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PSEUDO QP-INJECTIVE MODULES AND GENERALIZED PSEUDO QP-INJECTIVE MODULES

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Abstract. Let M be a right R-module with $S = End(M_R)$. Then M_R is called pseudo QP-injective (or PQP-injective for short) if every monomorphism from an M-cyclic submodule of M to M extends to an endomorphism of M . M_R is called generalized pseudo QP-injective (or $GPQP$-injective for short) if, for any $0 \neq s \in S$, there exists a positive integer n such that $s^n \neq 0$ and every monomorphism from $s^n M$ to M extends to an endomorphism of M. Characterizations and properties of the two classes of modules are studied. The two classes of modules with some additional conditions are studied, semisimple artinian rings are characterized by PQP-injective modules.

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1. Introduction

Throughout this paper, R denotes an associative ring with identity and all modules considered are unitary. Let M be a right R-module. Then we denote the injective hull of a module M by $E(M)$, the endomorphism ring of M by S, and the the Jacobson radical of S by $J(S)$ respectively. Let $X \subseteq M$ and $Y \subseteq S$, then we write $\mathbf{l}_S(X) = \{s \in S \mid sx = 0$, for all $x \in X\}$ and $\mathbf{r}_M(Y) = \{m \in M \mid ym = 0$, for all $y \in Y\}$.

Recall that a right R-module N is called M-cyclic [10, p41] if N is a homomorphic image of M, and M is called QP-injective [8] or semi-injective [10, p261] if for every M-cyclic submodule K of M, any R-homomorphism from K to M extends to an endomorphism of M, or equivalently, $\mathbf{l}_S(\mathrm{Ker}(s)) = Ss$. We also recall that a ring R is right MP-injective [12] if, for any $a \in R$, every monomorphism from aR to R extends to R; a ring R is right MGP-injective [12] if, for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and every monomorphism from $a^n R$ to R extends to R. In this paper, we generalize the concepts of QP-injective modules and MP-injective rings to pseudo QP-injective modules, and generalize the concepts of pseudo QP-injective modules and MGP-injective rings to generalized pseudo QP-injective modules, respectively, and give some interesting results on these modules.

2. PQP-injective Modules

We start with the following definition.

Definition 2.1. Let R be a ring and M, N be two right R-modules. Then N is called *M-cyclic* injective if every monomorphism from an M-cyclic submodule of M to M extends to a homomorphism of M to N, N is called *pseudo M-cyclic injective* if every monomorphism from an M-cyclic submodule of M to M extends to a homomorphism of M to N. A right R-module M is called *pseudo QP-injective* (or *PQP-injective* for short) if M is pseudo M-cyclic injective .

Clearly, a ring R is right MP-injective if and only if R_R is PQP-injective. We note that M-cyclic injective modules are called M-p-injective in [8], pseudo M-cyclic injective modules are called pseudo M-p-injective in [3], and pseudo QP-injective modules are called quasi-pseudo principally injective in [3].

Our following result extend the result of [12, Theorem 2.2]

Theorem 2.2. *The following conditions are equivalent for a module M_R with $S = End(M_R)$:*

(1) *M is PQP-injective.*
(2) *$Ker(s) = Ker(t), s, t,$ in S, implies that $Ss = St$.*

Proof. (1) $\Rightarrow$ (2) If $Ker(s) = Ker(t)$, then the mapping $f : sM \to tM$; $sm \mapsto tm$ is a monomorphism. Since M is PQP-injective, $f = s'\cdot$ for some $s' \in S$, and so $t = s's$. This implies that $St \subseteq Ss$. Similarly, $Ss \subseteq St$.

(2) $\Rightarrow$ (1) Let $f : sM \to M$ be monic. Then $Ker(s) = Ker(fs)$. By (2), $Ss = S(fs)$, thus $fs = s's$ for some $s' \in S$. Hence $f = s'\cdot$, as required. □

Recall that a module M is called C_2 if every submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M.

