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ON A SUBCLASS OF SEMISTAR GOING-DOWN DOMAINS

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Abstract. Let D be an integral domain and let $\star$ be a semistar operation on D. In this paper, we define the class of $\star$-quasi-going-up domains, a notion dual to the class of $\star$-going-down domains. It is shown that the class of $\star$-quasi-going-up domains is a proper subclass of $\star$-going-down domains and that every Prüfer-$\star$-multiplication domain is a $\star$-quasi-going-up domain. Next, we prove that the $\star$-Nagata ring $\mathrm{Na}(D,\star)$, is a quasi-going-up domain if and only if D is a $\widetilde{\star}$-quasi-going-up and a $\widetilde{\star}$-quasi-Prüfer domain. Several new characterizations are given for $\star$-going-down domains. We also define the universally $\star$-going-down domains, and then, give new characterizations of Prüfer-$\star$-multiplication domains.

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1. Introduction

Throughout this note, D denotes a (commutative integral) domain with identity and K denotes the quotient field of D. In [18], A. J. Hetzel introduced and studied a concept dual to going-down domains [4], [7], namely, quasi-going-up domains. By characterizing quasi-going-up domains as a particular type of going-down domains, he showed that, in addition to Prüfer domains, the pseudo-valuation domains of Hedstrom and Houston [17], are examples of quasi-going-up domains.

Let $\overline{\mathcal{F}}(D)$ denote the set of all nonzero D-submodules of K. As in [23], a *semistar operation on* D is a function $\star : \overline{\mathcal{F}}(D) \to \overline{\mathcal{F}}(D)$, $E \mapsto E^\star$, such that, for all $x \in K$, $x \neq 0$, and for all $E, F \in \overline{\mathcal{F}}(D)$, the following three properties hold: $(xE)^\star = xE^\star$; $E \subseteq F$ implies that $E^\star \subseteq F^\star$; $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$. Perhaps the most familiar (semi)star operations, d_D and v, are given by $E^{d_D} := E$ and $E^v := (E^{-1})^{-1}$ for all $E \in \overline{\mathcal{F}}(D)$. Introduced in part to generalize the notion of star operations (in the sense of [20, Section 32]), semistar operations have been shown in several articles to permit a finer study and new classification of domains in many respects.

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For instance, semistar-theoretic analogues of the classical notions of Noetherian and Prüfer domains have been introduced: see [11] and [10] for the basics on $\star$-Noetherian domains and Prüfer-$\star$-multiplication domains, respectively. In [8] and [9] the authors introduced and studied the concept of a $\star$-going-down (for short $\star$-GD) domain. They showed that every Prüfer-$\star$-multiplication domain and every domain of $\star$-dimension at most 1 is a $\star$-GD domain. The purpose of this paper is to define and to study a dual notion for $\star$-GD domain. So (as A. J. Hetzel wrote in the introduction of his paper [18]) it is natural to consider the semistar analogue of going-up property GU [21, Page 28] or the semistar analogue of lying-over property LO [21, Page 28]. But as Proposition 2.4 shows these notions are not suitable for our purpose. Nevertheless there are weaker notions of GU and LO properties. These are the quasi-lying-over property QLO [18, p. 419] and the quasi-going-up property QGU [18, p. 423] of D. E. Dobbs and M. Fontana. We next recall these notions. Let $D \subseteq T$ be an extension of domains. Recall that $D \subseteq T$ is said to satisfy *quasi-going-up property* (for short QGU) if, whenever $P_0 \subseteq P$ are prime ideals of D such that $PT \neq T$, and Q_0 is a prime ideal of T such that $Q_0 \cap D = P_0$, there exists a prime ideal Q of T such that $Q_0 \subseteq Q$ and $Q \cap D = P$. Also recall that $D \subseteq T$ satisfies *quasi-lying-over property* (for short QLO) if, whenever P is a prime ideal of D such that $PT \neq T$, there exists at least one prime ideal Q of T such that $Q \cap D = P$. In Section 2 we consider the QGU property and introduce the class of $\star$-quasi-going-up (for short $\star$-QGU) domains as a dual notion of $\star$-GD domains. In Theorem 2.7 we give several new characterization of $\star$-GD domains, and that the notion of $\star$-QLO domains are the same things as $\star$-GD domains, but (by Example 2.10) $\star$-QGU domains are a proper subclass of $\star$-GD domains. We also characterize $\star$-QGU domains and prove that $\mathrm{Na}(D, \star)$ is a quasi-going-up domain if and only if D is a $\widetilde{\star}$-QGU and a $\widetilde{\star}$-quasi-Prüfer domain. As an application, we give a new characterization of P$\star$MDs. In Section 3 we prove that a domain D is a $\widetilde{\star}$-QGU domain if and only if D/P is a $(\star/P)$-QGU domain for each $P \in \mathrm{QSpec}^{\widetilde{\star}}(D) \cup \{0\}$. In the last section we study the universal properties of $\star$-GD domains and $\star$-QGU domains, and again give new characterizations of P$\star$MDs.

In the reminder of the introduction, we collect some background about semistar operations. (For additional background, the reader is invited to consult papers such as [15] or [8].) As before, we suppose given a semistar operation $\star$ on a domain D. A nonzero ideal I of D is said to be a *quasi-$\star$-ideal of* D if $I^\star \cap D = I$; a *quasi-$\star$-prime (ideal of* D) if I is a prime quasi-$\star$-ideal of D; and a *quasi-$\star$-maximal (ideal of* D) if I is maximal in the set of all proper quasi-$\star$-ideals of D. Each quasi-$\star$-maximal ideal is a prime ideal. We denote by $\mathrm{QMax}^\star(D)$ (resp., $\mathrm{QSpec}^\star(D)$) the set of all quasi-$\star$-maximal ideals (resp., quasi-$\star$-prime ideals) of D. Associated to $\star$ is a semistar operation, $\star_f$, on D defined by $E^{\star_f} := \cup F^\star$, where the union is taken over all finitely generated $F \subseteq E$, for all $E \in \overline{\mathcal{F}}(D)$. Note that $\star_f$ is *of finite type*, in the sense that $(\star_f)_f = \star_f$. It was shown in [12, Lemma 4.20] that if $D^\star \neq K$, then each proper quasi-$\star_f$-ideal of D is contained in a quasi-$\star_f$-maximal ideal of D.

As above, $\star$ denotes a given semistar operation on a domain D. Let X be an indeterminate over K, the quotient field of D. For each $h \in K[X]$, let $c_D(h)$ denote the content of the polynomial h, i.e., the fractional ideal of D generated by the coefficients of h. If $N_\star := \{g \in D[X] \mid g \neq 0 \text{ and } c_D(g)^\star = D^\star\}$, then $N_\star = D[X] \setminus \bigcup\{P[X] | P \in \text{QMax}^{\star_f}(D)\}$ is a saturated multiplicatively closed subset of $D[X]$. The ring of fractions $\text{Na}(D, \star) := D[X]_{N_\star}$ is called the $\star$-*Nagata ring of* D *with respect to* $\star$. Note that $\text{Na}(D, d_D)$ coincides with the classical Nagata domain $D(X)$ (as in, for instance, [20, Section 33] and [15]).

