International Electronic Journal of Algebra
Volume 13 (2013) 23-39

ON ORTHOGONAL (σ,τ)-DERIVATIONS IN SEMIPRIME Γ-RINGS

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Received: 6 March 2012; Revised: 3 December 2012
Communicated by S. Tariq Rizvi

Abstract. Let M be a Γ-ring and σ,τ be endomorphisms of M. An additive mapping $d : M \longrightarrow M$ is called a *(σ,τ)-derivation* if $d(x\alpha y) = d(x)\alpha\sigma(y)+\tau(x)\alpha d(y)$ holds for all $x,y \in M$ and $\alpha \in \Gamma$. An additive mapping $F : M \longrightarrow M$ is called a *generalized (σ,τ)-derivation* if there exists a (σ,τ)-derivation $d : M \longrightarrow M$ such that $F(x\alpha y) = F(x)\alpha\sigma(y) + \tau(x)\alpha d(y)$ holds for all $x,y \in M$ and $\alpha \in \Gamma$. In this paper, some known results on orthogonal derivations and orthogonal generalized derivations of semiprime Γ-rings are extended to orthogonal (σ,τ)-derivations and orthogonal generalized (σ,τ)-derivations. Moreover, we present some examples which demonstrate that the restrictions imposed on the hypotheses of some of our results are not superfluous.

Mathematics Subject Classification (2010): 16W25, 16N60
Keywords: semiprime Γ-ring, derivation, orthogonal derivation, orthogonal (σ,τ)-derivation, orthogonal generalized derivation, orthogonal generalized (σ,τ)-derivation

1. Introduction

The study of Γ-ring goes back to Nobusawa [10] and further generalized by Barnes [6]. Following [6], a Γ-ring is a pair (M,Γ), where M and Γ are additive abelian groups for which there exists a map from $M\times\Gamma\times M \to M$ (the image of (a,γ,b) will be denoted by $a\gamma b$ for all $a,b\in M$ and $\gamma\in\Gamma$) satisfying (i) $(a+b)\alpha c = a\alpha c + b\alpha c$, (ii) $a(\alpha+\beta)b = a\alpha b + a\beta b$, (iii) $a\alpha(b+c) = a\alpha b + a\alpha c$ and (iv) $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a,b,c\in M$ and $\alpha,\beta\in\Gamma$. A Γ-ring M is said to be prime if $x\Gamma M\Gamma y = \{0\}$ implies $x=0$ or $y=0$ and M is said to be semiprime if $x\Gamma M\Gamma x = \{0\}$ implies $x=0$. M is said to be 2-torsionfree if $2x=0$ implies $x=0$ for all $x\in M$. For any $x,y\in M$ and $\alpha\in\Gamma$, the symbol $[x,y]_\alpha$ stands for the commutator $x\alpha y - y\alpha x$. If $x\alpha y\beta z = x\beta y\alpha z$ holds for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$,

This research is partially supported by a Major Research Project funded by U.G.C. (Grant No. 39-37/2010(SR)).

then commutator satisfies the following identities: $[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x\alpha[y, z]_\beta$ and $[x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y\alpha[x, z]_\beta$.

Following [9], an additive mapping $d : M \longrightarrow M$ is called a derivation if $d(x\alpha y) = d(x)\alpha y + x\alpha d(y)$ holds for all $x, y \in M$ and $\alpha \in \Gamma$. In [8], the notion of derivation has been extended to generalized derivation. An additive mapping $F : M \longrightarrow M$ is called a generalized derivation if there exists a derivation $d : M \longrightarrow M$ such that $F(x\alpha y) = F(x)\alpha y + x\alpha d(y)$ holds for all $x, y \in M$ and $\alpha \in \Gamma$. Two additive maps $d,\ g : M \longrightarrow M$ are called orthogonal if $d(x)\Gamma M\Gamma g(y) = \{0\} = g(y)\Gamma M\Gamma d(x)$ holds for all $x, y \in M$. In [3], Ashraf and Jamal introduced the notion of orthogonality for two derivations on Γ-rings, and established several necessary and sufficient conditions for derivations d and g to be orthogonal. Further in [4], they introduced orthogonal generalized derivation in Γ-rings and obtained some results concerning orthogonal generalized derivations. Some related papers on this subject can be found in [2], [7], [8], [11] and [12], where further references can be looked.

The objective of this paper is to extend the existing notions of derivations and generalized derivations in Γ-rings. Let σ and τ be endomorphisms of M. Motivated by the concepts of (σ, τ)-derivation and generalized (σ, τ)-derivation in rings (viz., [1] and [5]), the notions of (σ, τ)-derivation and generalized (σ, τ)-derivation in Γ-rings are defined as follows: an additive mapping $d : M \longrightarrow M$ is called a (σ, τ)-derivation if $d(x\alpha y) = d(x)\alpha\sigma(y) + \tau(x)\alpha d(y)$ holds for all $x, y \in M$ and $\alpha \in \Gamma$. Call an additive map F of M, a generalized (σ, τ)-derivation if there exists a (σ, τ)-derivation d of M such that $F(x\alpha y) = F(x)\alpha\sigma(y) + \tau(x)\alpha d(y)$ holds for all $x, y \in M$ and $\alpha \in \Gamma$. Clearly, the notion of generalized (σ, τ)-derivation includes those of (σ, τ)-derivation when $F = d$, of derivation when $F = d$, and $\sigma = \tau = I_M$, the identity map on M, and of generalized derivation, which is the case when $\sigma = \tau = I_M$. Note that, a generalized (I_M, I_M)-derivation is just a generalized derivation. It is clear that every generalized derivation is a generalized (σ, τ)-derivation with $\sigma = \tau = I_M$, the identity map on M, but the converse need not be true in general. The following example shows that the notion of a generalized (σ, τ)-derivation in fact generalizes that of a generalized derivation.