Corollary 2.3. [3, Proposition 2.8] *Every PQP-injective module is C_2.*

Proof. Let M_R be PQP-injective with $S = End(M_R)$. If $Ker(s) = Ker(e)$, where $s \in S, e^2 = e \in S$, then by Theorem 2.2, we have $Ss = Se$. Hence M_R is C_2 by [15, Theorem 3]. □

Theorem 2.4. *Let M be a right R- module with $S = End(M_R)$.*

(1) *If S is right MP-injective, then M is PQP-injective.*
(2) *If M is PQP-injective and M generates $Ker(s)$ for each $s \in S$, then S is right MP-injective.*

Proof. (1) Let $Ker(s) = Ker(t)$. Then $\mathbf{r_S}(s) = \mathbf{r_S}(t)$. Since S is right MP-injective, $Ss = St$. Hence M is PQP-injective.

(2) Assume that $\mathbf{r_S}(s) = \mathbf{r_S}(t)$. Since M generates $Ker(s)$, $Ker(s) = \sum_{a\in A} a(M)$ for some subset A of S, and so $sa = 0$ for each $a \in A$. Hence $ta = 0$ for each $a \in A$ and then $tKer(s) = 0$. This shows that $Ker(s) \subseteq Ker(t)$. Similarly, $Ker(t) \subseteq Ker(s)$. Thus, $Ker(s) = Ker(t)$. Since M is PQP-injective, $Ss = St$. Therefore, S is right MP-injective. □

Our following results extend the results of [8, Theorem 2.8(1)-(3)].

Theorem 2.5. *Let M_R be PQP-injective with $S = End(M_R)$ and let $s, t \in S$.*

(1) *If tM embeds in sM, then St is an image of Ss.*
(2) *If $tM \cong sM$, then $St \cong Ss$.*

Proof. (1) If $\sigma : tM \to sM$ is a monomorphism, then $\sigma = u\cdot$ for some $u \in S$ by the PQP-injectivity of M. Let $as = 0$. Then $sM \subseteq Ker(a)$, and so $autM = a\sigma(tM) \subseteq a(sM) = 0$. Now we define $\varphi : Ss \to St$ by $as \mapsto aut$. Then φ is a left S-homomorphism. Since σ is monic, $Ker(ut) = Ker(t)$. This follows that $S(ut) = St$ as M is PQP-injective. Thus φ is epic.

(2) If $\sigma : tM \to sM$ is an isomorphism, then by (1), φ is epic. If $aut = 0, a \in S$, then $a\sigma(tM) = 0$, and hence $asM = 0$, i.e., $as = 0$. This shows that φ is an isomorphism. □

For a module M_R, a submodule X of M is called a kernel submodule if $X = ker(f)$ for some $f \in End(M_R)$.

Theorem 2.6. *Let M_R be PQP-injective with $S = End(M_R)$. If M_R satisfies ACC on kernel submodules, then S is right perfect.*

Proof. If $s_i \in S, i = 1, 2, \cdots$ and $Ss_1 \supseteq Ss_2 \supseteq \cdots$, then $Ker(s_1) \subseteq Ker(s_2) \subseteq \cdots$. By hypothesis, there exists a natural number n such that $Ker(s_n) = Ker(s_{n+1}) = \cdots$. By Theorem 2.2, $Ss_n = Ss_{n+1} = \cdots$, and hence S is right perfect. □

Corollary 2.7. *If M_R is QP-injective with $S = End(M_R)$, then S is right perfect if and only if M_R satisfies ACC on kernel submodules.*

Proof. Since QP-injective module is PQP-injective, by Theorem 2.6, we need only to prove the necessity. Suppose that $s_i \in S, i = 1, 2, \cdots$ such that $Ker(s_1) \subseteq Ker(s_2) \subseteq \cdots$, then $\mathbf{l}_S(Ker(s_1)) \supseteq \mathbf{l}_S(Ker(s_2)) \supseteq \cdots$. Then since M_R is QP-injective, by [8, Theorem 2.10], we have $Ss_1 \supseteq Ss_2 \supseteq \cdots$. Since S is right perfect, there exists a natural number n such that $Ss_n = Ss_{n+1} = \cdots$, so $Ker(s_n) = Ker(s_{n+1}) = \cdots$. This shows that M_R satisfies ACC on kernel submodules. □

Theorem 2.8. *Let $M_1 \oplus M_2$ be a PQP-injective module and $\sigma : M_1 \to M_2$ be a monomorphism. Then σ splits and M_1 is QP-injective.*

