International Electronic Journal of Algebra
Volume 15 (2014) 66-76

A FEW COMMENTS ON MATLIS DUALITY

Waqas Mahmood

Received: 11 April 2013; Revised: 3 December 2013
Communicated by Abdullah Harmancı

Dedicated to the memory of Professor Efraim P. Armendariz

Abstract. For a Noetherian local ring $(R, \mathfrak{m})$ with $\mathfrak{p} \in \operatorname{Spec}(R)$, we denote the R-injective hull of $R/\mathfrak{p}$ by $E_R(R/\mathfrak{p})$. We show that it has an $\hat{R}^{\mathfrak{p}}$-module structure, and there is an isomorphism $E_R(R/\mathfrak{p}) \cong E_{\hat{R}^{\mathfrak{p}}}(\hat{R}^{\mathfrak{p}}/\mathfrak{p}\hat{R}^{\mathfrak{p}})$, where $\hat{R}^{\mathfrak{p}}$ stands for the $\mathfrak{p}$-adic completion of R. Moreover, for a complete Cohen-Macaulay ring R, the module $D(E_R(R/\mathfrak{p}))$ is isomorphic to $\hat{R}_{\mathfrak{p}}$ provided that $\dim(R/\mathfrak{p}) = 1$, where $D(\cdot)$ denotes the Matlis dual functor $\operatorname{Hom}_R(\cdot, E_R(R/\mathfrak{m}))$. Here, $\hat{R}_{\mathfrak{p}}$ denotes the completion of $R_{\mathfrak{p}}$ with respect to the maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$. These results extend those of Matlis (see [11]) shown in the case of the maximal ideal $\mathfrak{m}$.

Mathematics Subject Classification (2010): 13D45
Keywords: Matlis duality, injective hull, local cohomology, flat covers

1. Introduction

Throughout this paper R is a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and the residue field $k = R/\mathfrak{m}$. We denote the (contravaraint) Hom-functor $\operatorname{Hom}_R(\cdot, E_R(R/\mathfrak{p}))$ by $\vee$, that is $M^\vee := \operatorname{Hom}_R(M, E_R(R/\mathfrak{p}))$ for an R-module M and $E_R(R/\mathfrak{p})$ is a fixed R-injective hull of $R/\mathfrak{p}$ where $\mathfrak{p} \in \operatorname{Spec}(R)$. Moreover we denote the Matlis dual functor by $D(\cdot) := \operatorname{Hom}_R(\cdot, E_R(k))$. Also $\hat{R}_{\mathfrak{p}}$ (resp. $\hat{R}^{\mathfrak{p}}$) stands for the completion of $R_{\mathfrak{p}}$ (resp. of R) with respect to the maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$ (resp. with respect to the prime ideal $\mathfrak{p}$).

Our main goal is to give the structure of $E_R(R/\mathfrak{p})$ as an $\hat{R}^{\mathfrak{p}}$-module. In particular we will show the following result (see Theorem 3.1):

Theorem. *Let $(R, \mathfrak{m})$ be a ring and $\mathfrak{p} \in \operatorname{Spec}(R)$. Then $E_R(R/\mathfrak{p})$ admits the structure of an $\hat{R}^{\mathfrak{p}}$-module and $E_R(R/\mathfrak{p}) \cong E_{\hat{R}^{\mathfrak{p}}}(\hat{R}^{\mathfrak{p}}/\mathfrak{p}\hat{R}^{\mathfrak{p}})$ as $\hat{R}^{\mathfrak{p}}$-modules.*

In the case of the maximal ideal $\mathfrak{m}$, Matlis has shown (see [11]) that the injective hull $E_R(k)$ of the residue field k has the structure of $\hat{R}$-module and it is isomorphic to the $\hat{R}$-module $E_{\hat{R}}(k)$. Here we will extend this result to an arbitrary prime ideal

$\mathfrak{p}$, that is $E_R(R/\mathfrak{p})$ admits the structure of an $\hat{R}^{\mathfrak{p}}$-module and it is isomorphic to the $\hat{R}^{\mathfrak{p}}$-module $E_{\hat{R}^{\mathfrak{p}}}(\hat{R}^{\mathfrak{p}}/\mathfrak{p}\hat{R}^{\mathfrak{p}})$.