Also associated to $\star$ is a semistar operation, $\widetilde{\star}$, on D, which is defined by $E^{\widetilde{\star}} := \cap\{ED_M | M \in \text{QMax}^{\star_f}(D)\}$ for all $E \in \overline{\mathcal{F}}(D)$. A semistar operation $\star$ is said to be *stable* if $(E \cap F)^\star = E^\star \cap F^\star$ for all E, $F \in \overline{\mathcal{F}}(D)$. For any semistar operation $\star$, it is known that $\widetilde{\star}$ is a stable semistar operation of finite type [12, Lemma 4.1(3), Corollary 3.9]; and moreover that $\text{Na}(D, \widetilde{\star}) = \text{Na}(D, \star_f) = \text{Na}(D, \star)$.

Let $\star$ be a semistar operation on a domain D. The $\star$-*Krull dimension of* D is defined as

$$\star\text{-}\dim(D) := \sup\left\{ n \;\middle|\; \begin{array}{c} (0) = P_0 \subset P_1 \subset \cdots \subset P_n \text{ where } P_i \text{ is a} \\ \text{quasi-}\star\text{-prime ideal of } D \text{ for } 1 \leq i \leq n \end{array} \right\}.$$

It is known (see [11, Lemma 2.11]) that

$$\widetilde{\star}\text{-}\dim(D) = \sup\{\text{ht}(P) \mid P \text{ is a quasi-}\widetilde{\star}\text{-prime ideal of } D\}.$$

As a final piece of background, we recall that an *overring of* D is any ring T such that $D \subseteq T \subseteq K$. We denote the integral closure of a domain D in its quotient field by D'.

2. $\star$-Quasi-going-up Domains

In [8] and [9] the authors defined and studied the notion of $\star$-going-down domain as semistar-theoretic version of the more known notion of going-down domains [4], [7]. Let $D \subseteq T$ be an extension of domains. Let $\star$ and $\star'$ be semistar operations on D and T, respectively. As in [8], we say that $D \subseteq T$ satisfies the $(\star, \star')$-GD if, whenever $P_0 \subset P$ are quasi-$\star$-prime ideals of D and Q is a quasi-$\star'$-prime ideal of T such that $Q \cap D = P$, there exists a quasi-$\star'$-prime ideal Q_0 of T such that $Q_0 \subseteq Q$ and $Q_0 \cap D = P_0$. A domain D together with a semistar operation $\star$ on D is called a $\star$-*going-down domain* (for short $\star$-GD domain) if, for each overring T of D the ring extension $D \subseteq T$ satisfies the $(\star, d_T)$-GD property. It is clear that a domain D is a d_D-GD domain if and only if D is a going-down domain (in the sense of [4]). The following proposition is a new characterization of $\star$-GD domains (c.f. [8, Theorem 3.13]).

Proposition 2.1. *Let D be a domain and $\star$ a semistar operation on D. Then the following conditions are equivalent:*

(1) *D is a $\star$-GD domain;*
(2) *$D \subseteq V$ satisfies $(\star, d_V)$-GD for each valuation overring V of D;*
(3) *$D \subseteq T$ satisfies $(\star, d_T)$-GD for each domain T containing D.*

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are trivial. For the implication (2) $\Rightarrow$ (3) we follow the method of [7, Theorem 1]. Let T be a domain containing D. Let $P \subset M$ be quasi-$\star$-prime ideals of D and $N \in \text{Spec}(T)$ such that $N \cap D = M$. Let W be a valuation overring of T centered on N. Let $V := W \cap K$. Thus V is a valuation overring of D. Since $D \subseteq V$ satisfies $(\star, d_V)$-GD and $V \subseteq W$ satisfies GD, then $D \subseteq W$ satisfies $(\star, d_W)$-GD. We thus obtain a prime ideal Q of W such that $Q \cap D = P$. Therefore $Q \cap T$ is contained in N and contracts to P in D. $\square$

We aim to define and to study a concept dual to the notion of "$\star$-GD domain". It is natural to consider the semistar versions of going-up property GU [21, Page 28] and of lying-over property LO [21, Page 28]. Now we have the following definition (See also [2, Lemma 2.15] for the notions of $\widetilde{\star}$-GU and $\widetilde{\star}$-LO).

Definition 2.2. Let $D \subseteq T$ be an extension of domains. Let $\star$ and $\star'$ be semistar operations on D and T, respectively. We say that $D \subseteq T$ *satisfies* $(\star, \star')$-GU if, whenever $P_0 \subseteq P$ are elements of $\text{QSpec}^{\star}(D) \cup \{0\}$, and Q_0 is an element of $\text{QSpec}^{\star'}(T) \cup \{0\}$ such that $Q_0 \cap D = P_0$, there exists an element Q of $\text{QSpec}^{\star'}(T) \cup \{0\}$ satisfying both $Q_0 \subseteq Q$ and $Q \cap D = P$. We say that $D \subseteq T$ *satisfies* $(\star, \star')$-LO if, whenever P is a quasi-$\star$-prime ideal of D, there exists at least one quasi-$\star'$-prime ideal Q of T such that $Q \cap D = P$.

Note that, in the notion of $(\star, \star')$-GU, we consider the prime ideals $\text{QSpec}^{\star}(D) \cup \{0\}$ of D and $\text{QSpec}^{\star'}(T) \cup \{0\}$ of T.

Lemma 2.3. *Let D be a domain and $\star$ a semistar operation on D. If $D \subseteq T$ satisfies the $(\star, d_T)$-GU property, then it is satisfies $(\star, d_T)$-LO property.*

Proof. Let $P \in \text{QSpec}^{\star}(D)$. Then $0 \subsetneq P$. Since $D \subseteq T$ satisfies the $(\star, d_T)$-GU property, there exists a prime ideal Q of T such that $Q \cap D = P$. Thus $D \subseteq T$ satisfies the $(\star, d_T)$-LO property. $\square$

Proposition 2.4. *Let D be a domain and $\star$ a semistar operation on D. Then the following conditions are equivalent:*

(1) *$D \subseteq T$ satisfies $(\star, d_T)$-GU for every overring T of D;*
(2) *$D \subseteq D[u]$ satisfies $(\star, d_{D[u]})$-GU for every $u \in K$;*

(3) *$D \subseteq T$ satisfies $(\star, d_T)$-LO for every overring T of D;*

(4) *$D \subseteq D[u]$ satisfies $(\star, d_{D[u]})$-LO for every $u \in K$;*

(5) $\mathrm{QSpec}^{\star}(D) = \emptyset$.