Proof. Clearly, the submodule $0 \oplus \sigma(M_1)$ of $M_1 \oplus M_2$ is a homomorphism image of $M_1 \oplus M_2$, and $\alpha : 0 \oplus \sigma(M_1) \to M_1 \oplus M_2$ given by $\alpha(0, \sigma(x)) = (x, 0), x \in M_1$, is a monomorphism. Since $M_1 \oplus M_2$ is PQP-injective, α can be extended to an endomorphism α^* of $M_1 \oplus M_2$. Let $\iota : M_2 \to M_1 \oplus M_2$ and $\pi : M_1 \oplus M_2 \to M_1$ be the natural injection and projection, respectively. Then $\tau = \pi\alpha^*\iota$ is such that $\tau\sigma = 1_{M_1}$. Hence σ splits. Let $M_2 = \sigma(M_1) \oplus N_1$. Then $M_1 \oplus M_2 = M_1 \oplus \sigma(M_1) \oplus N_1$, and so $N = M_1 \oplus \sigma(M_1)$ is PQP-injective by [3, Corollary 2.7]. Let K be any M_1-cyclic submodule of M_1 and $f : K \to M_1$ be an R-homomorphism. Then $K \oplus 0$ is an $M_1 \oplus M_2$-cyclic submodule of $M_1 \oplus M_2$, and the mapping $\beta : K \oplus 0 \to M_1 \oplus \sigma(M_1)$ given by $\beta(x, 0) = (x, \sigma f(x)), x \in K$, is a monomorphism. Hence it can be extended to an endomorphism γ of N. Let $q : M_1 \to N$ and $p : N \to \sigma(M_1)$ be natural injective and projection respectively. Then $\tau p\gamma q$ is an endomorphism of M_1 which extends f. Hence M_1 is QP-injective. □

Corollary 2.9. *If M is a right R-module such that $M \oplus M$ is PQP-injective, then M is QP-injective.*

Theorem 2.10. *The following statements are equivalent for a ring R:*

(1) *R is semisimple artinian.*
(2) *Every right R-module is QP-injective.*
(3) *Every right R-module is PQP-injective.*
(4) *For every right R-module M, $End(M_R)$ is a regular ring.*

Proof. (1) ⇒ (2) ⇒ (3) and (1) ⇒ (4) are trivial.

(4) ⇒ (3) Let M be any right R-module with $S = End(M_R)$. Then for any $s \in S$, by (3), there exists $t \in S$ such that $s = sts$. Write $e = st$. Then $e^2 = e$ and $sM = eM$, so sM is a direct summand of M. Thus M is PQP-injective.

(3) ⇒ (1) Let M be any right R-module. Since $M \oplus E(M)$ is PQP-injective, by Theorem 2.8, the inclusion map $M \to E(M)$ is split, and thus $M = E(M)$ is injective. Therefore R is semisimple artinian. □

Recall that a module M is called pseudo-injective [5] if every monomorphism from a submodule of M to M extends to an endomorphism of M. Clearly, pseudo-injective modules are PQP-injective. At the end of this section, we give an example of a module which is pseudo-injective (and hence PQP-injective) but not QP-injective.

Example 2.11. *Let Φ be an algebraically closed field and x, y be indeterminates. Let $B = \Phi(y)[x]$ be the hereditary simple principle ideal domain over the field of rational function $\Phi(y)$ where $xf - fx = df/dy, f \in \Phi(y)$. Let $M = B/x(x + y)(x + y - (1/y))B$. Then by [5, Example], M is pseudo-injective. Let $M_1 = xB/x(x + y)(x + y - (1/y))B$ and $M_2 = x(x + y)B/x(x + y)(x + y - (1/y))B$,* *it is easy to see that M_1 is an M-cyclic submodule of M, and by [5, Example], the natural homomorphism $\pi : M_1 \to (M_1/M_2) \cong M_2$ can not be extended to an endomorphism of M, so M is not QP-injective.*

3. $GPQP$-injective Modules

At first, we extend the concepts of PQP-injective modules and MGP-injective rings as following.

