^n(\mathscr{C})) \cup \{E(N)|N \in \xi_{\twoheadrightarrow}(\eta^n(\mathscr{C}))\}$.*

Lemma 2.13. *If $\mathscr{C} \subseteq R$-mod, then $\eta^{\infty}(\mathscr{C}) = \xi_{\twoheadrightarrow,E}(\mathscr{C})$.*

Proof. It is clear that $\mathscr{C} \subseteq \eta^{\infty}(\mathscr{C})$, we will show that $\eta^{\infty}(\mathscr{C})$ is closed under taking quotients and injective hulls. Let $M \in \eta^{\infty}(\mathscr{C})$ and $M \twoheadrightarrow N$ an epimorphism, then $M \in \eta^n(\mathscr{C})$ for some $n \in \mathbb{N}$, thus $N \in \xi_{\twoheadrightarrow}(\eta^n(\mathscr{C})) \subseteq \eta^{n+1}(\mathscr{C})$. Also $E(M) \in \eta^{n+1}(\mathscr{C})$. Therefore $\eta^{\infty}(\mathscr{C}) \in \mathscr{L}_{\{\twoheadrightarrow,E\}}$.
Now we prove that $\eta^{\infty}(\mathscr{C})$ is the least class of the classes closed under taking quotients and injective hulls containing $\mathscr{C}$. Consider $\mathscr{D} \in \mathscr{L}_{\{\twoheadrightarrow,E\}}$ such that $\mathscr{C} \subseteq \mathscr{D}$. We prove by induction that $\eta^n(\mathscr{C}) \subseteq \mathscr{D}$ for all $n \in \mathbb{N}$. For $n = 0$ we have $\eta^0(\mathscr{C}) =$

$\mathscr{C} \subseteq \mathscr{D}$. Assume $\eta^n(\mathscr{C}) \subseteq \mathscr{D}$ and $M \in \eta^{n+1}(\mathscr{C}) = \xi_{\twoheadrightarrow}(\eta^n(\mathscr{C})) \cup \{E(N) | N \in \xi_{\twoheadrightarrow}(\eta^n(\mathscr{C}))\}$. If $M \in \xi_{\twoheadrightarrow}(\eta^n(\mathscr{C}))$ then $M \in \mathscr{D}$ since $\mathscr{D}$ is closed under taking quotients. If $M \in \{E(N)|N \in \xi_{\twoheadrightarrow}(\eta^n(\mathscr{C}))$ we have that $M \in \mathscr{D}$ because $\mathscr{D}$ is closed under taking injective hulls and quotients. Therefore $\eta^{n+1}(\mathscr{C}) \subseteq \mathscr{D}$ and thus $\eta^\infty(\mathscr{C}) = \underset{n\in\mathbb{N}}{\cup} \eta^n(\mathscr{C}) \subseteq \mathscr{D}$. □

Remark 2.14. *If $\mathscr{C}$ is a class of R-modules, then $\xi_{\leq}\xi_{\twoheadrightarrow}(\mathscr{C}) = \{M|M$ is a subquotient of K for some $K \in \mathscr{C}\}$ and $\xi_{\twoheadrightarrow}\xi_{\leq}(\mathscr{C}) = \{M|M$ is a quotient of a submodule K of a module $N \in \mathscr{C}\}$.*

Proposition 2.15. *If $\mathscr{C}$ is a class of R-modules, then $\xi_{\leq}\xi_{\twoheadrightarrow}(\mathscr{C}) = \xi_{\twoheadrightarrow}\xi_{\leq}(\mathscr{C})$.*

Proof. An immediate application of push outs and pull backs. □

3. Some relations among the lattices $\mathscr{L}_{\{\leq,E\}}$, $\mathscr{L}_{\{\twoheadrightarrow,E\}}$ and $\mathscr{L}_{\{\leq,\oplus\}}$

Theorem 3.1. *The following assertions are equivalent:*

(1) $\mathscr{L}_{\{\leq,E\}} \subseteq \mathscr{L}_{\{\twoheadrightarrow,E\}}$.
(2) *For each left injective R-module I, if there exists an epimorphism $I \twoheadrightarrow K$, then there exists a monomorphism $K \rightarrowtail I$.*

Proof. (1) $\Rightarrow$ (2) Assume $\mathscr{L}_{\{\leq,E\}} \subseteq \mathscr{L}_{\{\twoheadrightarrow,E\}}$. If $I \twoheadrightarrow K$ is an epimorphism, then by hypothesis $\xi_{\leq,E}(I) \in \mathscr{L}_{\{\twoheadrightarrow,E\}}$, then $K \in \xi_{\leq,E}(I)$. Thus there exists a monomorphism $K \rightarrowtail I$.