Moreover, we know that $\operatorname{Hom}_R(E_R(R/\mathfrak{p}), E_R(R/\mathfrak{q})) = 0$ for any $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}(R)$ such that $\mathfrak{p} \in V(\mathfrak{q})$ and $\mathfrak{p} \neq \mathfrak{q}$. It was proven by Enochs (see [5, p. 183]) that the module $E_R(R/\mathfrak{q})^\vee$ has the following decomposition:

$$E_R(R/\mathfrak{q})^\vee \cong \prod T_{\mathfrak{p}'}.$$

where $\mathfrak{p}' \in \operatorname{Spec}(R)$ and $T_{\mathfrak{p}'}$ denotes the completion of a free $R_{\mathfrak{p}'}$-module with respect to $\mathfrak{p}' R_{\mathfrak{p}'}$-adic completion. Here we prove the following result (see Theorem 3.3):

Theorem. *Let $(R, \mathfrak{m})$ be a complete Cohen-Macaulay ring of dimension n. Suppose that $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim(R/\mathfrak{p}) = 1$. Then there is an isomorphism*

$$D(E_R(R/\mathfrak{p})) \cong \hat{R}_{\mathfrak{p}}.$$

Recently Schenzel has shown (see [17, Theorem 1.1]) that if $\mathfrak{p}$ is a one dimensional prime ideal, then $D(E_R(R/\mathfrak{p})) \cong \hat{R}_{\mathfrak{p}}$ if and only if $R/\mathfrak{p}$ is complete.

Furthermore Enochs (see [5, p. 183]) has shown that the module $E_R(R/\mathfrak{q})^\vee$ is a flat cover of some cotorsion module. In the end of the Section 3 we will show the following result (see Theorem 3.7):

Theorem. *Let $(R, \mathfrak{m})$ be a ring and $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}(R)$ such that $\mathfrak{p} \in V(\mathfrak{q})$. Then $E_R(R/\mathfrak{q})^\vee$ is a flat precover of $(R/\mathfrak{q})^\vee$.*

2. Preliminaries

In this section we will fix the notation of the paper and summarize a few preliminaries and auxiliary results. Notice that the following two propositions are known for the maximal ideal $\mathfrak{m}$ we will extend them to any prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$.

Proposition 2.1. *Let $(R, \mathfrak{m})$ be a ring and $\mathfrak{p} \in \operatorname{Spec}(R)$. Then we have:*

(a) $E_R(R/\mathfrak{p}) \cong E_{R_{\mathfrak{q}}}(R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}})$ *for all* $\mathfrak{q} \in V(\mathfrak{p})$.

(b) *Suppose that M is an R-module and $\mathfrak{p} \in \operatorname{Supp}_R(M)$. Then $M^\vee$ has a structure of an $R_{\mathfrak{p}}$-module.*

Proof. For the proof of the statement (a) we refer to [12, Theorem 18.4]. We only prove the last claim. For this purpose note that $E_R(R/\mathfrak{p})$ is an $R_{\mathfrak{p}}$-module. Now we define $(\frac{r}{s} \cdot f)(m) := \frac{r}{s} f(m)$ where $\frac{r}{s} \in R_{\mathfrak{p}}, m \in M$ and $f \in M^\vee$. Then it defines the structure of $M^\vee$ to be an $R_{\mathfrak{p}}$-module which completes the proof. □

Proposition 2.2. *Let* $(R, \mathfrak{m})$, $(S, \mathfrak{n})$ *be local rings and* $R \to S$ *be a surjective local homomorphism i.e.* $S = R/I$ *for some ideal* $I \subseteq R$. *Suppose that* M *is an* R-*module and* $\mathfrak{p} \in \mathrm{Spec}(R)$. *Then the following are true:*

(a) *Suppose that* N *is an* S-*module. Then for any* $\mathfrak{p} \in V(I)$

$$N^\vee \cong \mathrm{Hom}_S(N, E_S(S/\mathfrak{p}S)).$$

(b) *For all* $\mathfrak{q} \in V(\mathfrak{p})$ *we have*

$$M^\vee \cong \mathrm{Hom}_{R\mathfrak{q}}(M_\mathfrak{q}, E_{R_\mathfrak{q}}(R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q})).$$

Proof. For the proof see [9, Example 3.6 and Excercise 13]. □

Lemma 2.3. *For a local ring* R *let* M, N *be any* R-*modules. Then for all* $i \in \mathbb{Z}$ *the following hold:*

(1) $\mathrm{Ext}^i_R(N, D(M)) \cong D(\mathrm{Tor}^R_i(N, M))$.
(2) *If in addition* N *is finitely generated, then*

$$D(\mathrm{Ext}^i_R(N, M)) \cong \mathrm{Tor}^R_i(N, D(M)).$$

(3) $\mathrm{Hom}_R(\varinjlim M_n, M) \cong \varprojlim \mathrm{Hom}_R(M_n, M)$, *where* $\{M_n : n \in \mathbb{N}\}$ *is a direct system of* R-*modules.*

Proof. For the proof see [9, Example 3.6] and [18]. □

In order to prove the next results we need a few more preparations. Let $I \subseteq R$ be an ideal and for $i = 1, \ldots, s$ let $\mathfrak{q}_\mathfrak{i}$ belongs to a minimal primary decomposition of the zero ideal in R where $\mathrm{Rad}(\mathfrak{q}_\mathfrak{i}) = \mathfrak{p}_\mathfrak{i}$. Then we denote $u(I)$ by the intersection of all $\mathfrak{p}_\mathfrak{i}$-primary components $\mathfrak{q}_\mathfrak{i}$ such that $\dim_R(R/(\mathfrak{p}_\mathfrak{i} + I)) > 0$. Moreover we denote the functor of the global transform by $T(\cdot) := \varinjlim \mathrm{Hom}_R(\mathfrak{m}^\alpha, \cdot)$. Also note that the local cohomology functor with respect to I is denoted by $H^i_I(\cdot), i \in \mathbb{Z}$, see [3] for its definition and applications in local algebra.