Definition 3.1. Let R be a ring. A right R-module M is called *generalized pseudo QP-injective* (or *$GPQP$-injective* for short) if for any $0 \neq s \in S$, there exists a positive integer n such that $s^n \neq 0$ and any right R-monomorphism from s^nM to M extends to an endomorphism of M.

It is obvious that PQP-injective modules are $GPQP$-injective, and that a ring R is right MGP-injective if and only if R_R is $GPQP$-injective.

Theorem 3.2. *The following conditions are equivalent for a module M_R with $S = End(M_R)$:*

(1) *M is $GPQP$-injective.*
(2) *For any $0 \neq s \in S$, there exists $n > 0$ such that $s^n \neq 0$ and $t \in Ss^n$ in case $Ker(s^n) = Ker(t)$.*

Proof. (1) $\Rightarrow$ (2). Let $0 \neq s \in S$. Since M is $GPQP$-injective, there exists a positive integer n, such that $s^n \neq 0$ and every monomorphism from s^nM to M extends to M. Suppose that $Ker(s^n) = Ker(t)$. Then $f : s^nM \to M; s^nm \mapsto tm$ is a monomorphism, which extends to an endomorphism g of M, so $tm = f(s^nm) = g(s^nm) = (gs^n)m$ for every $m \in M$. Therefore, $t = gs^n \in Ss^n$.

(2) $\Rightarrow$ (1). For any $0 \neq s \in S$. By (2), there exists $n > 0$ such that $s^n \neq 0$ and $t \in Ss^n$ for any $t \in S$ with $Ker(s^n) = Ker(t)$. Let $f : s^nM \to M$ be monic. Then $Ker(s^n) = Ker(fs^n)$, and so $fs^n = us^n$ for some $u \in S$. This follows that $f = u\cdot$, as required. □

Proposition 3.3. *Every direct summand of a $GPQP$-injective module is $GPQP$-injective.*

Proof. Let $M = M_1 \oplus M_2$ be $GPQP$-injective. Write $S = End(M)$ and $S_1 = End(M_1)$. Let e_i be the projection from M to M_i, ι_i be the inclusion from M_i to M, $i = 1, 2$. Then $M_1 = e_1M$. For any $0 \neq s_1 \in S_1$, let $s = s_1e_1$. Then $s \neq 0$. By the $GPQP$-injectivity of M, there exists a positive integer n such that $s^n \neq 0$ and every monomorphism from s^nM to M extends to M. Note that $s^n = \iota_1 s_1^n e_1$, we have $s_1^n \neq 0$. Now let $f : s_1^nM_1 \to M_1$ be any monomorphism. Then $g : s^nM \to M$ defined by $g(\iota_1 s_1^n e_1 x) = \iota_1 f s_1^n e_1 x$ is a monomorphism, so f

extends to an endomorphism h of M. Write $\varphi = e_1 h \iota_1$. Then $\varphi \in S_1$ and φ extends f. Hence, M_1 is $GPQP$-injective. □

Theorem 3.4. *Let M be a right R-module with $S = End(M_R)$.*

(1) *If S is right MGP-injective, then M is $GPQP$-injective.*
(2) *If M is $GPQP$-injective and M generates $Ker(s)$ for each $s \in S$, then S is right MGP-injective.*

Proof. (1) Let $0 \neq s \in S$. Since S is right MGP-injective, there exists a positive integer n such that $s^n \neq 0$ and $t \in Ss^n$ whenever $\mathbf{r}(s^n) = \mathbf{r}(t)$. Suppose $Ker(s^n) = Ker(t)$. Then $\mathbf{r}(s^n) = \mathbf{r}(t)$, so $t \in Ss^n$. Hence M is $GPQP$-injective by Theorem 3.2.

(2) Let $0 \neq s \in S$. Since M_R is $GPQP$-injective, there exists a positive integer n such that $s^n \neq 0$ and $t \in Ss^n$ whenever $Ker(s^n) = Ker(t)$. Assume that $\mathbf{r}(s^n) = \mathbf{r}(t)$. Since M generates $Ker(s^n)$, $Ker(s^n) = \sum_{a \in A} a(M)$ for some subset A of S, and so $s^n a = 0$ for each $a \in A$. Hence $ta = 0$ for each $a \in A$ and hence $tKer(s^n) = 0$. It follows that $Ker(s^n) \subseteq Ker(t)$. Similarly, $Ker(t) \subseteq Ker(s^n)$. Thus, $Ker(s^n) = Ker(t)$, and so $t \in Ss^n$. Therefore, S is right MGP-injective. □

Recall that a module M is said to be co-Hopfian (resp., Hopfian) if every monic (resp., surjective) endomorphism of M is an automorphism. A module M is said to be directly finite if M is not isomorphic to a proper summand of itself. A ring R is said to be directly finite (or Dedekind finite) if $ab = 1$ implies $ba = 1$. It is known that a module M is directly finite if and only if its endomorphism ring is directly finite [6, Proposition 1.25].