(2) $\Rightarrow$ (1) Let us take $\mathscr{C} \in \mathscr{L}_{\{\leq,E\}}$, it suffices to prove that $\mathscr{C}$ is closed under taking quotients. If $M \in \mathscr{C}$ and $f : M \twoheadrightarrow N$ is an epimorphism, consider the following commutative diagram:

$$\begin{array}{ccccc} Kerf & \hookrightarrow & M & \overset{f}{\twoheadrightarrow} & N \\ & & \big\downarrow & & \big\downarrow h \\ Kerf & \hookrightarrow & E(M) & \overset{\pi}{\twoheadrightarrow} & \frac{E(M)}{Kerf}. \end{array}$$

By hypothesis, there exists a monomorphism $\frac{E(M)}{Kerf} \rightarrowtail E(M)$ with $E(M) \in \mathscr{C}$ since $M \in \mathscr{C}$. So that $\frac{E(M)}{Kerf} \in \mathscr{C}$. It remains to prove that h is a monomorphism. Let us take $x \in Kerh$ and let $m \in M$ be such that $f(m) = x$. Then we have that $\pi(m) = h(f(m)) = h(x) = 0$, hence $m \in Kerf$, thus $0 = f(m) = x$ proving that $Kerh = \{0\}$. □

Theorem 3.2. $\mathscr{L}_{\{\twoheadrightarrow,E\}} \subseteq \mathscr{L}_{\{\leq,E\}}$ *if the following condition holds for each injective left R-module I: If there exists a monomorphism $K \rightarrowtail I$ then there exists an epimorphism $I \twoheadrightarrow K$.*

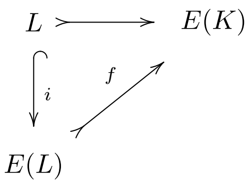
Proof. Let $\mathscr{C} \in \mathscr{L}_{\{\twoheadrightarrow,E\}}$, it suffices to prove that $\mathscr{C}$ is closed under taking submodules. Let us take $M \in \mathscr{C}$ and let $N \rightarrowtail M$ be a monomorphism, then there exists an epimorphism $f : E(M) \twoheadrightarrow N$. Then $N \in \mathscr{C}$ since $\mathscr{C}$ is closed under taking quotients and injective hulls. □

Theorem 3.3. *For a ring R, the following conditions are equivalent:*

(1) $\mathscr{L}_{\{\leq,E\}} = \mathscr{L}_{\{\twoheadrightarrow,E\}}$.
(2) *For each left injective R-module I we have:*
There exists a monomorphism $K \rightarrowtail I$ if and only if there exists an epimorphism $I \twoheadrightarrow K$.

Proof. (2) ⇒ (1) It follows from Theorems 3.1 and 3.2.

(1) ⇒ (2) From Theorem 3.1 we have that, if there exists an epimorphism $I \twoheadrightarrow K$ then there also exists a monomorphism $K \rightarrowtail I$. Now if there exists a monomorphism $K \rightarrowtail I$ we can change I for $E(K)$. If $L \in \xi_{\twoheadrightarrow}(E(K))$, then there exists an epimorphism $E(K) \twoheadrightarrow L$. Let $E(L)$ be the injective hull of L. As we pointed out at the beginning then there exists a monomorphism $L \rightarrowtail E(K)$ which extends to



with f being a monomorphism. Then $E(L)$ is isomorphic to direct summand of $E(K)$; therefore $E(L) \in \xi_{\twoheadrightarrow}(E(K))$. Then $\xi_{\twoheadrightarrow}(E(K)) \in \mathscr{L}_{\{\twoheadrightarrow,E\}}$ and $\xi_{\twoheadrightarrow}(E(K))$ contains $E(K)$, thus $\xi_{\twoheadrightarrow}(E(K)) = \xi_{\twoheadrightarrow,E}(E(K))$. By Remark 2.3 we have that $\xi_{\leq,E}(E(K)) = \xi_{\twoheadrightarrow,E}(E(K))$, now as $K \in \xi_{\leq,E}(E(K))$ we get $K \in \xi_{\twoheadrightarrow}(E(K))$. □