Lemma 2.4. *Let* R *be a local ring and* $I \subseteq R$ *be an ideal. Then there is an exact sequence:*

$$0 \to \hat{R}^I/u(I\hat{R}^I) \to \varprojlim T(R/I^s) \to \varprojlim H^1_\mathfrak{m}(R/I^s) \to 0,$$

where $\hat{R}^I$ *denotes the* I-*adic completion of* R.

Proof. Note that for each $\alpha \in \mathbb{N}$ there is an exact sequence

$$0 \to \mathfrak{m}^\alpha \to R \to R/\mathfrak{m}^\alpha \to 0.$$

For $s \in \mathbb{N}$ apply the functor $\operatorname{Hom}_R(\cdot, R/I^s)$ to this sequence. Then it induces the following exact sequence

$$0 \to \operatorname{Hom}_R(R/\mathfrak{m}^\alpha, R/I^s) \to R/I^s \to \operatorname{Hom}_R(\mathfrak{m}^\alpha, R/I^s) \to \operatorname{Ext}^1_R(R/\mathfrak{m}^\alpha, R/I^s) \to 0.$$

Now take the direct limit of this we again get an exact sequence

$$0 \to H^0_{\mathfrak{m}}(R/I^s) \to R/I^s \to T(R/I^s) \to H^1_{\mathfrak{m}}(R/I^s) \to 0$$

for each $s \in \mathbb{N}$. It induces the following two short exact sequences

$$0 \to H^0_{\mathfrak{m}}(R/I^s) \to R/I^s \to R/I^s : \mathfrak{m} \to 0 \text{ and}$$

$$0 \to R/I^s : \mathfrak{m} \to T(R/I^s) \to H^1_{\mathfrak{m}}(R/I^s) \to 0.$$

Note that the inverse systems at the left hand side of these short exact sequences satisfy the Mittag-Leffler condition. So if we take the inverse limit to them, then the resulting sequences will be exact also (see [1, Proposition 10.2]). If we combined these resulting sequences, we get the following exact sequence

$$0 \to \varprojlim H^0_{\mathfrak{m}}(R/I^s) \to \hat{R}^I \to \varprojlim T(R/I^s) \to \varprojlim H^1_{\mathfrak{m}}(R/I^s) \to 0.$$

But by [16, Lemma 4.1] there is an isomorphism $\varprojlim H^0_{\mathfrak{m}}(R/I^s) \cong u(I\hat{R}^I)$. Then the exactness of the last sequence provides the required statement. $\square$

In the next context we need the definition of the canonical module. To do this we recall the Local Duality Theorem (see [8]). Let $(S, \mathfrak{n})$ be a local Gorenstein ring of dimension t and N be a finitely generated R-module where $R = S/I$ for some ideal $I \subseteq S$. Then It was shown by Grothendieck (see [8]) that there is an isomorphism

$$H^i_{\mathfrak{m}}(N) \cong \operatorname{Hom}_R(\operatorname{Ext}^{t-i}_S(N, S), E)$$

for all $i \in \mathbb{N}$ (see [8]). For a slight extension of the Local Duality to an arbitrary module M over a Cohen-macaulay local ring see [14, Lemma 3.1]. Note that a more general result was proved by Hellus (see [10, Theorem 6.4.1]). Now we are able to define the canonical module as follows:

Definition 2.5. With the notation of the above Local Duality Theorem we define

$$K_N := \operatorname{Ext}^{t-r}_S(N, S), \dim(N) = r$$

as the *canonical module of* N. It was introduced by Schenzel (see [15]) as the generalization of the canonical module of a Cohen-Macaulay ring (see e.g. [2]).

Corollary 2.6. *With the notation of Lemma 2.4, suppose that* I *is a one dimensional ideal. Then the following are true:*

(a) *There is an exact sequence*

$$0 \to \hat{R}^I/u(I\hat{R}^I) \to \oplus_{i=1}^s \hat{R}_{\mathfrak{p}_i} \to \varprojlim H^1_{\mathfrak{m}}(R/I^s) \to 0,$$

where $\mathfrak{p}_i \in \operatorname{Ass}_R(R/I)$ *such that* $\dim(R/\mathfrak{p}_i) = 1$ *for* $i = 1, \ldots, s$.