Moreover if $\star = \star_f$, the above statements are also equivalent to:

(6) *D is a field or $\star = e$ (i.e. $E^e = K$ for all $E \in \overline{\mathcal{F}}(D)$).*

Proof. (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4), (5) $\Rightarrow$ (1) and (5) $\Rightarrow$ (3) are trivial.

(1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) follow by Lemma 2.3.

(4) $\Rightarrow$ (5) Suppose the contrary. Thus there exists a quasi-$\star$-prime ideal P of D. Choose $0 \neq p \in P$. Then the pair $D \subseteq D[\frac{1}{p}]$ does not satisfy $(\star, d_{D[\frac{1}{p}]})$-LO (since $PD[\frac{1}{p}] = D[\frac{1}{p}]$) which is a contradiction.

Therefore the statements $(1)-(5)$ are equivalent. Now assume that $\star = \star_f$. The implication (6) $\Rightarrow$ (5) is obvious and for (5) $\Rightarrow$ (6) suppose that (5) holds and (6) fails. Hence $D \neq K$ and $\star \neq e$. Then $D^{\star} \neq K$; hence by [12, Lemma 4.20], we have $\mathrm{QSpec}^{\star_f}(D) \neq \emptyset$ which is a contradiction. □

As Proposition 2.4 makes clear, for a given domain D and a semistar operation $\star$ on D, the property of "$D \subseteq T$ satisfies $(\star, d_T)$-GU (resp. $(\star, d_T)$-LO) for every overring T of D" implies that $\mathrm{QSpec}^{\star}(D) = \emptyset$. In particular, if $\star = \star_f$, then we have D is a field or $\star = e$. So we dispense with the notion that a "$\star$-LO domain" or a "$\star$-GU domain" could be a desirable dual concept to a "$\star$-GD domain".

In [6], D. E. Dobbs and M. Fontana defined the notions of quasi-going-up and quasi-lying-over properties. We now define the semistar analogue of these notions and make use of these as dual notion for "$\star$-GD domain".

Definition 2.5. Let $D \subseteq T$ be an extension of domains. Let $\star$ and $\star'$ be semistar operations on D and T, respectively. We say that $D \subseteq T$ *satisfies* $(\star, \star')$-QGU if, whenever $P_0 \subseteq P$ are elements of $\mathrm{QSpec}^{\star}(D) \cup \{0\}$ such that $PT \neq T$, and Q_0 is an element of $\mathrm{QSpec}^{\star'}(T) \cup \{0\}$ such that $Q_0 \cap D = P_0$, there exists an element Q of $\mathrm{QSpec}^{\star'}(T) \cup \{0\}$ satisfying both $Q_0 \subseteq Q$ and $Q \cap D = P$. We say that $D \subseteq T$ *satisfies* $(\star, \star')$-QLO if, whenever P is a quasi-$\star$-prime ideal of D such that $PT \neq T$, there exists at least one quasi-$\star'$-prime ideal Q of T such that $Q \cap D = P$.

The above definition generalizes the classical QGU (resp. QLO) property in the following sense. If $D \subseteq T$ are domains, then $D \subseteq T$ satisfies (d_D, d_T)-QGU (resp. (d_D, d_T)-QLO) if and only if $D \subseteq T$ satisfies QGU (resp. QLO). It is clear that if $D \subseteq T$ satisfies the $(\star, \star')$-QGU (resp. $(\star, \star')$-QLO) property, then it is satisfies the $(\star, \star')$-GU (resp. $(\star, \star')$-LO) property. The proof of the following lemma is the same as Lemma 2.3.

Lemma 2.6. *Let D be a domain and $\star$ a semistar operation on D. If $D \subseteq T$ satisfies the $(\star, d_T)$-QGU property, then it is satisfies $(\star, d_T)$-QLO property.*

In the following theorem, we give several new characterizations of $\star$-GD domains in terms of $(\star, \star')$-QGU and $(\star, \star')$-QLO properties. The special case of $\star = d_D$ is contained in [18, Theorem 2.5].

Theorem 2.7. *Let D be a domain and $\star$ a semistar operation on D. Then the following conditions are equivalent:*

(1) *$D \subseteq T$ satisfies $(\star, d_T)$-QGU for every quasilocal domain T containing D;*
(2) *$D \subseteq T$ satisfies $(\star, d_T)$-QGU for every quasilocal overring T of D;*
(3) *$D \subseteq T$ satisfies $(\star, d_T)$-QGU for every quasilocal treed overring T of D;*
(4) *$D \subseteq V$ satisfies $(\star, d_V)$-QGU for every valuation overring V of D;*
(5) *$D \subseteq T$ satisfies $(\star, d_T)$-QLO for every domain T containing D;*
(6) *$D \subseteq T$ satisfies $(\star, d_T)$-QLO for every overring T of D;*
(7) *$D \subseteq T$ satisfies $(\star, d_T)$-QLO for every quasilocal domain T containing D;*
(8) *$D \subseteq T$ satisfies $(\star, d_T)$-QLO for every quasilocal treed domain T containing D;*
(9) *$D \subseteq V$ satisfies $(\star, d_V)$-QLO for every valuation overring V of D;*
(10) *$D \subseteq D[u]$ satisfies $(\star, d_{D[u]})$-QLO for every $u \in K$;*
(11) *D is a $\star$-GD domain.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial.

(1) $\Rightarrow$ (5) Suppose (1) and fix a domain T containing D. Let $P \in \text{QSpec}^\star(D)$ such that $PT \neq T$. Then there exists a valuation overring V of T such that $PV \neq V$. Thus, by considering $0 \subseteq P$ in $\text{QSpec}^\star(D) \cup \{0\}$, there exists $\mathfrak{Q} \in \text{Spec}(V)$ such that $\mathfrak{Q} \cap D = P$. Set $Q := \mathfrak{Q} \cap T$. Then we have $Q \in \text{Spec}(T)$ such that $Q \cap D = P$. Therefore, $D \subseteq T$ satisfies $(\star, d_T)$-QLO.

(5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (9) are trivial.

(9) $\Rightarrow$ (11) It is true by Proposition 2.1 and the same argument as in the proof of [18, Theorem 2.5, part (9) $\Rightarrow$ (11)].

(11) $\Rightarrow$ (1) It is true by Proposition 2.1 and the same argument as in the proof of [18, Theorem 2.5, part (11) $\Rightarrow$ (1)].

(4) $\Rightarrow$ (9) It is true by Lemma 2.6.

(11) $\Rightarrow$ (10) Follows by (11) $\Rightarrow$ (6) by the above work.