Example 1.1. *Let R be any ring, and let* $M = \left\{ \begin{pmatrix} a & x \\ b & y \\ c & z \end{pmatrix} \Big| \ a, b, c, x, y, z \in R \right\}$, $\Gamma = \left\{ \begin{pmatrix} l & 0 & m \\ 0 & 0 & 0 \end{pmatrix} \Big| \ l, m \in R \right\}$. *Then M is a Γ-ring. Further, the mappings*

$\sigma,\ \tau : M \longrightarrow M$ defined by

$$\sigma\begin{pmatrix} a & x \\ b & y \\ c & z \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & 0 \\ c & 0 \end{pmatrix}, \quad \tau\begin{pmatrix} a & x \\ b & y \\ c & z \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \\ c & 0 \end{pmatrix} \quad \text{for all} \begin{pmatrix} a & x \\ b & y \\ c & z \end{pmatrix} \in M$$

are endomorphisms of M. Next, define the map $d : M \longrightarrow M$ such that

$$d\begin{pmatrix} a & x \\ b & y \\ c & z \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for all} \begin{pmatrix} a & x \\ b & y \\ c & z \end{pmatrix} \in M.$$

Clearly, d is a (σ,τ)-derivation but not a derivation on M. Moreover, consider the map $F : M \longrightarrow M$ defined as

$$F\begin{pmatrix} a & x \\ b & y \\ c & z \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for all} \begin{pmatrix} a & x \\ b & y \\ c & z \end{pmatrix} \in M.$$

Then F is a generalized (σ,τ)-derivation on M induced by d. However, F is not a generalized derivation on M.

Throughout the present paper, M is always a 2-torsionfree semiprime Γ-ring while σ and τ are automorphisms of M. The generalized (σ,τ)-derivation F with an associated (σ,τ)-derivation d of M will be denoted by (F,d).

2. Orthogonal (σ,τ)-Derivations

We begin with the following lemmas which are essential in developing the proof of our theorems.

Lemma 2.1. ([11, Lemma 3]) *Let M be a Γ-ring and a,b be the elements of M. Then the following conditions are equivalent:*

(*i*) *$a\alpha M\beta b = \{0\}$ for all $\alpha,\beta \in \Gamma$.*
(*ii*) *$b\alpha M\beta a = \{0\}$ for all $\alpha,\beta \in \Gamma$.*
(*iii*) *$a\alpha M\beta b + b\alpha M\beta a = \{0\}$ for all $\alpha,\beta \in \Gamma$.*

If any one of the condition is fulfilled, then $a\gamma b = b\gamma a = 0$ for all $\gamma \in \Gamma$.

Lemma 2.2. ([3, Lemma 2.2]) *Let M be a semiprime Γ-ring. Suppose that additive mapping f and h of M into itself satisfy $f(x)\Gamma M\Gamma h(x) = \{0\}$ for all $x \in M$. Then $f(x)\Gamma M\Gamma h(y) = \{0\}$ for all $x,y \in M$.*

Lemma 2.3. *Let M be a Γ-ring, and d, g be (σ,τ)-derivations of M. Then d and g are orthogonal if and only if $d(x)\alpha g(y)+g(x)\alpha d(y)=0$ for all $x,y\in M$ and $\alpha\in\Gamma$.*

Proof. Assume that

$$d(x)\alpha g(y)+g(x)\alpha d(y)=0 \text{ for all } x,y\in M \text{ and } \alpha\in\Gamma. \tag{2.1}$$

Replacing y by $y\beta x$ in (2.1) and using it, we obtain

$$d(x)\alpha\tau(y)\beta g(x)+g(x)\alpha\tau(y)\beta d(x)=0 \text{ for all } x,y\in M \text{ and } \alpha,\beta\in\Gamma. \tag{2.2}$$

Since τ is an automorphism of M and using Lemma 2.1, we get $d(x)\alpha y_1\beta g(x)=0$ for all $x,y_1\in M$ and $\alpha,\beta\in\Gamma$. Application of Lemma 2.2 yields that $d(x)\alpha y_1\beta g(z)=0$ for all $x,y_1,z\in M$ and $\alpha,\beta\in\Gamma$, and hence in view of Lemma 2.1, d and g are orthogonal.

Conversely, if d and g are orthogonal, then $d(x)\alpha z\beta g(y)=0=g(y)\alpha z\beta d(x)$ for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$. Therefore by Lemma 2.1, $d(x)\alpha g(y)=0=g(x)\alpha d(y)$ for all $x,y\in M$ and $\alpha\in\Gamma$. This implies that $d(x)\alpha g(y)+g(x)\alpha d(y)=0$ for all $x,y\in M$ and $\alpha\in\Gamma$. This completes the proof. □

Theorem 2.4. *Let M be a Γ-ring such that $x\alpha y\beta z=x\beta y\alpha z$ for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$. Further, suppose d and g are (σ,τ)-derivations of M such that $d\sigma=\sigma d$, $d\tau=\tau d$. Then d and g are orthogonal if and only if $d(x)\alpha g(x)=0$ for all $x\in M$ and $\alpha\in\Gamma$.*

Proof. Suppose that $d(x)\alpha g(x)=0$ for all $x\in M$ and $\alpha\in\Gamma$. Linearizing this relation, we get

$$d(x)\alpha g(y)+d(y)\alpha g(x)=0 \text{ for all } x,y\in M \text{ and } \alpha\in\Gamma. \tag{2.3}$$

Replacing y by $y\beta z$ in (2.3), we get

$$\begin{aligned} 0 &= d(x)\alpha g(y\beta z)+d(y\beta z)\alpha g(x) \\ &= d(x)\alpha g(y)\beta\sigma(z)+d(x)\alpha\tau(y)\beta g(z)+d(y)\beta\sigma(z)\alpha g(x)+\tau(y)\beta d(z)\alpha g(x). \end{aligned}$$

In view of (2.3), we have $d(x)\alpha g(y)=-d(y)\alpha g(x)$ and $d(z)\alpha g(x)=-d(x)\alpha g(z)$, and hence the above expression reduces to

$$d(y)\beta[\sigma(z),g(x)]_\alpha=[\tau(y),d(x)]_\alpha\beta g(z) \text{ for all } x,y,z\in M \text{ and } \alpha,\beta\in\Gamma. \tag{2.4}$$

Replacing y by $\tau^{-1}(d(x))$ in (2.4), we obtain

$$d(\tau^{-1}(d(x)))\beta[\sigma(z),g(x)]_\alpha=0 \text{ for all } x,z\in M \text{ and } \alpha,\beta\in\Gamma.$$

This implies that

$$\tau^{-1}(d^2(x))\beta[z_1,g(x)]_\alpha=0 \text{ for all } x,z_1\in M \text{ and } \alpha,\beta\in\Gamma. \tag{2.5}$$