Theorem 3.5. *Let M_R be a $GPQP$-injective module. Then the following statements are equivalent:*

(1) *$S/J(S)$ is directly finite.*
(2) *M is co-Hopfian.*
(3) *S is directly finite.*
(4) *M is directly finite.*

Proof. (1) ⇒ (2) Let $s : M_R \to M_R$ be monic. Then $s \neq 0$ and $Ker(s) = 0$. Since M is $GPQP$-injective, there exists $n > 0$ such that $s^n \neq 0$ and every monomorphism $s^n M \to M$ extends to M. In particular, the monomorphism $g : s^n M \to M, s^n x \mapsto x$ extends to M. So, $1 = ts^n$ for some $t \in S$, and hence $\overline{1} = \overline{ts^n}$ in $\overline{S} := S/J(S)$. By (1), we have $\overline{s^n t}$=$\overline{1}$. Write $1 = s^n t + j$, where $j \in J(S)$. Then $s^n t(1 - j)^{-1} = 1$ and so s is surjective, showing that s is an isomorphism.

(2) ⇒ (3) Let $st = 1$, where $s, t \in S$. Then t is a monic endomorphism of M and, by (2), t is an isomorphism. So $ts = 1$.

(3) $\Rightarrow$ (1) Let $\bar{s}\bar{t}=\bar{1}$ in $S/J(S)$. Then $st = 1 + j$ for some $j \in J(S)$, and hence $1 = (1 + j)^{-1}st$. By (3), $1 = t(1 + j)^{-1}s$. It follows that $\bar{1}=\bar{t}\bar{s}$.

(3) $\Leftrightarrow$ (4) By [6, Proposition 1.25]. □

Our following result extend the result of [3, Proposition 2.24.]

Corollary 3.6. *Let M be a GPQP-injective Hopfian module. Then it is co-Hopfian.*

Proof. Since M is Hopfian, S is directly finite. And so M is co-Hopfian by Theorem 3.5 . □

Recall that a module M_R is called GC_2 [11] if every submodule of M that is isomorphic to M is itself a direct summand of M. A ring R is called right Min-C_2 [7] if every simple right ideal of R that is isomorphic to a direct summand of R is itself a direct summand of R. We call a module M_R Min-C_2 if every simple submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M. According to Wisbauer [10], a module M is called a self-generator if it generates all its submodules.

Theorem 3.7. *Let M_R be GPQP-injective with $S = End(M_R)$. Then*

(1) *M_R is GC_2.*
(2) *M_R is Min-C_2 .*
(3) *for any $s \in S$, if $s(M)$ is a simple submodule of M, then Ss is a minimal left ideal of S.*
Furthermore, if M_R is a self-generator, then
(4) *$J(S) = W(S)$, where $W(S) = \{s \in S \mid \mathrm{Ker}(s) \subseteq^{ess} M\}$.*

Proof. (1) Let $s \in S$ with $Ker(s) = 0$. Then $Ker(s^k) = 0$ for each positive integer k. Since M is $GPQP$-injective, there exists a positive integer n such that $s^n \neq 0$ and every monomorphism from s^nM to M extents to M. Define $f : s^nM \to M; s^nx \mapsto x$. Then f is a monomorphism, and hence it extends to an endomorphism g of M. Thus $x = f(s^nx) = g(s^nx)$ for each $x \in M$, and so $1 = gs^n$. It follows that $S = Ss$. Therefore, M is GC_2 by [15, Theorem 4].