(10) $\Rightarrow$ (11) We modify the proof given in [18]. Suppose that the assertion fails. Then, by Proposition 2.1 there exists a valuation overring V of D, such that the extension $D \subseteq V$ does not satisfy the $(\star, d_V)$-GD property. Then there exist quasi-$\star$-prime ideals $P \subset P_1$ of D and a prime ideal Q_1 of V such that $Q_1 \cap D = P_1$ and no $Q \in \text{Spec}(V)$ satisfies both $Q \subset Q_1$ and $Q \cap D = P$. Therefore, $D \subseteq V$ does not satisfy GD. Applying [21, Exercise 37, page 44], we find $Q \in \text{Spec}(V)$ such that Q is minimal over PV, $Q \subseteq Q_1$, and $Q \cap D \neq P$. As V is quasilocal treed, Q is the radical of PV. Thus, choosing $r \in (Q \cap D) \backslash P$ leads to an equation $r^m = \sum p_i w_i$ for some $p_i \in P$, $w_i \in V$ and $m \geq 1$.

Now, the primes of V are linearly ordered by inclusion and, by a result of Prekowitz [22, Page 29], we may relabel the p_j such that, for each i, p_1 divides a power of p_i (with quotient in V). Raising the above equation to a suitably high power, say the t-th, gives an element w in V such that $r^{mt} = p_1 w$. Since $PD[w] \subseteq Q \cap D[w]$, we have $PD[w] \neq D[w]$. Thus, by hypothesis, there exists $Q_0 \in \mathrm{Spec}(D[w])$ such that $Q_0 \cap D = P$. Therefore, $p_1 w \in Q_0$, whence $r \in Q_0$, whence $r \in P$. But $r \notin P$ a contradiction. □

A. J. Hetzel, in [18], introduced and studied the notion of quasi-going-up domains (rings). A domain D is said to be a *quasi-going-up domain* (for short a QGU domain) if $D \subseteq T$ satisfies the quasi-going-up property for each overring T of D. As a semistar analogue we define:

Definition 2.8. Let D be a domain and $\star$ a semistar operation on D. Then D is said to be a $\star$-*quasi-going-up domain* (for short, a $\star$-QGU *domain*) if, for every overring T of D, the extension $D \subseteq T$ satisfies $(\star, d_T)$-QGU.

In the same way, one can define the $\star$-QLO domains. As the above theorem shows a $\star$-QLO domain is precisely the same as a $\star$-GD domain.

Let D be a domain and $\star$ a semistar operation on D. Recall from [8] that D is said to be a $\star$-*treed domain* if $\mathrm{QSpec}^{\star}(D)$, as a poset under inclusion, is a tree; i.e., if no quasi-$\star$-prime ideal of D contains incomparable quasi-$\star$-prime ideals of D. It is shown in [8, Theorem 3.6] that a $\star$-GD domain is a $\star$-treed domain.

Corollary 2.9. *If D is a $\star$-*QGU*-domain, then D is a $\star$-*GD*-domain, and hence is a $\star$-treed domain.*

Note that the converse of the above corollary does not longer true. In fact, in [18, Example 2.14], A. J. Hetzel gave an example of a d_D-GD domain which is not d_D-QGU domain. As another example we have:

Example 2.10. *Let V be a DVR, with maximal ideal N, dominating a two-dimensional local Noetherian domain D, with maximal ideal M* [3]*, and let $\star$ be a semistar operation on D defined by $E^{\star} = EV$ for each $E \in \overline{\mathcal{F}}(D)$. Then, clearly, $\star = \star_f$ and the only quasi-$\star$-prime ideal of D is M, since if P is a nonzero prime ideal of D, then $P^{\star} = PV = N^k$ for some integer $k \geq 1$. Thus, if we assume that P is quasi-$\star$-prime ideal of D, then we would have $P = PV \cap D = N^k \cap D \supseteq M^k$, which implies that $P = M$. Therefore, in this case, $\star$-$\dim(D) = 1$; so that D is a $\star$-*GD *domain by* [8, Proposition 3.2 (e)]. *Now since D is Noetherian and* $\dim(D) = 2$, *then D is not a* QGU *domain by* [18, Corollary 2.8]. *Hence by* [18, Corollary 2.8] *there exists an overring T of D such that there does not exist a prime ideal of T contracting to M at D. So the extension $D \subseteq T$ does not satisfy the* $(\star, d_T)$-QGU *(resp.* $(\star, d_T)$-QLO*) property considering the prime ideals* $0 \subseteq M$

in $\mathrm{QSpec}^{\star}(D) \cup \{0\}$ *(resp. the prime ideal* M *in* $\mathrm{QSpec}^{\star}(D)$*). Thus* D *is not a* $\star$-QGU *domain.*

Remark 2.11. *According to* [21, Page 45, Exercise 38], *if the extension* $D \subseteq T$ *satisfies the* GD *property, then it is satisfies the* QLO *property. Example 2.10 shows that in the semistar case this is not true, that is, there exists an extension* $D \subseteq T$ *of integral domains with a semistar operation* $\star$ *on* D *such that* $D \subseteq T$ *satisfies* $(\star, d_T)$-GD *property, but it does not satisfy the* $(\star, d_T)$-QLO *property.*

Although Example 2.10 shows that there is a (Noetherian) domain D and a (finite type) semistar operation $\star$ on D such that $\star$-$\dim(D) = 1$ and D is not a $\star$-QGU domain, we have the following result.

Corollary 2.12. *Let* D *be a* $\widetilde{\star}$*-Noetherian domain. Then* D *is a* $\widetilde{\star}$-QGU*-domain if and only if* $\widetilde{\star}$-$\dim(D) \leq 1$.

Proof. The "if" assertion is valid even without the "$\widetilde{\star}$-Noetherian" hypothesis using [18, Corollary 2.8]. For the "only if" part note that, by Corollary 2.9, D is a $\widetilde{\star}$-treed domain; so that $\widetilde{\star}$-$\dim(D) \leq 1$ by [9, Proposition 2.4]. □

The following proposition shows that the class of $\widetilde{\star}$-QGU domains is well behavior with respect to localization of quasi-$\widetilde{\star}$-prime ideals.

Proposition 2.13. *Let* D *be a domain and* $\star$ *a semistar operation on* D*. Then the following conditions are equivalent:*

(1) D *is a* $\widetilde{\star}$-QGU *domain;*
(2) D_P *is a quasi-going-up domain for all* $P \in \mathrm{QSpec}^{\widetilde{\star}}(D)$*;*
(3) D_M *is a quasi-going-up domain for all* $M \in \mathrm{QMax}^{\widetilde{\star}}(D)$.

Proof. (1) ⇒ (2) Suppose (1). Our task is to show that if $P \in \mathrm{QSpec}^{\widetilde{\star}}(D)$ and T is an overring of D_P, then $D_P \subseteq T$ satisfies QGU. Let $P_0D_P \subset P_1D_P$ be prime ideals of D_P such that $P_1T \neq T$ and Q_0 a prime ideal of T such that $Q_0 \cap D_P = P_0D_P$. We must find a prime ideal Q_1 of T such that $Q_0 \subseteq Q_1$ and $Q_1 \cap D_P = P_1D_P$. It is enough to find a prime ideal Q_1 of T such that $Q_0 \subseteq Q_1$ and $Q_1 \cap D = P_1$. By (1), the ring extension $D \subseteq T$ satisfies $(\widetilde{\star}, d_T)$-QGU. Therefore, it is enough to observe (via [12, Lemma 4.1 and Remark 4.5]) that P_0 and P_1 are elements of $\mathrm{QSpec}^{\widetilde{\star}}(D) \cup \{0\}$. (since they are contained in P).