(b) *Suppose in addition that* R *is complete Cohen-Macaulay. Then there is an exact sequence*

$$0 \to R/u(I) \to \oplus_{i=1}^s \hat{R}_{\mathfrak{p}_i} \to D(H^{n-1}_I(K_R)) \to 0.$$

Proof. Note that for the proof of the statement (a) it will be enough to prove the following isomorphism

$$\varprojlim T(R/I^s) \cong \oplus_{i=1}^s \hat{R}_{\mathfrak{p}_i}$$

(by Lemma 2.4). Since $\dim(R/I) = 1$ it implies that there exists an element $x \in \mathfrak{m}$ such that x is a parameter of R/I^s for all $s \in \mathbb{N}$. Then it induces the following isomorphism

$$T(R/I^s) \cong R_x/I^sR_x \text{ for all } s \in \mathbb{N}.$$

Now suppose that $S = \cap_{i=1}^s (R \setminus \mathfrak{p}_i)$. Then $x \in S$ and by Local Global Principal for each $s \in \mathbb{N}$ there is an isomorphism

$$R_x/I^sR_x \cong R_S/I^sR_S.$$

Since R_S is a semi local ring, there is an isomorphism

$$R_S/I^sR_S \cong \oplus_{i=1}^s R_{\mathfrak{p}_i}/I^sR_{\mathfrak{p}_i} \text{ for all } s \in \mathbb{N}.$$

(by Chinese Remainder Theorem). Since $\dim(R/\mathfrak{p}_i) = 1$ for $i = 1, \ldots, s$, it follows that $\operatorname{Rad}(IR_{\mathfrak{p}_i}) = \mathfrak{p}_i R_{\mathfrak{p}_i}$. By passing to the inverse limit of the last isomorphism we have

$$\varprojlim T(R/I^s) \cong \oplus_{i=1}^s \hat{R}_{\mathfrak{p}_i},$$

which is the required isomorphism. To prove (b), note that the canonical module of K_R of R exists (since R is complete Cohen-Macaulay). So by [14, Lemma 3.1] there is an isomorphism

$$D(H^i_{\mathfrak{m}}(R/I^s)) \cong \operatorname{Ext}^{n-i}_R(R/I^s, K_R)$$

for each $i \in \mathbb{N}$. Since $H^i_{\mathfrak{m}}(R/I^s)$ is an Artinian R-module and R is complete, by Matlis Duality, its double Matlis dual is itself. That is for each $s \in \mathbb{N}$ there is an isomorphism

$$H^i_{\mathfrak{m}}(R/I^s) \cong D(\operatorname{Ext}^{n-i}_R(R/I^s, K_R)).$$

By passing to the inverse limit of this isomorphism induces the following isomorphism

$$\varprojlim H^i_{\mathfrak{m}}(R/I^s) \cong \varprojlim D(\operatorname{Ext}^{n-i}_R(R/I^s, K_R)).$$

Now take the Matlis dual of the isomorphism $H^{n-i}_I(K_R) \cong \varinjlim \operatorname{Ext}^{n-i}_R(R/I^s, K_R)$. It induces that

$$\varprojlim H^i_{\mathfrak{m}}(R/I^s) \cong D(H^{n-i}_I(K_R))$$

(by Lemma 2.3 (3)). For $i = 1$, by the exact sequence in (a), we can get the exact sequence of (b) (since R is complete). This finishes the proof of the Corollary. □

3. On Matlis Duality

In this section, we will prove one of the main results. Actually this result was proved by Matlis for the maximal ideal $\mathfrak{m}$. Here we will extend this to any prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$.

Theorem 3.1. *Let $(R, \mathfrak{m})$ be a ring and $\mathfrak{p} \in \operatorname{Spec}(R)$. Then $E_R(R/\mathfrak{p})$ admits the structure of an $\hat{R}^{\mathfrak{p}}$-module and $E_R(R/\mathfrak{p}) \cong E_{\hat{R}^{\mathfrak{p}}}(\hat{R}^{\mathfrak{p}}/\mathfrak{p}\hat{R}^{\mathfrak{p}})$.*

Proof. If $x \in E_R(R/\mathfrak{p})$, then by [11] $\operatorname{Ass}_R(Rx) = \{\mathfrak{p}\}$. It follows that some power of $\mathfrak{p}$ annihilates x.

Now let $\hat{r} \in \hat{R}^{\mathfrak{p}}$. Then $\hat{r} = (r_1 + \mathfrak{p}, \ldots, r_n + \mathfrak{p}^n, \ldots)$ such that $r_{n+1} - r_n \in \mathfrak{p}^n$ where $r_i + \mathfrak{p} \in R/\mathfrak{p}^i$ for all $i \geq 1$. Since $x \in E_R(R/\mathfrak{p})$, then there exists a fixed $n \in \mathbb{N}$ such that $\mathfrak{p}^n x = 0$. Choose $r \in R$ such that $\hat{r} - r \in \mathfrak{p}^n$ (by definition of the completion). Define $\hat{r}x = rx$. Then it is clear that this gives a well-defined $\hat{R}^{\mathfrak{p}}$-module structure to $E_R(R/\mathfrak{p})$ which agrees with its R-module structure. Since $E_R(R/\mathfrak{p})$ is an essential extension of $R/\mathfrak{p}$ as an R-module then it necessarily is also an essential extension of $\hat{R}^{\mathfrak{p}}/\mathfrak{p}\hat{R}^{\mathfrak{p}}$ as an $\hat{R}^{\mathfrak{p}}$-module. So $E_R(R/\mathfrak{p}) \subseteq E_{\hat{R}^{\mathfrak{p}}}(\hat{R}^{\mathfrak{p}}/\mathfrak{p}\hat{R}^{\mathfrak{p}})$.