Replacing z_1 by $z\gamma s$ in (2.5) and using Lemma 2.2 and relation (2.5), we obtain

$$\tau^{-1}(d^2(x))\beta z\gamma[s,g(y)]_\alpha = 0 \text{ for all } s,x,y,z \in M \text{ and } \alpha,\beta,\gamma \in \Gamma. \tag{2.6}$$

Replacing x by $x\delta u$ in (2.6) and using it, we get

$$2(d(x)\delta d(\tau^{-1}(\sigma(u)))\beta z\gamma[s,g(y)]_\alpha) = 0 \text{ for all } s,u,x,y,z \in M \text{ and } \alpha,\beta,\gamma,\delta \in \Gamma.$$

Putting $u = \sigma^{-1}(\tau(u))$ in above and using the fact that M is 2-torsionfree, we find that

$$d(x)\delta d(u)\beta z\gamma[s,g(y)]_\alpha = 0 \text{ for all } s,u,x,y,z \in M \text{ and } \alpha,\beta,\gamma,\delta \in \Gamma. \tag{2.7}$$

Substituting $x\alpha_1 t$ for x in (2.7) and using it, we find that

$$d(x)\alpha_1\sigma(t)\delta d(u)\beta z\gamma[s,g(y)]_\alpha = 0 \text{ for all } s,t,u,x,y,z \in M \text{ and } \alpha_1,\alpha,\beta,\gamma,\delta \in \Gamma.$$

The above expression yields that

$$d(x)\beta z\gamma[s,g(y)]_\alpha\alpha_1 M\delta d(x)\beta z\gamma[s,g(y)]_\alpha = \{0\} \text{ for all } s,x,y,z \in M \text{ and } \alpha_1,\alpha,\beta,\gamma,\delta \in \Gamma.$$

Semiprimeness of M implies that

$$d(x)\beta z\gamma[s,g(y)]_\alpha = 0 \text{ for all } s,x,y,z \in M \text{ and } \alpha,\beta,\gamma \in \Gamma,$$

and hence

$$d(x)\alpha z\gamma[d(x),g(y)]_\alpha = 0 \text{ for all } x,y,z \in M \text{ and } \alpha,\gamma \in \Gamma. \tag{2.8}$$

Replacing z by $g(y)\beta z$, we get

$$d(x)\alpha g(y)\beta z\gamma[d(x),g(y)]_\alpha = 0 \text{ for all } x,y,z \in M \text{ and } \alpha,\beta,\gamma \in \Gamma. \tag{2.9}$$

Also, from (2.8), we have

$$g(y)\alpha d(x)\beta z\gamma[d(x),g(y)]_\alpha = 0 \text{ for all } x,y,z \in M \text{ and } \alpha,\beta,\gamma \in \Gamma. \tag{2.10}$$

Subtracting (2.10) from (2.9), we get

$$[d(x),g(y)]_\alpha\beta M\gamma[d(x),g(y)]_\alpha = \{0\} \text{ for all } x,y \in M \text{ and } \alpha,\beta,\gamma \in \Gamma.$$

Semiprimeness of M yields that $[d(x),g(y)]_\alpha = 0$ for all $x,y \in M$ and $\alpha \in \Gamma$. That is, $d(x)\alpha g(y) = g(y)\alpha d(x)$ for all $x,y \in M$ and $\alpha \in \Gamma$. Thus, (2.3) can be written as $d(x)\alpha g(y) + g(x)\alpha d(y) = 0$ for all $x,y \in M$ and $\alpha \in \Gamma$. By Lemma 2.3, d and g are orthogonal.

Conversely, suppose that d and g are orthogonal. Then $d(x)\beta M\gamma g(x) = \{0\}$ for all $x \in M$ and $\beta,\gamma \in \Gamma$. Therefore, $d(x)\alpha g(x) = 0$ for all $x \in M$ and $\alpha \in \Gamma$ by Lemma 2.1. □

Theorem 2.5. *Let M be a Γ-ring. Suppose d and g are (σ,τ)-derivations of M such that $d\sigma=\sigma d$, $g\sigma=\sigma g$, $d\tau=\tau d$, $g\tau=\tau g$. Then the following conditions are equivalent:*

(i) *d and g are orthogonal.*

(ii) $dg=0$.

(iii) $gd=0$.

(iv) $dg+gd=0$.

(v) *dg is a (σ^2,τ^2)-derivation of M.*

Proof. $(ii)\Leftrightarrow(i)$. Assume $dg=0$. Then for any $x,y\in M$ and $\gamma\in\Gamma$, our hypotheses yields that

$$\begin{aligned}0 &= dg(x\alpha y)\\ &= dg(x)\alpha\sigma^2(y)+\tau(g(x))\alpha d(\sigma(y))+d(\tau(x))\alpha\sigma(g(y))+\tau^2(x)\alpha dg(y)\\ &= \tau(g(x))\alpha d(\sigma(y))+d(\tau(x))\alpha\sigma(g(y)).\end{aligned}$$

Since σ,τ are automorphisms of M and using the fact that $g\tau=\tau g$, $g\sigma=\sigma g$, we find that

$$g(x_1)\alpha d(y_1)+d(x_1)\alpha g(y_1)=0 \text{ for all } x_1,y_1\in M \text{ and } \alpha\in\Gamma.$$

Hence d and g are orthogonal in view of Lemma 2.3.

Conversely, suppose that d and g are orthogonal. Then $d(x)\alpha y\beta g(z)=0$ for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$. Thus for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$, we have

$$\begin{aligned}0 &= d(d(x)\alpha y\beta g(z))\\ &= d^2(x)\alpha\sigma(y)\beta\sigma(g(z))+\tau(d(x))\alpha d(y)\beta\sigma(g(z))+\tau(d(x))\alpha\tau(y)\beta dg(z)\\ &= \tau(d(x))\alpha\tau(y)\beta dg(z).\end{aligned}$$

Noting that $d\tau=\tau d$ and τ is an automorphism of M, we obtain

$$d(x_1)\alpha y_1\beta dg(z_1)=0 \text{ for all } x_1,y_1,z_1\in M \text{ and } \alpha,\beta\in\Gamma.$$

Replacing x_1 by $g(z_1)$ in the last expression, we get $dg(z_1)\alpha y_1\beta dg(z_1)=0$ for all $y_1,z_1\in M$ and $\alpha,\beta\in\Gamma$. Semiprimeness of M yields that $dg(z_1)=0$ for all $z_1\in M$. Thus, we conclude that $dg=0$.