(2) ⇒ (3) is trivial.

(3) ⇒ (1) Let T be an overring of D. Suppose that $P_0 \subset P$ are elements of $\mathrm{QSpec}^{\widetilde{\star}}(D) \cup \{0\}$ such that $PT \neq T$, and Q_0 is a prime ideal of T such that $Q_0 \cap D = P_0$. We must find a prime ideal Q of T such that both $Q_0 \subseteq Q$ and $Q \cap D = P$. Choose a quasi-$\widetilde{\star}$-maximal ideal M of D which contains P. It is enough to find a prime ideal Q of T such that $Q_0 \subseteq Q$ and $Q \cap D_M = PD_M$. This can be

done thanks to (3), as the ring extension $D_M \subseteq T_{D\setminus M}$ satisfies QGU, and noting that $PT_{D\setminus M} \neq T_{D\setminus M}$. $\square$

The special case of $\star = d_D$ is contained in [18, Theorem 2.10].

Theorem 2.14. *Let D be a domain and $\star$ a semistar operation on D. Then D is a $\widetilde{\star}$-QGU domain if and only if D is a $\widetilde{\star}$-GD domain and $(D_P)'$ is a valuation domain for each $P \in \mathrm{QSpec}^{\widetilde{\star}}(D) \setminus \mathrm{QMax}^{\widetilde{\star}}(D)$.*

Proof. ($\Rightarrow$) By Corollary 2.9 we have D is a $\widetilde{\star}$-GD domain. Assume that $P \in \mathrm{QSpec}^{\widetilde{\star}}(D) \setminus \mathrm{QMax}^{\widetilde{\star}}(D)$ and choose a quasi-$\widetilde{\star}$-maximal ideal M of D containing P. Then D_M is a QGU-domain by Proposition 2.13. Since PD_M is a non-maximal prime ideal of D_M, then $((D_M)_{PD_M})' = (D_P)'$ is a valuation domain by [18, Theorem 2.10].

($\Leftarrow$) Let $M \in \mathrm{QMax}^{\widetilde{\star}}(D)$. Then D_M is a going-down domain by [9, Proposition 2.5]. Now let $\mathcal{P} := PD_M$ be a non-maximal prime ideal of D_M for some $P \in \mathrm{Spec}(D)$. Then $P(\subsetneq M)$ is a quasi-$\widetilde{\star}$-prime ideal of D which is not quasi-$\widetilde{\star}$-maximal. Then $(D_P)' = ((D_M)_{\mathcal{P}})'$ is a valuation domain by the hypothesis. Consequently D_M is a QGU-domain by [18, Theorem 2.10]. Now the proof is complete by Proposition 2.13. $\square$

Let $\star$ be a semistar operation on an integral domain D. We now consider which overrings T of the domain D are sufficient to test the $(\star, d_T)$-QGU property in order to guarantee that D is a $\star$-QGU domain. Recall from [16] that an overring T of D is a $\star$-*overring* of D provided for each $F \in f(D)$ we have $F^\star \subseteq FT$ (or equivalently $F^\star T = FT$). It is observed [16, Lemma 4.2 (6)] that a Bézout overring B of D is an $\star$-overring of D if and only if $B = B^{\star_f}$. For the case where $\star = d_D$ of the following theorem see [18].

Theorem 2.15. *Let D be a domain and $\star$ a semistar operation on D. Then the following conditions are equivalent:*

(1) *D is a $\widetilde{\star}$-QGU domain;*
(2) *$D \subseteq D[u,v]$ satisfies $(\widetilde{\star}, d_{D[u,v]})$-QGU for each u and v in K;*
(3) *$D \subseteq B$ satisfies $(\widetilde{\star}, d_B)$-QGU for each Bézout $\widetilde{\star}$-overring B of D;*
(4) *$D \subseteq B$ satisfies $(\widetilde{\star}, d_B)$-QGU for each Bézout $\widetilde{\star}$-overring B of D with at most two maximal ideals.*

Proof. (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are trivial.

(4) $\Rightarrow$ (1) Assume (4). Let $M \in \mathrm{QMax}^{\widetilde{\star}}(D)$. Using Proposition 2.13, we only have to show that D_M is a QGU-domain. Let B be a Bézout overring of D_M with at most two maximal ideals. Note that $B \subseteq B^{\widetilde{\star}} = \bigcap\{BD_M | M \in \mathrm{QMax}^{\star_f}(D)\} \subseteq B$, hence $B = B^{\widetilde{\star}}$. Therefore, using [16, Lemma 4.2 (6)], B is a Bézout $\widetilde{\star}$-overring of D. Thus $D \subseteq B$ satisfies $(\widetilde{\star}, d_B)$-QGU by the hypothesis. Now let $P_0 D_M \subset P_1 D_M$

be prime ideals of D_M such that $P_1B \neq B$ and Q_0 be a prime ideal of B satisfying $Q_0 \cap D_M = P_0D_M$. Therefore $P_0 \subset P_1$ are elements of $\mathrm{QSpec}^{\widetilde{\star}}(D) \cup \{0\}$ such that $P_1B \neq B$ and Q_0 be a prime ideal of B satisfying $Q_0 \cap D = P_0$. Hence there exists a prime ideal Q_1 of B satisfying both $Q_0 \subseteq Q_1$ and $Q_1 \cap D = P_1$. Thus $Q_1 \cap D_M = P_1D_M$. Consequently D_M is a QGU-domain by [18, Theorem 4.1].

(2) $\Rightarrow$ (1) Let $P \in \mathrm{QSpec}^{\widetilde{\star}}(D)$. Again it is enough by Proposition 2.13 to show that D_P is a QGU-domain. To this end, fix $u, v \in K$. Let $P_0D_P \subset P_1D_P$ be prime ideals of D_P such that $P_1D_P[u, v] \neq D_P[u, v]$ and Q_0 be a prime ideal of $D_P[u, v]$ satisfying $Q_0 \cap D_P = P_0D_P$. Hence $P_1D[u, v] \neq D[u, v]$. Since $P_0 \subset P_1 \subseteq P$, P_0 and P_1 are elements of $\mathrm{QSpec}^{\widetilde{\star}}(D) \cup \{0\}$ (via [12, Lemma 4.1 and Remark 4.5]). By the hypothesis there exists a prime ideal Q_1 of $D[u, v]$ such that $Q_0 \subseteq Q_1$ and $Q_1 \cap D = P_1$. Hence $Q_1D_P[u, v] \cap D_P = P_1D_P$. It is shown that $D_P \subseteq D_P[u, v]$ satisfies the QGU property for all $u, v \in K$. Consequently, by [18, Theorem 4.4], D_P is a QGU-domain as desired. $\square$

Now we show that when the Nagata ring $D(X)$ is a QGU domain. Recall a domain D is called a *quasi-Prüfer domain* if it has Prüferian integral closure (c.f. [13, Section 6.5]).