To show that $E_R(R/\mathfrak{p})$ is an injective hull of $\hat{R}^{\mathfrak{p}}/\mathfrak{p}\hat{R}^{\mathfrak{p}}$ as an $\hat{R}^{\mathfrak{p}}$-module, it is enough to prove that $E_R(R/\mathfrak{p}) = E_{\hat{R}^{\mathfrak{p}}}(\hat{R}^{\mathfrak{p}}/\mathfrak{p}\hat{R}^{\mathfrak{p}})$. To do this it suffices to see that $E_{\hat{R}^{\mathfrak{p}}}(\hat{R}^{\mathfrak{p}}/\mathfrak{p}\hat{R}^{\mathfrak{p}})$ is an essential extension of $R/\mathfrak{p}$ as an R-module since $E_R(R/\mathfrak{p})$ is an extension of $R/\mathfrak{p}$ as an R-module (see [11]).

Let $x \in E_{\hat{R}^{\mathfrak{p}}}(\hat{R}^{\mathfrak{p}}/\mathfrak{p}\hat{R}^{\mathfrak{p}})$ and choose an element $\hat{r} \in \hat{R}^{\mathfrak{p}}$ such that $0 \neq \hat{r}x \in \hat{R}^{\mathfrak{p}}/\mathfrak{p}\hat{R}^{\mathfrak{p}}$. By the argument used in the beginning of the proof applied to $\hat{R}^{\mathfrak{p}}$, $E_{\hat{R}^{\mathfrak{p}}}(\hat{R}^{\mathfrak{p}}/\mathfrak{p}\hat{R}^{\mathfrak{p}})$ and x there is an $n \in \mathbb{N}$ such that $\mathfrak{p}^n x = 0$. Choose $r \in R$ such that $\hat{r} - r \in \mathfrak{p}^n$. Then $\hat{r}x = rx \in R/\mathfrak{p} \cong \hat{R}^{\mathfrak{p}}/\mathfrak{p}\hat{R}^{\mathfrak{p}}$ (see [1]) and $rx \neq 0$. Hence $E_{\hat{R}^{\mathfrak{p}}}(\hat{R}^{\mathfrak{p}}/\mathfrak{p}\hat{R}^{\mathfrak{p}})$ is an essential extension of $R/\mathfrak{p}$ as an R-module and $E_R(R/\mathfrak{p}) = E_{\hat{R}^{\mathfrak{p}}}(\hat{R}^{\mathfrak{p}}/\mathfrak{p}\hat{R}^{\mathfrak{p}})$ which completes the proof. □

Remark 3.2. (1) *If* $\mathfrak{p} \in \operatorname{Spec}(R)$*, then by Theorem 3.1 for any finitely generated* R*-module* M *we have*

$$M^\vee \cong \operatorname{Hom}_R(M, \operatorname{Hom}_{\hat{R}^\mathfrak{p}}(\hat{R}^\mathfrak{p}, E_{\hat{R}^\mathfrak{p}}(\hat{R}^\mathfrak{p}/\mathfrak{p}\hat{R}^\mathfrak{p}))).$$

By Lemma 2.3 the later module is isomorphic to $\operatorname{Hom}_{\hat{R}^\mathfrak{p}}(\hat{M}^\mathfrak{p}, E_{\hat{R}^\mathfrak{p}}(\hat{R}^\mathfrak{p}/\mathfrak{p}\hat{R}^\mathfrak{p}))$*. Here we use that* $\hat{M}^\mathfrak{p} \cong M \otimes_R \hat{R}^\mathfrak{p}$*. By Proposition 2.1 it implies that* $M^\vee$ *has an* $\hat{R}^\mathfrak{p}$*-module structure.*

(2) *Since* $\operatorname{Supp}_R(E_R(R/\mathfrak{p})) = V(\mathfrak{p})$*, where* $\mathfrak{p} \in \operatorname{Spec}(R)$ *(by Proposition 2.1(a)),* $E_R(R/\mathfrak{p})$ *is p-torsion. So from* [13, Remark A.30] *the natural homomorphism*

$$E_R(R/\mathfrak{p}) \to E_R(R/\mathfrak{p}) \otimes_R \hat{R}^\mathfrak{p}.$$

is an isomorphism. But $E_R(R/\mathfrak{p})$ *is isomorphic to the module* $E_{\hat{R}^\mathfrak{p}}(\hat{R}^\mathfrak{p}/\mathfrak{p}\hat{R}^\mathfrak{p})$ *and it admits the structure of an* $\hat{R}^\mathfrak{p}$*-module (see Theorem 3.1). Moreover the above natural homomorphism is compatible with this structure. It implies that the* $\mathfrak{p}$*-adic completion of* R *commutes with the injective hull of* $R/\mathfrak{p}$*.*