(2) Let K be a simple submodule of M and $K \cong eM$ for some $e^2 = e \in S$. Then $K = seM$ for some $s \in S$ with $Ker(se) = Ker(e)$. Since M is $GPQP$-injective, there exists a positive integer n such that $(se)^n \neq 0$ and every monomorphism from $(se)^nM$ to M extends to an endomorphism of M. But K is simple, $K = (se)^nM$. Now let $f : K \to M; sem \mapsto em$. Then f is a monomorphism, hence it extends to an endomorphism t of M. Thus, $em = f(sem) = tsem$ for all $m \in M$ and then $e = tse$, which shows that $(set)^2 = set$. Note that $se = setse$, so $K = seM = setM$ is a direct summand.

(3) Suppose that $s(M)$ is simple. For any $0 \neq ts \in Ss$, since M_R is $GPQP$-injective, there exists a positive integer n such that $(ts)^n \neq 0$ and any R-monomorphism from $(ts)^n M$ to M extends to an endomorphism of M. Now we define $\varphi : s(M) \to (ts)^n M$ such that $\varphi(sm) = (ts)^n m$ for all $m \in M$. Then φ is an isomorphism. Let $i : s(M) \to M$ be the inclusion map and let $\psi = i\varphi^{-1}$. Then ψ is a monomorphism from $(ts)^n M$ to M with $\psi((ts)^n m) = sm$ for all $m \in M$, and so there exists $u \in S$ such that $u(ts)^n m = sm$ for all $m \in M$. It means that $u(ts)^n = s$ and then $Ss = S(ts)$. Therefore, Ss is minimal.

(4) Since M is GC_2, $W(S) \subseteq J(S)$ by [15, Corollary 6]. Conversely, let $s \in J(S)$, then we will show that $s \in W(S)$. If not, then there exists a nonzero submodule K of M such that $Ker(s) \cap K = 0$. Since M is a self-generator, $K = \sum_{a \in A} a(M)$ for some subset A of S. Take a $0 \neq t \in A$. Then $st \neq 0$. But since M is $GPQP$-injective, there exists a positive integer n such that $(st)^n \neq 0$ and $u \in S(st)^n$ for any $u \in S$ with $Ker(st)^n = Ker(u)$. Now let $u = t(st)^{n-1}$. Then $Ker(st)^n = Ker(u)$, and so $u = v(st)^n$ for some $v \in S$. Thus $(1 - vs)u = 0$, which implies that $u = 0$ because $1 - vs$ is invertible. Hence $(st)^n = su = 0$, a contradiction. □

Corollary 3.8. *Let R be a right MGP-injective ring. Then*

(1) *R is right GC_2.*
(2) *R is right Min-C_2 .*
(3) *for any $a \in R$, if aR is a minimal right ideal of R, then Ra is a minimal left ideal of R.*
(4) *$J(R) = Z_r$.*
(5) *$Soc(R_R) \subseteq Soc(_R R)$.*

Recall that a ring S is called left Kasch [9] if every simple left S-module embeds in $_S S$, equivalently, $\mathbf{r}_S(T) \neq 0$ for every maximal left ideal T of S. The concept of left Kasch rings was generalized to modules in [1]. Following [1], a module $_S M$ is said to be Kasch provided that every simple module in $\sigma[M]$ embeds in M, where $\sigma[M]$ is the category consisting of all M-subgenerated left S-modules. We call a module $_S M$ strongly Kasch [13] if every simple left S-module embeds in M.

Theorem 3.9. *For a nonzero left S-module $_S M$, the following are equivalent:*

(1) *$_S M$ is strongly Kasch.*
(2) *$Hom(N, M) \neq 0$ for every finitely generated nonzero left S-module N.*
(3) *$Hom(N, M) \neq 0$ for every cyclic nonzero left S-module N.*
(4) *$\mathbf{r}_M(I) \neq 0$ for every left ideal I of S that not equals to S.*
(5) *$\mathbf{r}_M(T) \neq 0$ for every maximal left ideal T of S.*
(6) *$\mathbf{l}_S\mathbf{r}_M(T) = T$ for every maximal left ideal T of S.*
(7) *For every maximal left ideal T of S, there exists a subset X of M such that $T = \mathbf{l}_S(X)$.*

(8) *$E(M)$ is a cogenerator.*
(9) *$Hom(N, E(M)) \neq 0$ for every nonzero left S-module N.*
(10) *$E(M)$ is strongly Kasch.*

Proof. (1) $\Rightarrow$ (2) Let N be any finitely generated nonzero left S-module. Then there exists a simple factor module N'. Since M is strongly Kasch, $Hom(N', M) \neq 0$, and so $Hom(N, M) \neq 0$.