Theorem 2.16. *Let D be a domain. Then $D(X)$ is a* QGU *domain if and only if D is a* QGU *and a quasi-Prüfer domain.*

Proof. ($\Rightarrow$) Since $D(X)$ is a QGU domain then by [18, Corollary 2.6], it is a GD domain. Therefore by [1, Corollary 2.12], D is a GD domain and also a quasi-Prüfer domain. Note that the contraction map $\mathrm{Spec}(D(X)) \to \mathrm{Spec}(D)$ is a bijection by [1, Theorem 2.7]. Now let $P \in \mathrm{Spec}(D)\setminus \mathrm{Max}(D)$. Hence we have $\mathcal{P} := PD(X) \in \mathrm{Spec}(D(X))\setminus \mathrm{Max}(D(X))$. Thus $(D(X)_{\mathcal{P}})' = (D_P(X))' = (D_P)'(X)$ is a valuation domain. Consequently $(D_P)'$ is a valuation domain. Therefore D is a QGU domain by [18, Theorem 2.10].

($\Leftarrow$) Since D is a GD and a quasi-Prüfer domain, then $D(X)$ is a GD domain by [1, Corollary 2.12]. Now let $\mathcal{P} \in \mathrm{Spec}(D(X))\setminus \mathrm{Max}(D(X))$. There exists a prime ideal $P \in \mathrm{Spec}(D)\setminus \mathrm{Max}(D)$ such that $\mathcal{P} = PD(X)$. Using [18, Theorem 2.10] we have $(D_P)'$ is a valuation domain. Hence $(D_P)'(X) = (D(X)_{\mathcal{P}})'$ is a valuation domain. Therefore $D(X)$ is a QGU domain by [18, Theorem 2.10]. $\square$

Recall from [2] that D is said to be a *$\star$-quasi-Prüfer domain*, in case, if Q is a prime ideal in $D[X]$, and $Q \subseteq P[X]$, for some $P \in \mathrm{QSpec}^{\star}(D)$, then $Q = (Q \cap D)[X]$. This notion is the semistar analogue of the classical notion of the quasi-Prüfer domains. By [2, Corollary 2.4], D is a $\star_f$-quasi-Prüfer domain if and only if D is a $\widetilde{\star}$-quasi-Prüfer domain.

Corollary 2.17. *Let D be a domain and $\star$ be a semistar operation on D. Then* $\mathrm{Na}(D,\star)$ *is a* QGU *domain if and only if D is a* $\widetilde{\star}$-QGU *and a $\widetilde{\star}$-quasi-Prüfer domain.*

Proof. ($\Rightarrow$) Since $\mathrm{Na}(D,\star)$ is a QGU-domain, it is a going-down domain by [18, Corollary 2.6]. Hence D is a $\widetilde{\star}$-GD and a $\widetilde{\star}$-quasi-Prüfer domain by [9, Theorem 2.6]. Now let $P \in \mathrm{QSpec}^{\widetilde{\star}}(D) \setminus \mathrm{QMax}^{\widetilde{\star}}(D)$. Then $P\,\mathrm{Na}(D,\star)$ is a non-maximal prime ideal of $\mathrm{Na}(D,\star)$, since the canonical contraction map $\mathrm{Spec}(\mathrm{Na}(D,\star)) \to \mathrm{QSpec}^{\widetilde{\star}}(D) \cup \{0\}$ is a bijection by [9, Lemma 2.1]. Therefore $(\mathrm{Na}(D,\star)_{P\,\mathrm{Na}(D,\star)})' = (D_P(X))' = (D_P)'(X)$ is a valuation domain by [18, Theorem 2.10]. Hence $(D_P)'$ is a valuation domain. Consequently D is a $\widetilde{\star}$-QGU domain by Theorem 2.14.

($\Leftarrow$) We will show that the localization of $\mathrm{Na}(D,\star)$ at any of its maximal ideals $\mathcal{M}$ is a QGU-domain. By [15, Proposition 3.1 (3)], $\mathcal{M} = M\,\mathrm{Na}(D,\star)$ for some $M \in \mathrm{QMax}^{\widetilde{\star}}(D)$. Since D is a $\widetilde{\star}$-quasi-Prüfer domain, [2, Theorem 2.16] gives that $\mathrm{Na}(D,\star)$ is a quasi-Prüfer domain. Hence its overring $\mathrm{Na}(D,\star)_{\mathcal{M}} = D[X]_{M[X]} = D_M(X)$ is also a quasi-Prüfer domain by [2, Theorem 1.1]. Another appeal to [2, Theorem 1.1] yields that D_M itself is a quasi-Prüfer domain. Thus, to show that $D_M(X)$ is a QGU-domain (and thereby finish the proof), it suffices, by Theorem 2.16, to prove that D_M is a QGU-domain. This, in turn, follows from Proposition 2.13. □

Let $\star$ be a semistar operation on a domain D. As in [14] and [10] (cf. also [19] for the case of a star operation), D is called a *Prüfer $\star$-multiplication domain* (for short, a $P \star MD$) if each finitely generated ideal of D is $\star_f$-invertible; i.e., if $(II^{-1})^{\star_f} = D^\star$ for all $I \in f(D)$. When $\star = v$, we recover the classical notion of a $PvMD$; when $\star = d$, the identity (semi)star operation, we recover the notion of a Prüfer domain.

Corollary 2.18. *Let D be a domain and $\star$ a semistar operation on D. Then the following conditions are equivalent:*

(1) *D is a P$\star$MD;*
(2) *$D^{\widetilde{\star}}$ is integrally closed and* $\mathrm{Na}(D,\star)$ *is a* QGU *domain;*
(3) $\mathrm{Na}(D,\star)$ *is an integrally closed* QGU *domain;*

Proof. (1) $\Rightarrow$ (3) If D is a P$\star$MD, then [14, Theorem 3.1] ensures that $\mathrm{Na}(D,\star)$ is a Prüfer domain and, hence, an integrally closed QGU-domain.

(3) $\Rightarrow$ (2) If $\mathrm{Na}(D,\star)$ is integrally closed, so is $\mathrm{Na}(D,\star) \cap K = D^{\widetilde{\star}}$.