It is well known that $\operatorname{Hom}_R(E_R(R/\mathfrak{p}), E_R(R/\mathfrak{q})) = 0$ for any $\mathfrak{p} \in V(\mathfrak{q})$ and $\mathfrak{p} \neq \mathfrak{q}$. Moreover Enochs has proved that

$$E_R(R/\mathfrak{q})^\vee \cong \prod T_{\mathfrak{p}'}.$$

where $\mathfrak{p}' \in \operatorname{Spec}(R)$ and $T_{\mathfrak{p}'}$ denotes the completion of a free $R_{\mathfrak{p}'}$-module with respect to $\mathfrak{p}' R_{\mathfrak{p}'}$-adic completion. Here we will show the following theorem:

Theorem 3.3. *Let* $(R, \mathfrak{m})$ *be a complete Cohen-Macaulay ring and* $\mathfrak{q} \in \operatorname{Spec}(R)$ *be a one dimensional prime ideal. Then there is an isomorphism*

$$D(E_R(R/\mathfrak{q})) \cong \hat{R}_\mathfrak{q}.$$

Proof. Since R is complete Cohen-Macaulay, K_R exists, and we have

$$H^i_\mathfrak{q}(K_R) = 0 \text{ for all } i < \operatorname{height}(\mathfrak{q}) = n - 1.$$

This is true, because K_R is a maximal Cohen-Macaulay R-module of finite injective dimension and $\operatorname{Supp}_R(K_R) = \operatorname{Spec}(R)$. Let $E^{\cdot}_R(K_R)$ be a minimal injective resolution of K_R. Then by [7, Theorem 1.1], we have

$$E^{\cdot}_R(K_R)^i \cong \bigoplus_{\operatorname{height}(\mathfrak{p})=i} E_R(R/\mathfrak{p}).$$

Moreover, $\Gamma_\mathfrak{q}(E_R(R/\mathfrak{p})) = 0$ for all $\mathfrak{p} \notin V(\mathfrak{q})$ and $\Gamma_\mathfrak{q}(E_R(R/\mathfrak{p})) = E_R(R/\mathfrak{p})$ for all $\mathfrak{p} \in V(\mathfrak{q})$. Then apply $\Gamma_\mathfrak{q}$ to $E^{\cdot}_R(K_R)$. It induces the following exact sequence

$$0 \to H^{n-1}_\mathfrak{q}(K_R) \to E_R(R/\mathfrak{q}) \to E_R(k) \to H^n_\mathfrak{q}(K_R) \to 0.$$

Applying Matlis dual to this sequence yields the following exact sequence

$$0 \to D(H^n_{\mathfrak{q}}(K_R)) \to R \to D(E_R(R/\mathfrak{q})) \to D(H^{n-1}_{\mathfrak{q}}(K_R)) \to 0. \tag{1}$$

Here we use that $D(E_R(k)) \cong R$ (since R is complete). By the proof of Corollary 2.6 (b) and [16, Lemma 4.1], there are isomorphisms

$$D(H^n_{\mathfrak{q}}(K_R)) \cong \varprojlim H^0_{\mathfrak{m}}(R/\mathfrak{q}^s) \cong u(\mathfrak{q}).$$

So the exact sequence (1) provides the following exact sequence

$$0 \to R/u(\mathfrak{q}) \to D(E_R(R/\mathfrak{q})) \to D(H^{n-1}_{\mathfrak{q}}(K_R)) \to 0. \tag{2}$$

Now the module $D(E_R(R/\mathfrak{q}))$ is an $R_{\mathfrak{q}}$-module, so there is a natural homomorphism $R_{\mathfrak{q}} \to D(E_R(R/\mathfrak{q}))$. Then tensoring with $R_{\mathfrak{q}}/\mathfrak{q}^s R_{\mathfrak{q}}$, this homomorphism induces the following homomorphism

$$R_{\mathfrak{q}}/\mathfrak{q}^s R_{\mathfrak{q}} \to D(E_R(R/\mathfrak{q}))/\mathfrak{q}^s D(E_R(R/\mathfrak{q}))$$

for each $s \in \mathbb{N}$. Now take the inverse limit of it to get

$$\hat{R}_{\mathfrak{q}} \to \varprojlim D(E_R(R/\mathfrak{q}))/\mathfrak{q}^s D(E_R(R/\mathfrak{q})).$$