$(iii)\Leftrightarrow(i)$. Proof is similar as $(ii)\Leftrightarrow(i)$.

$(iv)\Leftrightarrow(i)$. Suppose $dg+gd=0$. Then for all $x,y\in M$ and $\alpha\in\Gamma$, we have

$$\begin{aligned}0 &= (dg+gd)(x\alpha y)\\ &= (dg+gd)(x)\alpha\sigma^2(y)+2(g(\tau(x))\alpha d(\sigma(y))+d(\tau(x))\alpha g(\sigma(y)))+\tau^2(x)\alpha(dg+gd)(y)\\ &= 2(g(\tau(x))\alpha d(\sigma(y))+d(\tau(x))\alpha g(\sigma(y))).\end{aligned}$$

Since M is 2-torsionfree and σ, τ are automorphisms of M, we conclude that

$$g(x_1)\alpha d(y_1) + d(x_1)\alpha g(y_1) = 0 \text{ for all } x_1, y_1 \in M \text{ and } \alpha \in \Gamma.$$

Hence d and g are orthogonal by Lemma 2.3.

Conversely, suppose that d and g are orthogonal. Then $dg = 0$ and $gd = 0$ by part (ii) and (iii). Hence, $dg + gd = 0$.

$(v) \Leftrightarrow (i)$. Suppose dg is a (σ^2, τ^2)-derivation on M. That is,

$$dg(x\alpha y) = dg(x)\alpha\sigma^2(y) + \tau^2(x)\alpha dg(y) \text{ for all } x, y \in M \text{ and } \alpha \in \Gamma. \tag{2.11}$$

Also, we have

$$dg(x\alpha y) = dg(x)\alpha\sigma^2(y) + \tau(g(x))\alpha d(\sigma(y)) + d(\tau(x))\alpha\sigma(g(y)) + \tau^2(x)\alpha dg(y). \tag{2.12}$$

Comparing (2.11) and (2.12), we get

$$\tau(g(x))\alpha d(\sigma(y)) + d(\tau(x))\alpha\sigma(g(y)) = 0 \text{ for all } x, y \in M \text{ and } \alpha \in \Gamma.$$

Since $g\tau = \tau g,\ g\sigma = \sigma g$ and σ, τ are automorphisms of M, so we have

$$g(x_1)\alpha d(y_1) + d(x_1)\alpha g(y_1) = 0 \text{ for all } x_1, y_1 \in M \text{ and } \alpha \in \Gamma.$$

In view of Lemma 2.3, we conclude that d and g are orthogonal.

Conversely, suppose that d and g are orthogonal. By (ii), we obtain $dg = 0$. Thus, dg is a (σ^2, τ^2)-derivation on M. □

The following example shows that the hypothesis of semiprimeness in Theorem 2.5 is essential.

Example 2.6. *Let R be any 2-torsionfree ring and let* $M = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \,\middle|\, a, b, c \in R \right\}$, $\Gamma = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \,\middle|\, x, y \in R \right\}$. *Then M is a 2-torsionfree Γ-ring. It can be easily seen that M is not semiprime. Take $\sigma = \tau = I_M$, where I_M is the identity map on M. Define the maps $d,\ g : M \longrightarrow M$ such that*

$$d\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix},\ g\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix} \quad \textit{for all} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in M.$$

Then it is straightforward to check that d and g are (σ, τ)-derivations on M. Also, d and g are orthogonal, and dg is a (σ^2, τ^2)-derivation on M. However, $dg \neq 0$, $gd \neq 0$ and $dg + gd \neq 0$.

3. Orthogonal Generalized (σ,τ)-Derivations

Two generalized derivations (F,d) and (G,g) of M are called orthogonal if $F(x)\Gamma M\Gamma G(y)=\{0\}=G(y)\Gamma M\Gamma F(x)$ holds for all $x,y\in M$. Recently, Ashraf and Jamal in [4] obtained some necessary and sufficient conditions for two generalized derivations to be orthogonal. In the present section, our objective is to generalizes their results in more general setting for semiprime Γ-rings. We begin with the following lemma.

Lemma 3.1. *Suppose that two generalized (σ,τ)-derivations (F,d) and (G,g) of M are orthogonal. Then following relations hold:*

- (i) *$F(x)\alpha G(y)=G(x)\alpha F(y)=0$, and hence $F(x)\alpha G(y)+G(x)\alpha F(y)=0$ for all $x,y\in M$ and $\alpha\in\Gamma$.*
- (ii) *d and G are orthogonal and $d(x)\alpha G(y)=G(y)\alpha d(x)=0$ for all $x,y\in M$ and $\alpha\in\Gamma$.*
- (iii) *g and F are orthogonal and $g(x)\alpha F(y)=F(y)\alpha g(x)=0$ for all $x,y\in M$ and $\alpha\in\Gamma$.*
- (iv) *d and g are orthogonal.*
- (v) *If $F\sigma=\sigma F,\ F\tau=\tau F,\ G\sigma=\sigma G,\ G\tau=\tau G$ and $d\sigma=\sigma d,\ d\tau=\tau d,\ g\sigma=\sigma g,\ g\tau=\tau g$, then $dG=Gd=0,\ gF=Fg=0$ and $FG=GF=0$.*

Proof. (i). By the hypothesis, we have $F(x)\alpha z\beta G(y)=0$ for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$. Application of Lemma 2.1 yields that $F(x)\gamma G(y)=0=G(y)\gamma F(x)$. Therefore, $F(x)\gamma G(y)+G(y)\gamma F(x)=0$ for all $x,y\in M$ and $\gamma\in\Gamma$.
(ii). By (i), we have $F(x)\alpha G(y)=0$ and $F(x)\beta z\gamma G(y)=0$ for all $x,y,z\in M$ and $\alpha,\beta,\gamma\in\Gamma$. Hence

$$\begin{aligned}0&=F(z\beta x)\alpha G(y)\\&=F(z)\beta\sigma(x)\alpha G(y)+\tau(z)\beta d(x)\alpha G(y)\\&=\tau(z)\beta d(x)\alpha G(y).\end{aligned}$$