(2) $\Rightarrow$ (1) Assume (2). Since $\mathrm{Na}(D,\star)$ is a QGU-domain, Corollary 2.17 yields that D is a $\widetilde{\star}$-quasi-Prüfer domain. As it is also the case that $D^{\widetilde{\star}}$ is integrally closed, [2, Corollary 2.17] gives (1). □

3. Semistar-QGU domains and factor domains

Let D be a domain with quotient field K, let X be an indeterminate over D, let $\star$ be a semistar operation on D, and let P be a quasi-$\star$-prime ideal of D. Set

$$\mathcal{S}_P^\star := (D/P)[X] \setminus \bigcup \{(Q/P)[X] \mid Q \in \mathrm{QSpec}^{\star_f}(D) \text{ and } P \subseteq Q\}.$$

Clearly, $\mathcal{S}_P^\star$ is a multiplicatively closed subset of $(D/P)[X]$.

For all $E \in \overline{\mathcal{F}}(D/P)$, set

$$E^{\circlearrowleft_{\mathcal{S}_P^\star}} := E(D/P)[X]_{\mathcal{S}_P^\star} \cap (D_P/PD_P).$$

It is proved in [9, Theorem 3.2] that the mapping $\star/P := \circlearrowleft_{\mathcal{S}_P^\star} : \overline{\mathcal{F}}(D/P) \to \overline{\mathcal{F}}(D/P)$, $E \mapsto E^{\circlearrowleft_{\mathcal{S}_P^\star}}$, is a stable semistar operation of finite type on D/P; i.e., $\widetilde{\star/P} = \star/P$, $\mathrm{QMax}^{\star/P}(D/P) = \{Q/P \in \mathrm{Spec}(D/P) \mid Q \in \mathrm{QMax}^{\star_f}(D) \text{ and } P \subseteq Q\}$, $\widetilde{\star}/P = \star_f/P = \star/P$ and $d_D/P = d_{D/P}$.

Remark 3.1. *Let D be a domain and $\star$ a semistar operation on D. If in the construction of $\star/P$, we consider $P = 0$, then one can easily seen that $\star/P = \star/0 = \widetilde{\star}$ (cf.,* [15, Proposition 3.4 (3)]*).*

The next result uses/generalizes the fact that the class of quasi-going-up domains is stable under the formation of factor domains [18, Proposition 3.12].

Theorem 3.2. *Let D be a domain and $\star$ a semistar operation on D. Then D is a $\widetilde{\star}$-QGU domain if and only if D/P is a $(\star/P)$-QGU domain for each $P \in \mathrm{QSpec}^{\widetilde{\star}}(D) \cup \{0\}$.*

Proof. ($\Rightarrow$) Let $P \in \mathrm{QSpec}^{\widetilde{\star}}(D)$. By [9, Theorem 3.2 (a)], $\star/P = \widetilde{\star/P}$. Hence, by Proposition 2.13, D/P is a $(\star/P)$-QGU domain if and only if $(D/P)_{\mathcal{M}}$ is a quasi-going-up domain for each $\mathcal{M} \in \mathrm{QMax}^{\star/P}(D/P)$, that is (by [9, Theorem 3.2 (b)]), if and only if D_M/PD_M is a quasi-going-up domain whenever P is a subset of $M \in \mathrm{QMax}^{\widetilde{\star}}(D)$. Thus, by Proposition 2.13, our task is to prove that this condition holds for each $P \in \mathrm{QSpec}^{\widetilde{\star}}(D)$ if D_M is a quasi-going-up domain for all $M \in \mathrm{QMax}^{\widetilde{\star}}(D)$. This, in turn, is immediate since any factor domain of a quasi-going-up domain must be a quasi-going-up domain by [18, Proposition 3.12].

($\Leftarrow$) It is enough to consider $P = 0$ and noting Remark 3.1. □

Example 3.3. *Consider the domain $D = \mathbb{Q}[X, Y]$ which is not a quasi-going-up domain. In other words, D is not a $\widetilde{d_D}$-QGU domain. However, each $P \in \mathrm{QSpec}^{d_D}(D) = \mathrm{Spec}(D) \setminus \{0\}$ has height 1 or 2; so that D/P has Krull dimension at most 1 and, hence, is necessarily a (d_D/P)-QGU domain.*

Remark 3.4. [9, Corollary 3.3] *By the same proof as Theorem 3.2, one has D is a $\widetilde{\star}$-GD domain if and only if D/P is a $(\star/P)$-GD domain for each $P \in \mathrm{QSpec}^{\widetilde{\star}}(D) \cup \{0\}$.*

4. Universal properties

In this section, we introduce and explore the concept of "universally $\star$-GD domain" analogous to "universally going-down domains" [5, Page 426]. Recall that a (unital) homomorphism $R \to T$ of (commutative) rings is said to be a *universally* GD-*homomorphism* in case $S \to S \otimes_R T$ is a GD-homomorphism for each commutative R-algebra S. A domain D is a *universally* GD-*domain* if, for each overring T of D, the inclusion map $D \subseteq T$ is a universally GD-homomorphism. The most natural examples of such are the Prüfer domains. It is easy to see (cf. [6, Corollary 2.3]) that, in testing for a universally GD-domain, one may restrict to $S = D[X_1, \cdots, X_n]$ and, then, test the induced inclusion maps between polynomial rings $D[X_1, \cdots, X_n] \subseteq T[X_1, \cdots, X_n]$ for GD.

Let D be an integral domain with quotient field K, let X, Y be two indeterminates over D and let $\star$ be a semistar operation on D. Set $D_1 := D[X]$, $K_1 := K(X)$ and take the following subset of $\mathrm{Spec}(D_1)$.

$$\Theta_1^\star := \{Q_1 \in \mathrm{Spec}(D_1) |\ Q_1 \cap D = (0) \text{ or } (Q_1 \cap D)^{\star_f} \subsetneq D^\star\}.$$

Set $\mathfrak{S}_1^\star := D_1[Y] \backslash (\bigcup \{Q_1[Y] | Q_1 \in \Theta_1^\star\})$ and

$$E^{\circlearrowleft_{\mathfrak{S}_1^\star}} := E[Y]_{\mathfrak{S}_1^\star} \cap K_1, \text{ for all } E \in \overline{\mathcal{F}}(D_1).$$

It is proved in [24, Theorem 2.1] (see also [25]) that the mapping $\star[X] := \circlearrowleft_{\mathfrak{S}_1^\star}$: $\overline{\mathcal{F}}(D_1) \to \overline{\mathcal{F}}(D_1)$, $E \mapsto E^{\star[X]}$ is a stable semistar operation of finite type on $D[X]$, i.e., $\widetilde{\star[X]} = \star[X]$. It is also proved that $\widetilde{\star}[X] = \star_f[X] = \star[X]$, $d_D[X] = d_{D[X]}$ and $\mathrm{QSpec}^{\star[X]}(D[X]) = \Theta_1^\star \backslash \{0\}$. If $X_1, \cdots, X_r$ are indeterminates over D, for $r \geq 2$, we let

$$\star[X_1, \cdots, X_r] := (\star[X_1, \cdots, X_{r-1}])[X_r],$$

where $\star[X_1, \cdots, X_{r-1}]$ is a stable semistar operation of finite type on $D[X_1, \cdots, X_{r-1}]$. For an integer r, put $\star[r]$ to denote $\star[X_1, \cdots, X_r]$ and $D[r]$ to denote $D[X_1, \cdots, X_r]$.