On the other side, since $\operatorname{Supp}_R(E_R(R/\mathfrak{q})) = V(\mathfrak{q})$, the module $E_R(R/\mathfrak{q})$ is isomorphic to $\varinjlim \operatorname{Hom}_R(R/\mathfrak{q}^s, E_R(R/\mathfrak{q}))$. Then by Proposition 2.3 (3) there is an isomorphism

$$D(E_R(R/\mathfrak{q})) \cong \varprojlim D(\operatorname{Hom}_R(R/\mathfrak{q}^s, E_R(R/\mathfrak{q}))).$$

But again Proposition 2.3 (2) implies that the module $D(\operatorname{Hom}_R(R/\mathfrak{q}^s, E_R(R/\mathfrak{q})))$ is isomorphic to $D(E_R(R/\mathfrak{q}))/\mathfrak{q}^s D(E_R(R/\mathfrak{q}))$. Therefore, there is a natural homomorphism

$$\hat{R}_{\mathfrak{q}} \to D(E_R(R/\mathfrak{q})).$$

Since R is complete Cohen-Macaulay, by Corollary 2.6 (b), there is an exact sequence

$$0 \to R/u(\mathfrak{q}) \to \hat{R}_{\mathfrak{q}} \to D(H^{n-1}_{\mathfrak{q}}(K_R)) \to 0.$$

Then this sequence together with the sequence (2) induces the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \to & R/u(\mathfrak{q}) & \to & \hat{R}_{\mathfrak{q}} & \to & D(H^{n-1}_{\mathfrak{q}}(K_R)) & \to 0 \\
 & & || & & \downarrow & & || & \\
0 & \to & R/u(\mathfrak{q}) & \to & D(E_R(R/\mathfrak{q})) & \to & D(H^{n-1}_{\mathfrak{q}}(K_R)) & \to 0
\end{array}$$

Then by Five Lemma, there is an isomorphism $D(E_R(R/\mathfrak{q})) \cong \hat{R}_{\mathfrak{q}}$ which is the required isomorphism. □

Corollary 3.4. *Fix the notation of Theorem 3.3. Then the following hold:*

(a) *There is an exact sequence*

$$0 \to R/u(\mathfrak{q}) \to D(E_R(R/\mathfrak{q})) \to D(H^{n-1}_{\mathfrak{q}}(K(R))) \to 0.$$

(b) *Suppose in addition that R is a domain. Then there is an isomorphism*

$$D(H^{n-1}_{\mathfrak{q}}(K(R))) \cong \hat{R}_{\mathfrak{q}}/R.$$

Proof. Since R is complete, we apply Corollary 2.6 (b) for $I = \mathfrak{q}$. Then it implies that there is an exact sequence

$$0 \to R/u(\mathfrak{q}) \to \hat{R}_{\mathfrak{q}} \to D(H^{n-1}_{\mathfrak{q}}(K(R))) \to 0.$$

Then the statement (a) is easily follows from Theorem 3.3. Note that the statement (b) follows from the above short exact sequence. Recall that $u(\mathfrak{q}) = 0$ since R is a domain. $\square$

In the next context we need the following definition of flat covers.

Definition 3.5. Let M be an R-module and F any flat R-module. Then the linear map $\phi : F \to M$ is called a *flat cover* of M if the following conditions hold:

(i) For any flat R-module G the following sequence is exact

$$\operatorname{Hom}_R(G, F) \to \operatorname{Hom}_R(G, M) \to 0$$

(ii) If $\phi = \phi \circ f$ for some $f \in \operatorname{Hom}_R(F, F)$, then f is an automorphism of F.

Note that if condition (i) holds, then F is called a *flat precover*. Enochs proved in his paper (see [4, Theorem 3.1]) that if M has a flat precover, then it also admits a flat cover and it is unique up to isomorphism.

The following lemma is an easy consequence of chasing diagram (see [5, Lemma 1.1]).

Lemma 3.6. *Let M be an R-module and F, F' any flat R-modules. Then we have:*

(a) *If $\phi' : F' \to M$ is a flat precover of M and $\phi : F \to M$ is a flat cover of M such that $\phi' = \phi \circ f$ for some $f \in \operatorname{Hom}_R(F', F)$, then f is surjective and* $\ker(f)$ *is a direct summand of F'.*

(b) *If $\phi : F \to M$ is a flat precover of M, then it is a cover if and only if* $\ker(\phi)$ *contains no non-zero direct summand of F.*

Enochs (see [5, p. 183]) showed that the module $\operatorname{Hom}_R(E_R(R/\mathfrak{q}), E_R(R/\mathfrak{p}))$ is a flat cover of some cotorsion module. Here we will show the following result:

Theorem 3.7. *Let* $(R, \mathfrak{m})$ *be a ring and* $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}(R)$ *such that* $\mathfrak{p} \in V(\mathfrak{q})$*. Then* $E_R(R/\mathfrak{q})^\vee$ *is a flat precover of* $(R/\mathfrak{q})^\vee$*.*