(3) $\Rightarrow$ (4) Let I be any left ideal of S that not equals to S. By (3), $Hom(S/I, M) \neq 0$. Take a nonzero homomorphism φ from S/I to M, and let $m = \varphi(1 + I)$. Then $0 \neq m \in M$ and $Im = 0$. Hence, $\mathbf{r}_M(I) \neq 0$.

(5) $\Rightarrow$ (6) Let T be any maximal left ideal of S. Then by (5), $\mathbf{l}_S\mathbf{r}_M(T) \neq S$. Note that we always have $T \subseteq \mathbf{l}_S\mathbf{r}_M(T)$, so $\mathbf{l}_S\mathbf{r}_M(T) = T$ by the maximality of T.

(7) $\Rightarrow$ (1) Let T be a maximal left ideal of S. Then there exists $0 \neq x \in \mathbf{r}_M(T)$ by (7). Define $\varphi : S/T \to M$ by $s + T \mapsto sx$. Then φ is a left S-monomorphism.

(1) $\Rightarrow$ (8) Assume (1). Then every simple left S-module embeds in M and hence embeds in $E(M)$. By [2, Proposition 18.15], $E(M)$ is a cogenerator.

(2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5), (6) $\Rightarrow$ (7), and (8) $\Rightarrow$ (9) $\Rightarrow$ (10) $\Rightarrow$ (1) are clear. □

Lemma 3.10. *Let S be a left Kasch ring, and ${}_SM$ be a faithful module. Then ${}_SM$ is strongly Kasch.*

Proof. Let K be any maximal left ideal of S. Since S is left Kasch, $\mathbf{r}_S(K) \neq 0$. Choose $0 \neq s \in \mathbf{r}_S(K)$. Then $0 \neq sM \subseteq \mathbf{r}_M(K)$ for ${}_SM$ is faithful. So $\mathbf{r}_M(K) \neq 0$, and then ${}_SM$ is strongly Kasch. □

Proposition 3.11. *Let M be a right R-module with $S = End(M_R)$. If M_R is a self-generator, then S is left Kasch if and only if ${}_SM$ is strongly Kasch.*

Proof. By Lemma 3.10, we need only to prove the sufficiency. Assume that ${}_SM$ is strongly Kasch. Then for any maximal left ideal K of S, we have $\mathbf{r}_M(K) \neq 0$. Take $0 \neq m \in \mathbf{r}_M(K)$. Then $KmR = 0$. Since M_R is a self-generator, $mR = \sum_{t \in I} t(M)$ for some subset I of S. So $Kt = 0$ for some $0 \neq t \in I$ which implies that $\mathbf{r}_S(K) \neq 0$, and then S is left Kasch. □

Recall that a module M is called pseudo-injective [5] if every monomorphism from a submodule of M to M extends to an endomorphism of M.

Lemma 3.12. *Let M_R be a finitely cogenerated module with $Soc(M_R) \subseteq Soc({}_SM)$, where $S = End(M_R)$. Then the following statements are equivalent:*

(1) *${}_SM$ is strongly Kasch.*
(2) *M_R is C_2.*
(3) *M_R is GC_2.*
(4) *$W(S) \subseteq J(S)$.*

Proof. See the proof of [13, Theorem 6]. □

Following [14], we call a right R-module M minimal quasi-injective if every homomorphism from a simple submodule of M to M can be extended to an endomorphism of M.