Theorem 3.4. *If $\mathscr{L}_{\{\twoheadrightarrow,E\}} \subseteq \mathscr{L}_{\{\leq\}}$, then R is a quasi-Frobenius ring.*

Proof. We will show that each projective module is injective, a condition which is equivalent to R being quasi-Frobenius by the Faith-Walker Theorem (see [2], [6] or [8]). Let P be a projective R-module. As $\xi_{\twoheadrightarrow,E}(E(P)) \in \mathscr{L}_{\{\leq\}}$ then $P \in \xi_{\twoheadrightarrow,E}(E(P))$. Let $n \in \mathbb{N}$ be least such that $P \in \eta^n(E(P))$. Note that if $n = 0$ then $P \in \eta^0(E(P)) = \{E(P)\}$ and therefore P is an injective R-module. If $n > 0$ then $P \in \xi_{\twoheadrightarrow}(\eta^{n-1}(E(P))) \cup \{E(N) | N \in \xi_{\twoheadrightarrow}(\eta^{n-1}E(P))\}$. So just consider the case $P \in \xi_{\twoheadrightarrow}(\eta^{n-1}(E(P)))$, thus there exists $M_1 \in \eta^{n-1}(E(P)) = \xi_{\twoheadrightarrow}(\eta^{n-2}E(P)) \cup \{E(N) | N \in \xi_{\twoheadrightarrow}(\eta^{n-2}E(P))\}$ and an epimorphism $\pi_1 : M_1 \twoheadrightarrow P$, if M_1 is injective then π_1 splits because P is a projective R-module, thus P is an injective module.

If $M_1 \in \xi_{\twoheadrightarrow}(\eta^{n-2}E(P))$, then there exists $M_2 \in \eta^{n-2}(E(P)) = \xi_{\twoheadrightarrow}(\eta^{n-3}E(P)) \cup \{E(N) | N \in \xi_{\{\twoheadrightarrow\}}(\eta^{n-3}E(P))\}$ and an epimorphism $\pi_2 : M_2 \twoheadrightarrow M_1$, thus P is a quotient of M_2. If M_2 is injective module, as before we have finished. Repeating the argument we have a finite sequence of modules $M_1, \ldots, M_n$ and epimorphisms $\pi_{i+1} : M_{i+1} \twoheadrightarrow M_i$ with $M_{i+1} \in \xi_{\twoheadrightarrow}(\eta^{n-(i+1)}(E(P)))$ for $i \in \{1, \ldots, n-1\}$ and with $M_i \in \eta^{n-i}(E(P))$ for $i \in \{1, \ldots, n\}$. Then we have the sequence

$$M_n \xrightarrow{\pi_n} M_{n-1} \xrightarrow{\pi_{n-1}} \ldots \xrightarrow{\pi_1} P \ .$$

Where $M_n \in \eta^{n-n}(E(P)) = \eta^0(E(P)) = \{E(P)\}$, then P is a quotient of $E(P)$. Therefore P is injective. □

Theorem 3.5. *If $\mathscr{L}_{\{\leq,\oplus\}} \subseteq \mathscr{L}_{\{E\}}$, then R is a left V-ring and a left noetherian ring.*

Proof. Let $\mathscr{C} = \{M \mid M \text{is semisimple}\}$. Clearly $\mathscr{C}$ is a class of R-modules closed under taking submodules and direct sums, then $\mathscr{C}$ is closed under taking injective hulls. Hence $E(M)$ is semisimple for each $M \in \mathscr{C}$, this implies that M is a direct summand of its injective hull, so that $M = E(M)$. Thus every semisimple module is injective, therefore R is a left V-ring. We also have that $\bigoplus_{i\in\mathbb{N}} E(S_i) = \bigoplus_{i\in\mathbb{N}} S_i$ is semisimple and injective, therefore R is left noetherian. □