Since τ is an automorphism of M, the last expression yields that

$$d(x)\alpha G(y)\gamma M\beta d(x)\alpha G(y)=\{0\}\text{ for all }x,y\in M\text{ and }\alpha,\beta,\gamma\in\Gamma.$$

Thus, the semiprimeness of M forces that

$$d(x)\alpha G(y)=0\text{ for all }x,y\in M\text{ and }\alpha\in\Gamma.\tag{3.1}$$

Replacing x by $x\beta s$ in (3.1), we get

$$\begin{aligned}0&=d(x\beta s)\alpha G(y)\\&=d(x)\beta\sigma(s)\alpha G(y)+\tau(x)\beta d(s)\alpha G(y).\end{aligned}$$

Using (3.1) and the fact that σ is an automorphism of M, we obtain

$$d(x)\Gamma M\Gamma G(y)=\{0\} \text{ for all } x,y\in M.$$

Application of Lemma 2.1 yields that d and G are orthogonal, and hence $d(x)\alpha G(y)=G(y)\alpha d(x)=0$ for all $x,y\in M$, $\alpha\in\Gamma$.
(iii). Using similar approach as we have used in (ii).
(iv). By the assumption, we have $F(x)\alpha G(y)=0$ for all $x,y\in M$ and $\alpha\in\Gamma$. This implies that

$$\begin{aligned}0 &= F(x\beta z)\alpha G(y\gamma w)\\ &= (F(x)\beta\sigma(z)+\tau(x)\beta d(z))\alpha(G(y)\gamma\sigma(w)+\tau(y)\gamma g(w))\\ &= F(x)\beta\sigma(z)\alpha G(y)\gamma\sigma(w)+F(x)\beta\sigma(z)\alpha\tau(y)\gamma g(w)+\tau(x)\beta d(z)\alpha G(y)\gamma\sigma(w)\\ &\quad +\tau(x)\beta d(z)\alpha\tau(y)\gamma g(w).\end{aligned}$$

Using (ii) and (iii), we find that

$$\tau(x)\beta d(z)\alpha\tau(y)\gamma g(w)=0 \text{ for all } w,x,y,z\in M \text{ and } \alpha,\beta,\gamma\in\Gamma.$$

Since τ is an automorphism of M, so the last expression yields that

$$d(z)\alpha M\gamma g(w)\delta M\beta d(z)\alpha M\gamma g(w)=\{0\} \text{ for all } w,z\in M \text{ and } \alpha,\beta,\gamma,\delta\in\Gamma.$$

The semiprimeness of M forces that

$$d(z)\alpha M\gamma g(w)=\{0\} \text{ for all } w,z\in M \text{ and } \alpha,\gamma\in\Gamma.$$

Hence by Lemma 2.1, d and g are orthogonal.
(v). In view of (ii) d and G are orthogonal. Hence,

$$\begin{aligned}0 &= G(d(x)\alpha z\beta G(y))\\ &= Gd(x)\alpha\sigma(z)\beta\sigma(G(y))+\tau(d(x))\alpha g(z)\beta\sigma(G(y))+\tau(d(x))\alpha\tau(z)\beta g(G(y)).\end{aligned}$$

Since $d\tau=\tau d,\ G\sigma=\sigma G$ and d,g are orthogonal, so we obtain

$$Gd(x)\alpha z_1\beta G(y_1)=0 \text{ for all } x,y_1,z_1\in M \text{ and } \alpha,\beta\in\Gamma. \tag{3.2}$$

Replacing y_1 by $d(x)$ in (3.2) and using the semiprimeness of M, we get $Gd=0$. Similarly, since each of the equalities $d(G(x)\alpha z\beta d(y))=0,\ F(g(x)\alpha z\beta F(y))=0$, $g(F(x)\alpha z\beta g(y))=0,\ F(G(x)\alpha z\beta F(y))=0$ and $G(F(x)\alpha z\beta G(y))=0$ hold for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$, we conclude that $dG=Fg=gF=FG=GF=0$, respectively. □

In view of Theorem 2.5(ii) and Lemma 3.1, we have the following corollary:

Corollary 3.2. *Let (F,d) and (G,g) be orthogonal generalized (σ,τ)-derivations of M such that $F\sigma=\sigma F$, $F\tau=\tau F$, $G\sigma=\sigma G$, $G\tau=\tau G$ and $d\sigma=\sigma d$, $d\tau=\tau d$, $g\sigma=\sigma g$, $g\tau=\tau g$. Then dg is a (σ^2,τ^2)-derivation of M and $(FG,dg)=(0,0)$ is a generalized (σ^2,τ^2)-derivation of M.*

Theorem 3.3. *Suppose (F,d) and (G,g) are generalized (σ,τ)-derivations of M such that $F\sigma=\sigma F$, $F\tau=\tau F$, $G\sigma=\sigma G$, $G\tau=\tau G$ and $d\sigma=\sigma d$, $d\tau=\tau d$, $g\sigma=\sigma g$, $g\tau=\tau g$. Then (F,d) and (G,g) are orthogonal if and only if one of the following holds:*

(*i*) (*a*) $F(x)\gamma G(y)+G(x)\gamma F(y)=0$ *for all* $x,y\in M$ *and* $\gamma\in\Gamma$;
(*b*) $d(x)\gamma G(y)+g(x)\gamma F(y)=0$ *for all* $x,y\in M$ *and* $\gamma\in\Gamma$;
(*ii*) $F(x)\gamma G(y)=d(x)\gamma G(y)=0$ *for all* $x,y\in M$ *and* $\gamma\in\Gamma$;
(*iii*) $F(x)\gamma G(y)=0$ *for all* $x,y\in M$ *and* $\gamma\in\Gamma$ *and* $dG=dg=0$;
(*iv*) (FG,dg) *is a generalized* (σ^2,τ^2)*-derivation and* $F(x)\gamma G(y)=0$ *for all* $x,y\in M$ *and* $\gamma\in\Gamma$.