Definition 4.1. Let D be a domain and $\star$ a semistar operation on D. Then D is said to be a *universally $\star$-going-down domain* (for short, a *universally* $\star$-GD *domain*) if, for every overring T of D and every positive integer n, the extension $D[n] \subseteq T[n]$ satisfies $(\star[n], d_T[n])$-GD property.

Note that the notion of universally d_D-GD domain coincides with the "classical" notion of universally GD-domain.

Theorem 4.2. *Let D be a domain and $\star$ a semistar operation on D. Then the following conditions are equivalent:*

(1) *D is a universally $\widetilde{\star}$-GD domain;*
(2) *D_P is a universally going-down domain for all $P \in \mathrm{QSpec}^{\widetilde{\star}}(D)$;*
(3) *D_M is a universally going-down domain for all $M \in \mathrm{QMax}^{\widetilde{\star}}(D)$.*

Proof. (1) $\Rightarrow$ (2) Suppose (1). Let n be a positive integer and $P \in \text{QSpec}^{\widetilde{\star}}(D)$. Suppose that T is an overring of D_P. We have to show that $D_P[n] \subseteq T[n]$ satisfies the GD property by [6, Corollary 2.3]. Suppose that $L_0 \subseteq L$ are prime ideals of $D_P[n]$ and $Q \in \text{Spec}(T[n])$ such that $Q \cap D_P[n] = L$. But $D_P[n] = D[n]_{D \backslash P}$. So there exist $K_0, K \in \text{Spec}(D[n])$ such that $K_0 \subseteq K$, $K_0 D_P[n] = L_0$ and $K D_P[n] = L$. Since $K_0 \cap D \subseteq P$ and $K \cap D \subseteq P$, then we have $K_0, K \in \Theta_1^\star = \text{QSpec}^{\star[n]}(D[n]) \cup \{0\}$. On the other hand

$$Q \cap D[n] = (Q \cap D_P[n]) \cap D[n] = L \cap D[n] = K D_P[n] \cap D[n] = K.$$

Thus by the hypothesis there exists $Q_0 \in \text{Spec}(T[n])$ such that $Q_0 \subseteq Q$ and $Q_0 \cap D[n] = K_0$. So $Q_0 \cap D_P[n] = K_0 D_P[n] = L_0$. Therefore is D_P is a universally going-down domain.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Let T be an overring of D and n is a positive integer. We have to show that $D[n] \subseteq T[n]$ satisfies the $(\star[n], d_T[n])$-GD property. Suppose that $K_0 \subseteq K$ are elements of $\text{QSpec}^{\star[n]}(D[n])$ and $Q \in \text{Spec}(T[n])$ such that $Q \cap D[n] = K$. Set $P := K \cap D$. So that $P \in \text{QSpec}^{\widetilde{\star}}(D) \cup \{0\}$. Next choose a quasi-$\widetilde{\star}$-maximal ideal M of D containing P. Hence $D_M[n] \subseteq D_P[n]$, and since $D_M[n]$ is a universally GD-domain, then $D_P[n]$ is a universally GD-domain by [5, Proposition 2.2 (a)]. We have $K_0 D_P[n] \subseteq K D_P[n]$. Since $Q \cap D = (Q \cap D[n]) \cap D = K \cap D = P$, then $Q \cap (D \backslash P) = \emptyset$. Therefore $QT[n]_{D \backslash P} \in \text{Spec}(T[n]_{D \backslash P})$. Note that $QT[n]_{D \backslash P} \cap D_P[n] = K D_P[n]$. Then there exists $Q_0 T[n]_{D \backslash P} \in \text{Spec}(T[n]_{D \backslash P})$ contained $QT[n]_{D \backslash P}$ such that $Q_0 T[n]_{D \backslash P} \cap D_P[n] = K_0 D_P[n]$. Intersecting the preceding one with $D[n]$, we obtain that $Q_0 \cap D[n] = K_0$. $\square$

Corollary 4.3. *Let D be a domain and $\star$ a semistar operation on D. If D is a P$\star$MD, then D is a universally $\widetilde{\star}$-GD domain, hence, a universally $\star$-GD domain.*

Proof. Suppose that D is a P$\star$MD. Then for every $P \in \text{QSpec}^{\widetilde{\star}}(D)$, D_P is a valuation domain by [14, Theorem 3.1]. So D_P is a universally GD-domain by [5]. Thus D is a universally $\widetilde{\star}$-GD domain by Theorem 4.2. The last assertion is true since $\widetilde{\star} \leq \star$. $\square$

Corollary 4.4. *Let D be a domain and $\star$ a semistar operation on D. Then* $\text{Na}(D, \star)$ *is a universally* GD*-domain if and only if D is a universally $\widetilde{\star}$-*GD *domain.*

Proof. It is true by combining [1, Corollary 2.16 (a)] with Theorem 4.2. $\square$

In [18], A. J. Hetzel defined and studied the notion of universally quasi-going-up domains.

Definition 4.5. Let D be a domain and $\star$ a semistar operation on D. Then D is said to be a *universally $\star$-quasi-going-up domain* (for short, a *universally* $\star$-QGU *domain*) if, for every overring T of D and every positive integer n, the extension $D[n] \subseteq T[n]$ satisfies $(\star[n], d_T[n])$-QGU property.

Note that the notion of universally d_D-QGU domain coincides with the notion of universally quasi-going-up domain by [18, Proposition 2.6]. The proof of the following theorem is the same as Theorem 4.2; so we omit it.

Theorem 4.6. *Let D be a domain and $\star$ a semistar operation on D. Then the following conditions are equivalent:*

(1) *D is a universally $\widetilde{\star}$-QGU domain;*
(2) *D_P is a universally quasi-going-up domain for all $P \in \text{QSpec}^{\widetilde{\star}}(D)$;*
(3) *D_M is a universally quasi-going-up domain for all $M \in \text{QMax}^{\widetilde{\star}}(D)$.*

Corollary 4.7. *Let D be a domain and $\star$ a semistar operation on D. Assume that $D^{\widetilde{\star}}$ is integrally closed. Then the following conditions are equivalent:*

(1) *D is a P$\star$MD;*
(2) *D is a universally $\widetilde{\star}$-GD domain;*
(3) *D is a universally $\widetilde{\star}$-QGU domain.*

Proof. Let $P \in \text{QSpec}^{\widetilde{\star}}(D)$. Then D_P is an integrally closed domain by [9, Proposition 3.8]. Now D_P is a Prüfer (valuation) domain if and only if D is a universally GD-domain by [5, Corollary 2.3] (resp. is a universally quasi-going-up domain by [18, Corollary 6.5]). Therefore the result is clear by [14, Theorem 3.1] and Theorems 4.2 and 4.6. □

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