Proposition 3.6. *If I is an indecomposable injective left R-module, then $\xi_{\leq,E}(I)$ is an atom in $\mathscr{L}_{\{\leq,E\}}$.*

Proof. Let $0 \neq C \in \mathscr{C} \subseteq \xi_{\leq,E}(I)$, this implies that there exists a monomorphism $C \rightarrowtail I$ so that $E(C)$ is a direct summand of I, as I is indecomposable then $I \cong E(C)$; therefore $\mathscr{C} \supseteq \xi_{\leq,E}(C) = \xi_{\leq,E}(E(C)) = \xi_{\leq,E}(I)$. □

Theorem 3.7. *Let R be a left noetherian ring. Then the following assertions are equivalent for a class $\mathscr{C}$ of left R-modules:*

(1) *$\mathscr{C}$ is an atom in $\mathscr{L}_{\{\leq,E\}}$.*
(2) *There exists an indecomposable injective left R-module I such that $\mathscr{C} = \xi_{\leq,E}(I)$.*

Proof. (2) $\Rightarrow$ (1) It follows from Proposition 3.6.

(1) $\Rightarrow$ (2) Assume that $\mathscr{C}$ is an atom of $\mathscr{L}_{\{\leq,E\}}$ and let us take $C \in \mathscr{C}$. Then $E(C) \in \mathscr{C}$. As R is a left noetherian ring we have that there exists a family $\{I_\alpha\}_{\alpha\in J}$ of left indecomposable injective modules such that $\underset{\alpha\in J}{\oplus} I_\alpha = E(C)$. For $\alpha \in J$ we have that $\xi_{\leq,E}(I_\alpha) \subseteq \mathscr{C}$, but as $\mathscr{C}$ is an atom then $\xi_{\leq,E}(I_\alpha) = \mathscr{C}$. □

4. Artinian principal ideal rings

The following theorem contains some well known results about artinian principal ideal rings (i.e. left and right artinian, left and right principal ideal rings), we include them here for convenience.

Theorem 4.1. *The following properties are equivalent:*

(1) *R is an artinian principal ideal ring.*
(2) *R is a left principal ideal ring and a quasi-Frobenius ring.*
(3) *The injective hull and the projective cover of each (right or left) finitely generated R-module are isomorphic.*
(4) *For each left R-module M, ${}_Rsoc(M) \cong \frac{M}{JM}$ and for each right R-module N, $soc(N)_R \cong \frac{N}{NJ}$, where J denotes the Jacobson radical of the ring R.*
(5) *For each ideal I of R, R/I is quasi-Frobenius.*
(6) *$\mathscr{L}_{\{\leq\}} = \mathscr{L}_{\{\twoheadrightarrow\}}$.*

Proof. (1) ⇔ (2) See Faith [6], (1) ⇔ (3) ⇔ (4) see Boyle [3], (1) ⇔ (5) can be found in [7] and [2]. Finally (1) ⇔ (6) is a theorem given by Alvarado, Rincón, Ríos and can be found in [1]. □

Theorem 4.2. *For a ring R the following properties are equivalent:*

(1) *For each injective left R-module I, if there exists a monomorphism $K \rightarrowtail I$, then there exists an epimorphism $I \twoheadrightarrow K$.*
(2) *R is an artinian principal ideal ring.*
(3) *$\mathscr{L}_{\{\leq,E\}} = \mathscr{L}_{\{\twoheadrightarrow,E\}}$.*
(4) *$\mathscr{L}_{\{\leq\}} \subseteq \mathscr{L}_{\{\twoheadrightarrow\}}$.*
(5) *$\mathscr{L}_{\{\leq\}} \supseteq \mathscr{L}_{\{\twoheadrightarrow\}}$.*

Proof. (1) ⇒ (2) Let us consider the inclusion $R \rightarrowtail E(R)$. Then by hypothesis, there exists an epimorphism $g: E(R) \twoheadrightarrow R$. Since ${}_RR$ is a projective module, then $E(R) = Ker g \oplus R$. Therefore, R is left selfinjective. For each left ideal I of R there exists an epimorphism $f: R \twoheadrightarrow I$, therefore I is cyclic. Thus R is a left principal ideal ring in particular it is left noetherian. As R is a left noetherian and left selfinjective ring, R is a quasi-Frobenius ring. Thus condition 2) of Theorem 4.1 is fulfilled.