Proof. In view of Lemma 3.1, Corollary 3.2 and the orthogonality of (F,d) and $(G,g)\Rightarrow(i),(ii),(iii)$ and (iv). Now, we establish
$(i)\Rightarrow$ "(F,d) and (G,g) are orthogonal." By the hypothesis, we have

$$F(x)\gamma G(y)+G(x)\gamma F(y)=0 \text{ for all } x,y\in M \text{ and } \gamma\in\Gamma.$$

Replacing x by $x\alpha z$ in above, we find that

$$\begin{aligned} 0 &= F(x\alpha z)\gamma G(y)+G(x\alpha z)\gamma F(y)\\ &= F(x)\alpha\sigma(z)\gamma G(y)+\tau(x)\alpha d(z)\gamma G(y)+G(x)\alpha\sigma(z)\gamma F(y)+\tau(x)\alpha g(z)\gamma F(y).\end{aligned}$$

Using (b) in last expression, we get

$$F(x)\alpha\sigma(z)\gamma G(y)+G(x)\alpha\sigma(z)\gamma F(y)=0 \text{ for all } x,y,z\in M \text{ and } \alpha,\gamma\in\Gamma.$$

Since σ is an automorphism of M, the above relation can be rewritten as

$$F(x)\alpha z_1\gamma G(x)+G(x)\alpha z_1\gamma F(x)=0 \text{ for all } x,z_1\in M \text{ and } \alpha,\gamma\in\Gamma.$$

By Lemma 2.1, we conclude that $F(x)\alpha z_1\gamma G(x)=0$ and $G(x)\alpha z_1\gamma F(x)=0$ for all $x,z_1\in M$ and $\alpha,\gamma\in\Gamma$. Using Lemma 2.2, we have $F(x)\alpha z_1\gamma G(y)=0$ for all $x,y,z_1\in M$ and $\alpha,\gamma\in\Gamma$. Therefore, F and G are orthogonal, by Lemma 2.1.
$(ii)\Rightarrow$ "(F,d) and (G,g) are orthogonal." Given that $F(x)\gamma G(y)=0$. Putting $x\alpha z$ for x, we get

$$\begin{aligned} 0 &= F(x\alpha z)\gamma G(y)\\ &= F(x)\alpha\sigma(z)\gamma G(y)+\tau(x)\alpha d(z)\gamma G(y)\\ &= F(x)\alpha\sigma(z)\gamma G(y).\end{aligned}$$

Using Lemma 2.1 and the fact that σ is an automorphism of M, we conclude that (F,d) and (G,g) are orthogonal.

$(iii) \Rightarrow$ "(F,d) and (G,g) are orthogonal." By the assumption, we have

$$\begin{aligned} 0 &= dG(x\alpha y) \\ &= d(G(x)\alpha\sigma(y)+\tau(x)\alpha g(y)) \\ &= dG(x)\alpha\sigma^2(y)+\tau(G(x))\alpha d(\sigma(y))+d(\tau(x))\alpha\sigma(g(y))+\tau^2(x)\alpha dg(y) \\ &= \tau(G(x))\alpha d(\sigma(y))+d(\tau(x))\alpha\sigma(g(y)). \end{aligned}$$

Since $G\tau=\tau G,\ g\sigma=\sigma g$ and σ,τ are automorphisms of M, we have

$$G(x_1)\alpha d(y_1)+d(x_1)\alpha g(y_1)=0 \text{ for all } x_1,y_1\in M \text{ and } \alpha\in\Gamma.$$

Application of Lemma 2.5(iv) and Lemma 2.1 yields that

$$G(x_1)\alpha d(y_1)=0 \text{ for all } x_1,y_1\in M \text{ and } \alpha\in\Gamma.$$

Replacing x_1 by $x\beta z$ and using Lemma 2.5(iv) and Lemma 2.1, we obtain

$$G(x)\beta\sigma(z)\alpha d(y_1)=0 \text{ for all } x,y_1,z\in M \text{ and } \alpha,\beta\in\Gamma.$$

By Lemma 2.1, we have $d(y_1)\gamma G(x)=0$ for all $x,y_1\in M$ and $\gamma\in\Gamma$, which satisfies (ii). Therefore, (iii) implies that (F,d) and (G,g) are orthogonal.

$(iv) \Rightarrow$ "(F,d) and (G,g) are orthogonal." Since (FG,dg) is a generalized (σ^2,τ^2)-derivation and dg is a (σ^2,τ^2)-derivation, we have

$$FG(x\gamma y)=FG(x)\gamma\sigma^2(y)+\tau^2(x)\gamma dg(y) \text{ for all } x,y\in M \text{ and } \gamma\in\Gamma. \tag{3.3}$$

Also

$$FG(x\gamma y)=FG(x)\gamma\sigma^2(y)+\tau(G(x))\gamma d(\sigma(y))+F(\tau(x))\gamma\sigma(g(y))+\tau^2(x)\gamma dg(y). \tag{3.4}$$

Comparing (3.3) and (3.4), we get

$$\tau(G(x))\gamma d(\sigma(y))+F(\tau(x))\gamma\sigma(g(y))=0 \text{ for all } x,y\in M \text{ and } \gamma\in\Gamma.$$

Since σ,τ are automorphisms of M and noting that $G\tau=\tau G,\ g\sigma=\sigma g$, we have

$$G(x_1)\gamma d(y_1)+F(x_1)\gamma g(y_1)=0 \text{ for all } x_1,y_1\in M \text{ and } \gamma\in\Gamma. \tag{3.5}$$

Since, $F(x_1)\gamma G(y_1)=0$, we get

$$\begin{aligned} 0 &= F(x_1)\gamma G(y_1\alpha z_1) \\ &= F(x_1)\gamma G(y_1)\alpha\sigma(z_1)+F(x_1)\gamma\tau(y_1)\alpha g(z_1) \\ &= F(x_1)\gamma\tau(y_1)\alpha g(z_1). \end{aligned}$$

By Lemma 2.1, we have $g(z_1)\gamma F(x_1) = 0$ for all $x_1, z_1 \in M$ and $\gamma \in \Gamma$. Replace z_1 by $y_1\beta z_1$ to get

$$\begin{aligned} 0 &= g(y_1\beta z_1)\gamma F(x_1) \\ &= g(y_1)\beta\sigma(z_1)\gamma F(x_1) + \tau(y_1)\beta g(z_1)\gamma F(x_1) \\ &= g(y_1)\beta\sigma(z_1)\gamma F(x_1). \end{aligned}$$