Proposition 3.13. *Let M_R be a finitely cogenerated, minimal quasi-injective GC_2 module with $S = End(M_R)$. Then the following statements hold:*

(1) *${}_SM$ is strongly Kasch.*
(2) *M_R is C_2.*

Proof. Since M_R is minimal quasi-injective, by [14, Theorem 1.4], we have $Soc(M_R) \subseteq Soc({}_SM)$. Since M_R is GC_2 , by [15, Corollary 6], we have $W(S) \subseteq J(S)$. So the results follows immediately from Lemma 3.12. □

Corollary 3.14. *Let M_R be a finitely cogenerated pseudo-injective module. Then M_R is a C_2 module and ${}_SM$ is strongly Kasch.*

Proof. Since M_R is a pseudo-injective module, it is minimal quasi-injective and $GPQP$-injective, so M_R is a C_2 module and ${}_SM$ is strongly Kasch by Theorem 3.7(1) and Proposition 3.13. □

Corollary 3.15. *Let R be a right finitely cogenerated right MGP-injective ring. Then it is a left Kasch and right C_2 ring.*

Proof. Let R be a right MGP-injective ring. Then R_R is a minimal quasi-injective GC_2 module, so the result follows from Proposition 3.13. □

Recall that M is called weakly injective [4] if for every finitely generated submodule $N_R \subseteq E(M)$, we have $N \subseteq X_R \subseteq E(M)$ for some $X_R \cong M$.

Proposition 3.16. *Let M_R be a finitely generated module. Then M is injective if and only if it is a weakly injective GC_2 module.*

Proof. We need only to prove the sufficiency. Let $x \in E(M)$. Then there exists $X \subseteq E(M)$ such that $M + xR \subseteq X \cong M$, hence X is GC_2, and then M is a direct summand of X. But $M \subseteq^{ess} E(M)$, so $M \subseteq^{ess} X$. Thus $M = X$, and then $x \in M$. Therefore, $M = E(M)$ is injective. □

Corollary 3.17. *If M is a finitely generated module, then M is injective if and only if it is weakly injective and $GPQP$-injective.*

Let M be a right R-module with $S = End(M_R)$. Recall that a submodule K of M is called an annihilator submodule [13] if $K = \mathbf{r}_M(A)$ for some subset A of S.

Theorem 3.18. *Let M be a $PGQP$-injective module with $S = End(M_R)$.*

(1) *If M_R is finite dimensional, then S is semilocal.*

(2) *If M_R is a noetherian self-generator, then S is semiprimary.*

Proof. (1) By Theorem 3.7 (1), M_R satisfies GC_2. Since M_R has finite Goldie dimension, S is semilocal by [15, Corollary 12].

(2) If M_R is noetherian, then S is semilocal by (1). Since M has ACC on annihilator submodules, $W(S)$ is nilpotent by [13, Lemma 22]. But M_R is $GPQP$-injective and self-generated, $W(S) = J(S)$ by Theorem 3.7 (4). Hence S is semiprimary. □

Lemma 3.19. *Let M_R be a $GPQP$-injective module which is a self-generator with $S = End(M_R)$. If $s \notin W(S)$, then the inclusion $Ker(s) \subset Ker(s - sts)$ is strict for some $t \in S$.*

Proof. Since $s \notin W(S)$, $Ker(s)$ is not essential in M, so there exists a nonzero submodule K of M such that $Ker(s) \cap K = 0$. As M is a self-generator, there exists $0 \neq u \in S$ such that $uM \subseteq K$, then $su \neq 0$. By the $MGQP$-injectivity of M, there exists a positive integer n such that $(su)^n \neq 0$ and every monomorphism from $(su)^n M$ to M can be extended to M. In particular, the monomorphism $f : (su)^n M \to M$ given by $f((su)^n m) = u(su)^{n-1} m$ can be extended to an endomorphism of M. Thus, $u(su)^{n-1} = t(su)^n$ for some $t \in S$, so $u(su)^{n-1}M \subseteq Ker(1 - ts)$, and then $u(su)^{n-1}M \subseteq Ker(s - sts)$. But $u(su)^{n-1}M \nsubseteq Ker(s)$, hence the inclusion $Ker(s) \subset Ker(s - sts)$ is strict. □

Theorem 3.20. *Let M_R be a $GPQP$-injective module which is a self-generator with $S = End(M_R)$. Then the following statements are equivalent:*

(1) *S is right perfect.*

(2) *For any sequence $\{s_1, s_2, \cdots\} \subseteq S$, the chain $Ker(s_1) \subseteq Ker(s_2 s_1) \subseteq \cdots$ terminates.*

Proof. By Theorem 3.7(4), Lemma 3.19 and [13, Proposition 19]. □

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