(2) ⇒ (1) By Theorem 4.1 we have that $\mathscr{L}_{\{\leq\}} = \mathscr{L}_{\{\twoheadrightarrow\}}$ and that implies that $\xi_{\leq}(M) = \xi_{\twoheadrightarrow}(M)$ for each R-module M, in particular this holds for each injective module I.

(2) ⇒ (3) By Theorem 4.1 we have that $\mathscr{L}_{\{\leq\}} = \mathscr{L}_{\{\twoheadrightarrow\}}$ and this implies (3).

(3) $\Rightarrow$ (1) This follows from Theorem 3.3.

(5) $\Rightarrow$ (2) By Theorem 3.4 and (5) we obtain that R is a quasi-Frobenius ring. Notice that condition (5), which we are assuming, holds also for $\frac{R}{I}$-modules for each two-sided ideal I. Thus we conclude that $\frac{R}{I}$ is quasi-Frobenius for each two-sided ideal I. Now we use Theorem 4.1.

(2) $\Rightarrow$ (4) and (2) $\Rightarrow$ (5) follow from the fact that (2) is equivalent to $\mathscr{L}_{\{\leq\}} = \mathscr{L}_{\{\twoheadrightarrow\}}$ by Theorem 4.1.

(4) $\Rightarrow$ (2) We claim that $\xi_{\leq}(\{F \in R\text{-mod}|F \text{ is free }\})=R\text{-mod}$. Indeed, as each R-module M is a quotient of a free left R-module F and as $F \in \xi_{\leq}(\{F \in R\text{-mod}|F \text{ is free }\}) \in \mathscr{L}_{\twoheadrightarrow}$ then M belongs to this class. Thus each R-module ${}_RM$ is a submodule of a free module and this is equivalent to R being a quasi-Frobenius ring (see [7]). Thus R is a quasi-Frobenius ring. Observe that condition (4) holds also for $\frac{R}{I}$ for each two-sided ideal I. Thus $\frac{R}{I}$ is a quasi-Frobenius ring for each two-sided ideal I. Now use Theorem 4.1. $\square$

Recall that artinian modules are precisely the left R-modules such that all of their quotients are finitely cogenerated. As the class of finitely cogenerated left modules is closed under taking submodules and injective hulls, we have the following result.

Theorem 4.3. *If $\mathscr{L}_{\{\leq,E\}} \subseteq \mathscr{L}_{\{\twoheadrightarrow,E\}}$, then for each R-module M the following statements are equivalent:*

(1) *M is artinian.*
(2) *M is finitely cogenerated.*

Recall that a ring R is called left co-noetherian if the injective hull of each simple R-module is artinian [11].

Proposition 4.4. *If $\mathscr{L}_{\{\leq,E\}} \subseteq \mathscr{L}_{\{\twoheadrightarrow,E\}}$, then R is a left co-noetherian ring.*

Proof. If S is a simple left R-module, then S is finitely cogenerated, then also $E(S)$ is finitely cogenerated. By Theorem 4.3 we have that $E(S)$ is artinian, therefore R is co-noetherian. $\square$

Proposition 4.5. *If $\mathscr{L}_{\{\leq,E\}} \subseteq \mathscr{L}_{\{\twoheadrightarrow,E\}}$, then $E(R)$ is a cogenerator of R-mod.*