Since σ is an automorphism of M and using Lemma 2.1, we find that $F(x_1)\gamma g(y_1) = 0$ for all $x_1, y_1 \in M$ and $\gamma \in \Gamma$. Now from (3.5), we get $G(x_1)\gamma d(y_1) = 0$ for all $x_1, y_1 \in M$ and $\gamma \in \Gamma$. Putting $z_1\alpha y_1$ for y_1 in the last relation, we get

$$\begin{aligned} 0 &= G(x_1)\gamma d(z_1\alpha y_1) \\ &= G(x_1)\gamma d(z_1)\alpha\sigma(y_1) + G(x_1)\gamma\tau(z_1)\alpha d(y_1) \\ &= G(x_1)\gamma\tau(z_1)\alpha d(y_1). \end{aligned}$$

Since τ is an automorphism of M, the above expression forces that $G(x_1)\gamma z_2\alpha d(y_1) = 0$ for all $x_1, y_1, z_2 \in M$ and $\alpha, \gamma \in \Gamma$. Again using Lemma 2.1, we obtain $d(y_1)\gamma G(x_1) = 0$ for all $x_1, y_1 \in M$ and $\gamma \in \Gamma$. By (ii), (F, d) and (G, g) are orthogonal. □

Theorem 3.4. *Let (F, d) and (G, g) be generalized (σ, τ)-derivations of M such that $d\sigma = \sigma d$, $d\tau = \tau d$, $g\sigma = \sigma g$, $g\tau = \tau g$. Then the following conditions are equivalent:*

(i) (FG, dg) is a generalized (σ^2, τ^2)-derivation.

(ii) (GF, gd) is a generalized (σ^2, τ^2)-derivation.

(iii) F and g are orthogonal, and G and d are orthogonal.

Proof. $(i) \Rightarrow (iii)$. Suppose (FG, dg) is a generalized (σ^2, τ^2)-derivation. From (3.5), we have

$$G(x)\gamma d(y) + F(x)\gamma g(y) = 0 \text{ for all } x, y \in M \text{ and } \gamma \in \Gamma.$$

Replacing y by $y\beta z$, we obtain

$$\begin{aligned} 0 &= G(x)\gamma d(y\beta z) + F(x)\gamma g(y\beta z) \\ &= G(x)\gamma d(y)\beta\sigma(z) + G(x)\gamma\tau(y)\beta d(z) + F(x)\gamma g(y)\beta\sigma(z) + F(x)\gamma\tau(y)\beta g(z) \\ &= G(x)\gamma\tau(y)\beta d(z) + F(x)\gamma\tau(y)\beta g(z). \end{aligned}$$

Since τ is an automorphism of M, the above relation yields that

$$G(x)\gamma y_1\beta d(z) + F(x)\gamma y_1\beta g(z) = 0 \text{ for all } x, y_1, z \in M \text{ and } \beta, \gamma \in \Gamma. \quad (3.6)$$

Since dg is a (σ^2,τ^2)-derivation, so d and g are orthogonal by Theorem 2.5. Replacing y_1 by $g(z)\alpha y$ and using the orthogonality of d and g, we get

$$\begin{aligned} 0 &= G(x)\gamma g(z)\alpha y\beta d(z) + F(x)\gamma g(z)\alpha y\beta g(z) \\ &= F(x)\gamma g(z)\alpha y\beta g(z). \end{aligned}$$

Again replacing y by $y\delta F(x)$ and β by γ and using the semiprimeness of M, we obtain

$$F(x)\gamma g(z) = 0 \text{ for all } x,z \in M \text{ and } \gamma \in \Gamma. \tag{3.7}$$

Substituting $y\alpha z$ for z in (3.7), we find that

$$F(x)\gamma g(y)\alpha\sigma(z) + F(x)\gamma\tau(y)\alpha g(z) = 0 \text{ for all } x,y,z \in M \text{ and } \alpha,\gamma \in \Gamma.$$

Using (3.7) and the fact that τ is an automorphism of M, we get

$$F(x)\gamma y_1\alpha g(z) = 0 \text{ for all } x,y_1,z \in M \text{ and } \alpha,\gamma \in \Gamma.$$

Therefore by Lemma 2.1, F and g are orthogonal. Hence (3.6) becomes $G(x)\gamma y_1\beta d(z) = 0$ for all $x,y_1,z \in M$ and $\beta,\gamma \in \Gamma$. Thus, G and d are orthogonal.
$(iii) \Rightarrow (i)$. By the orthogonality of F and g, we have

$$F(x)\alpha y\beta g(z) = 0 \text{ for all } x,y,z \in M \text{ and } \alpha,\beta \in \Gamma. \tag{3.8}$$

Replacing x by $s\gamma x$, we get

$$\begin{aligned} 0 &= F(s\gamma x)\alpha y\beta g(z) \\ &= F(s)\gamma\sigma(x)\alpha y\beta g(z) + \tau(s)\gamma d(x)\alpha y\beta g(z) \\ &= \tau(s)\gamma d(x)\alpha y\beta g(z). \end{aligned}$$

Since τ is an automorphism of M and using the semiprimeness of M, we get $d(x)\alpha y\beta g(z) = 0$ for all $x,y,z \in M$ and $\alpha,\beta \in \Gamma$. By Lemma 2.1, d and g are orthogonal. Thus, by Theorem 2.5, dg is a (σ^2,τ^2)-derivation. Now, replacing y by $g(z)\gamma y\delta F(x)$ and β by α in (3.8), we get

$$F(x)\alpha g(z)\gamma y\delta F(x)\alpha g(z) = 0 \text{ for all } x,y,z \in M \text{ and } \alpha,\gamma,\delta \in \Gamma.$$

By the semiprimeness of M, we have $F(x)\alpha g(z) = 0$ for all $x,z \in M$ and $\alpha \in \Gamma$. Similarly, by the orthogonality of G and d, we have $G(x)\alpha d(z) = 0$ for all $x,z \in M$ and $\alpha \in \Gamma$. Thus,

$$FG(x\alpha y) = FG(x)\alpha\sigma^2(y) + \tau^2(x)\alpha dg(y) \text{ for all } x,y \in M \text{ and } \alpha \in \Gamma.$$

Hence (FG,dg) is a generalized (σ^2,τ^2)-derivation.
$(ii) \Leftrightarrow (iii)$. Using similar approach as we have used to prove $(i) \Leftrightarrow (iii)$. □

As an immediate consequence of above theorem we have the following:

Corollary 3.5. ([4, Theorem 2.2]) *Let (F,d) and (G,g) be generalized derivations of M. Then the following conditions are equivalent:*

(*i*) *(FG,dg) is a generalized derivation.*
(*ii*) *(GF,gd) is a generalized derivation.*
(*iii*) *F and g are orthogonal, and G and d are orthogonal.*

The following example shows that Theorem 3.4 does not hold for arbitrary Γ-rings.