Proof. Note that $E_R(R/\mathfrak{q})$ is as essential extension of $R/\mathfrak{q}$. Let F be any flat R-module. Then the inclusion map $R/\mathfrak{q} \otimes_R F \hookrightarrow E_R(R/\mathfrak{q}) \otimes_R F$ induces the following exact sequence

$$\operatorname{Hom}_R(E_R(R/\mathfrak{q}) \otimes_R F, E_R(R/\mathfrak{p})) \to \operatorname{Hom}_R(R/\mathfrak{q} \otimes_R F, E_R(R/\mathfrak{p})) \to 0.$$

By the adjunction formula (see Lemma 2.3), we conclude that the following homomorphism is surjective for any flat R-module F

$$\operatorname{Hom}_R(F, E_R(R/\mathfrak{q})^\vee) \to \operatorname{Hom}_R(F, (R/\mathfrak{q})^\vee),$$

which proves that $E_R(R/\mathfrak{q})^\vee$ is a flat precover of $(R/\mathfrak{q})^\vee$. □

Remark 3.8. *Note that if* R *is a complete local ring, then* R *is a flat cover of the residue field* k *(see* [6, Example 5.3.19]*).*

Problem 3.9. (1) Let R be a complete local ring and $\mathfrak{p} \in V(\mathfrak{q})$ and $\mathfrak{p} \neq \mathfrak{q}$. It would be of some interest to see whether $E_R(R/\mathfrak{q})^\vee$ is a flat cover of $(R/\mathfrak{q})^\vee$ or not?

(2) Note that it was shown in [17, Theorem 1.1] that if $\dim(R/\mathfrak{p}) = 1$, then $R/\mathfrak{p}$ is complete if and only if $D(E_R(R/\mathfrak{p})) \cong \hat{R}_\mathfrak{p}$. Let R be a complete local Cohen-Macaulay ring. Is it possible to generalize Theorem 3.3 for arbitrary prime ideals $\mathfrak{q} \subsetneq \mathfrak{p}$?

Acknowledgement. The author is grateful to the reviewer for suggestions which improve the manuscript.

References

[1] M. F. Atiyah and I. G. Macdonald, Introduction to Commutative Algebra, University of Oxford, 1969.

[2] W. Bruns and J. Herzog, Cohen-Macaulay Rings, Cambridge Univ. Press, 39, 1998.

[3] M. Brodmann and R. Sharp, Local Cohomology, An Algebraic Introduction with Geometric Applications, Cambridge Studies in Advanced Mathematics No. 60. Cambridge University Press, 1998.

[4] E. E. Enochs, *Injective and Flat Covers, Envelopes and Resolvents,* Israel J. Math., 39 (1981), 189-209.

[5] E. E. Enochs, *Flat Covers and Flat Cotorsion Modules*, Proc. Amer. Math. Soc., 92 (1984), 179-184.

[6] E. E. Enochs and O.M.G. Jenda, Relative Homological Algebra(de Gruyter Expositions in Mathematics, 30), Walter de Gruyter, Berlin, 2000.
[7] R. Fossum, H.-B. Foxby, B. Griffith and I. Reiten, *Minimal Injective Resolutions with Applications to Dualizig Modules and Gorenstein Modules*, Publ. Math. Inst. Hautes Etudues Sci., 45 (1976), 193-215.
[8] A. Grothendieck, Local Cohomology(Notes by R. Hartshorne), Lecture Notes in Math. vol.41, Springer, 1967.
[9] C. Huneke, *Lectures on Local Cohomology (with an Appendix by Amelia Taylor)*, Contemp. Math., 436 (2007), 51-100.
[10] M. Hellus, *Local Cohomology and Matils Duality*, arXiv:math/0703124v1.
[11] E. Matlis, *Injective Modules Over Noetherian Rings*, Pacific J. Math., 8 (1958), 511-528.
[12] H. Matsumura, Commutative Ring Theory, Cambridge Studies in Advanced Mathematics 8, Cambridge University Press, 1986.
[13] E. Miller, S. Iyengar, G. J. Leuschke, A. Leykin, C. Miller, A.K. Singh and U. Walther, Twenty Four Hours of Local Cohomology (Graduate Studies in Mathematics), American Mathematical Society, Vol. 87, 2007.
[14] W. Mahmood, *On Cohomologically Complete Intersections in Cohen-Macaulay Rings*, submitted.
[15] P. Schenzel, *On Birational Macaulayfications and Cohen-Macaulay Canonical Modules*, J. Algebra, 275 (2004), 751-770.
[16] P. Schenzel, *On Formal Local Cohomology and Connectedness*, J. Algebra, 315(2) (2007), 894-923.
[17] P. Schenzel, *A Note on the Matlis Dual of a Certain Injective Hull*, arXiv:1306.3311v1.
[18] C. Weibel, An Introduction to Homological Algebra, Cambridge Univ. Press, 1994.

Waqas Mahmood
Abdus Salam School of Mathematical Sciences,
Government College University,
Lahore, Pakistan
e-mail: waqassms@gmail.com