Proof. Let S a simple R-module. If $0 \neq x \in S$ then $Rx = S$. As we have an epimorphism $R \twoheadrightarrow Rx$ and $\xi_{\leq,E}(R) \in \mathscr{L}_{\{\leq,E\}} \subseteq \mathscr{L}_{\{\twoheadrightarrow,E\}}$, then by Theorem 3.1 there exists a monomorphism $Rx \rightarrowtail E(R)$. Therefore $E(R)$ contains a copy of each simple left R-module, thus $E(R)$ is an injective cogenerator. $\square$

Proposition 4.6. *If $\mathscr{L}_{\{\leq,E\}} \subseteq \mathscr{L}_{\{\twoheadrightarrow,E\}}$, then R is a left semiartinian ring.*

Proof. We show that each non zero R-module contains a simple submodule. If M is a non zero R-module, then the class $\xi_{\leq}(E(M))$ belongs to $\mathscr{L}_{\{\leq,E\}}$. As a consequence, a simple quotient S of a non zero cyclic submodule of M, also embeds in $E(M)$. Thus, such a simple quotient S embeds in M, inasmuch as M is an essential submodule of $E(M)$. □

In the proof of the following Theorem we adapt an idea of [5] which also uses a Lemma in [10]. Recall that a ring is left local when all of its simple left modules are isomorphic.

Theorem 4.7. *If R is a left local and left co-noetherian ring, then R satisfies the ascending chain condition for two-sided ideals.*

Proof. Since R is a left local ring then there exists just one simple left R-module S up to isomorphism. Then $E = E(S)$ is a cogenerator for R-mod. Also notice that it is left artinian by the actual hypothesis. Let $I_0 = 0 \subseteq I_1 \subseteq \dots$ be an ascending chain of two-sided ideals of R. Taking $L_i = \{x \in E | I_i x = 0\}$ we obtain a descending chain of left submodules of E, $L_0 = E \supseteq L_1 \supseteq L_2 \supseteq \dots$. As E is left artinian then there exists $i \in \mathbb{N}$ such that $L_{i+k} = L_i$ for all $k \geq 0$. We may identify L_j with $Hom_R\left(\frac{R}{I_j}, E\right)$ by letting correspond x in L_j to the homomorphism sending $1 + I_j$ into x. Since

$$0 \longrightarrow \frac{I_{j+1}}{I_j} \longrightarrow \frac{R}{I_j} \longrightarrow \frac{R}{I_{j+1}} \longrightarrow 0$$

is an exact sequence of left R-modules and as E is a left injective R-modules, we obtain an exact sequence of abelian groups

$$0 \longrightarrow Hom_R\left(\frac{R}{I_{j+1}}, E\right) \longrightarrow Hom_R\left(\frac{R}{I_j}, E\right) \longrightarrow Hom_R\left(\frac{I_{j+1}}{I_j}, E\right) \longrightarrow 0.$$

Then we get the exact sequence

$$0 \longrightarrow L_{j+1} \longrightarrow L_j \longrightarrow Hom_R\left(\frac{I_{j+1}}{I_j}, E\right) \longrightarrow 0.$$

In particular for $j = i$, we obtain $Hom_R(\frac{I_{i+1}}{I_i}, E) \cong \frac{L_i}{L_{i+1}} = 0$. As E is an injective cogenerator of R-mod, then $\frac{I_{i+1}}{I_i} = 0$. From this we obtain $I_{i+k} = I_i$ for all $k \geq 0$. Therefore R satisfies the ascending chain condition for two-sided ideals. □

In [4] Bronowitz and Teply proved that the rings for which all of its hereditary torsion theories are cohereditary are precisely the finite products of left local right perfect rings.

Theorem 4.8. *If $\mathscr{L}_{\{\leq,E\}} \subseteq \mathscr{L}_{\{\twoheadrightarrow,E\}}$, then R is a product of finitely many left local right perfect rings, and R satisfies the ascending chain condition for two-sided ideals.*

Proof. The hypothesis implies that each hereditary torsion theory is cohereditary, then by [4], R is a product of finitely many left local right perfect rings. By Proposition 4.4, R is a left co-noetherian ring. In view of Theorem 4.7, R satisfies the ascending chain condition for two-sided ideals. □

Remark 4.9. *Recall the Loewy sequence*

$$soc(R) \hookrightarrow soc_2(R) \hookrightarrow \cdots \hookrightarrow soc_n(R) \hookrightarrow \cdots$$

where $\frac{soc_{n+1}(R)}{soc_n(R)} = soc\left(\frac{R}{soc_n(R)}\right)$. *If $\mathscr{L}_{\{\leq,E\}} \subset \mathscr{L}_{\{\twoheadrightarrow,E\}}$ then R satisfies the ascending chain condition on two-sided ideals by Theorem 4.8. As a consequence, the ascending Loewy sequence for ${}_RR$ must stabilize with $soc_n(R) = soc_{n+1}(R)$ for some $n \in \mathbb{N}$. Since R is left semiartinian by Proposition 4.6, every nonzero left R-module has nonzero socle. Inasmuch as* $soc\left(\frac{R}{soc_n(R)}\right) = \frac{soc_{n+1}(R)}{soc_n(R)} = 0$, *this entails* $\frac{R}{soc_n(R)} = 0$, *whence* $R = soc_n(R)$.

Recall that if R is an arbitrary ring, M an arbitrary left R-module and $J = J(R)$, then $Jsoc(M) = 0$, and more generally, $J^n soc_n(M) = 0$. Hence if $soc_n(R) = R$, then $J^n = 0$.

Theorem 4.10. If $\mathscr{L}_{\{\leq,E\}} \subseteq \mathscr{L}_{\{\twoheadrightarrow,E\}}$, then R is a finite product of left local right and left perfect rings satisfying ascending chain condition for two-sided ideals.

Proof. By Remark 4.9 and the above observation, we have that J is a nilpotent ideal, thus J is left and right T-nilpotent. By Bass's Theorem [2] we have that R is right and left perfect. The remaining assertions are proved in Theorem 4.8. □

Remark 4.11. *As the referee points out, a commutative ring satisfying $\mathscr{L}_{\{\leq,E\}} \subseteq \mathscr{L}_{\{\twoheadrightarrow,E\}}$ must be artinian. Indeed, the ACC on ideals means that ${}_RR$ must have a finitely generated socle. Since R is left semiartinian its socle is essential and a module with finitely generated essential socle is finitely cogenerated and thus artinian by Theorem 4.3. We dont know if there exists a left non artinian ring satisfying $\mathscr{L}_{\{\leq,E\}} \subseteq \mathscr{L}_{\{\twoheadrightarrow,E\}}$.*

The authors thank the referee for the following example showing that the converse of Theorem 4.10 fails.

Example 4.12. *Let R denote the trivial extension of $\mathbb{Z}_2$ by V a two dimensional $\mathbb{Z}_2$-vector space. That is* $R = \left\{ \begin{pmatrix} a & (a,b) \\ 0 & a \end{pmatrix} : a, b, c \in \mathbb{Z}_2 \right\}$.
Notice that R is a commutative artinian local ring and also it is a $\mathbb{Z}_2$-algebra with 8 elements. Notice that $J(R) = soc(R)$ is essential in R. Let us observe that each simple module is isomorphic to $S = \frac{R}{J(R)}$. According to [9, 3.41] $E({}_RS) \cong E(\frac{R}{J(R)}) = Hom_{\mathbb{Z}_2}(R, \mathbb{Z}_2)$. Besides, $\frac{E({}_RS)}{S}$ is a singular module, thus it is annihilated by $J(R)$, hence $\frac{E({}_RS)}{S}$ is a four element semisimple module. Thus $\frac{E({}_RS)}{S} \cong S \oplus S$. For this reason, $\frac{E({}_RS)}{S}$ cannot embed in the uniform module $E({}_RS)$. Thus $\xi_{\leq,E}(S) \notin \mathscr{L}_{\{\twoheadrightarrow,E\}}$, consequently $\mathscr{L}_{\{\leq,E\}} \nsubseteq \mathscr{L}_{\{\twoheadrightarrow,E\}}$. With the same argument it can be seen that for a finite field F, the trivial extension of F by V an F-vector space of finite dimention n, $\mathscr{L}_{\{\leq,E\}} \subseteq \mathscr{L}_{\{\twoheadrightarrow,E\}}$ holds if and only if $n = 1$.

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