Example 3.6. *Let R be any 2-torsionfree ring and let* $M = \left\{ \begin{pmatrix} a \\ b \\ c \\ f \\ h \end{pmatrix} \;\middle|\; a,b,c,f,h \in R \right\}$, $\Gamma = \left\{ \begin{pmatrix} l & 0 & 0 & 0 & m \end{pmatrix} \mid l,m \in R \right\}$. *Then M is a 2-torsionfree Γ-ring which is not semiprime. Define the map $\sigma : M \longrightarrow M$ such that* $\sigma \begin{pmatrix} a \\ b \\ c \\ f \\ h \end{pmatrix} = \begin{pmatrix} a \\ c \\ b \\ f \\ h \end{pmatrix}$. *Clearly, σ is an automorphism of M and take $\tau = I_M$, where I_M is the identity map of M. Next, define the maps $d,\ g : M \longrightarrow M$ such that*

$$d \begin{pmatrix} a \\ b \\ c \\ f \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ f \\ 0 \end{pmatrix}, \; g \begin{pmatrix} a \\ b \\ c \\ f \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ c \\ b \\ 0 \\ 0 \end{pmatrix} \;\; \text{for all} \begin{pmatrix} a \\ b \\ c \\ f \\ h \end{pmatrix} \in M.$$

It can be easily verified that d and g are (σ,τ)-derivations of M such that $d\sigma = \sigma d,\ d\tau = \tau d,\ g\sigma = \sigma g,\ g\tau = \tau g$. Now, consider the maps $F,\ G : M \longrightarrow M$ such that

$$F \begin{pmatrix} a \\ b \\ c \\ f \\ h \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; G \begin{pmatrix} a \\ b \\ c \\ f \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ h \end{pmatrix} \;\; \text{for all} \begin{pmatrix} a \\ b \\ c \\ f \\ h \end{pmatrix} \in M.$$

It can be easily check that (F,d) and (G,g) are generalized (σ,τ)-derivations of M. Also, (FG,dg) and (GF,gd) are generalized (σ^2,τ^2)-derivations of M but neither F and g are orthogonal nor G and d are orthogonal.

Corollary 3.7. *Let (F,d) be generalized (σ,τ)-derivation of M. If $F(x)\gamma F(y)=0$ for all $x,y\in M$ and $\gamma\in\Gamma$, then $F=d=0$.*

Proof. Notice that $F(x)\gamma F(y)=0$ for all $x,y\in M$ and $\gamma\in\Gamma$. Replacing y by $y\beta z$, we get

$$\begin{aligned}0 &= F(x)\gamma F(y\beta z)\\ &= F(x)\gamma F(y)\beta\sigma(z)+F(x)\gamma\tau(y)\beta d(z)\\ &= F(x)\gamma\tau(y)\beta d(z).\end{aligned}$$

Since τ is an automorphism of M and using Lemma 2.1, we have $d(z)\gamma F(x)=0$ for all $x,z\in M$ and $\gamma\in\Gamma$. Now, replacing x by $x\alpha z$, we get

$$\begin{aligned}0 &= d(z)\gamma F(x\alpha z)\\ &= d(z)\gamma F(x)\alpha\sigma(z)+d(z)\gamma\tau(x)\alpha d(z)\\ &= d(z)\gamma\tau(x)\alpha d(z).\end{aligned}$$

By the semiprimeness of M, we get $d(z)=0$ for all $z\in M$. Therefore, $d=0$. Again

$$\begin{aligned}0 &= F(x\gamma z)\alpha F(y)\\ &= F(x)\gamma\sigma(z)\alpha F(y)+\tau(x)\gamma d(z)\alpha F(y)\\ &= F(x)\gamma\sigma(z)\alpha F(y).\end{aligned}$$

In particular, we have

$$F(x)\gamma z_1\alpha F(x)=0 \text{ for all } x,z_1\in M \text{ and } \alpha,\gamma\in\Gamma.$$

Using the semiprimeness of M, we get $F(x)=0$ for all $x\in M$ and hence $F=0$. □

We conclude our paper with the following example which shows that the hypothesis of semiprimeness is crucial in above result.

Example 3.8. *Let R be any 2-torsionfree ring and* $M=\left\{\begin{pmatrix}a\\b\\c\\f\end{pmatrix}\;\middle|\; a,b,c,f\in R\right\}$,

$\Gamma=\left\{\begin{pmatrix}0 & x & 0 & 0\end{pmatrix}\mid x\in R\right\}$. *Then M is a 2-torsionfree Γ-ring which is not semiprime. Define the mappings $\sigma,\ \tau: M\longrightarrow M$ such that*

$$\sigma\begin{pmatrix}a\\b\\c\\f\end{pmatrix}=\begin{pmatrix}c\\b\\a\\f\end{pmatrix},\ \tau\begin{pmatrix}a\\b\\c\\f\end{pmatrix}=\begin{pmatrix}f\\b\\c\\a\end{pmatrix}\ \text{for all}\ \begin{pmatrix}a\\b\\c\\f\end{pmatrix}\in M.$$

Clearly, σ and τ are automorphisms of M. Next, define the map $d : M \longrightarrow M$ such that

$$d \begin{pmatrix} a \\ b \\ c \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ c \\ f \end{pmatrix} \text{ for all } \begin{pmatrix} a \\ b \\ c \\ f \end{pmatrix} \in M.$$

It can be easily verified that d is a (σ, τ)-derivation of M. Further, consider the map $F : M \longrightarrow M$ such that

$$F \begin{pmatrix} a \\ b \\ c \\ f \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ for all } \begin{pmatrix} a \\ b \\ c \\ f \end{pmatrix} \in M.$$

Then it is straightforward to check that F is a generalized (σ, τ)-derivation of M. Moreover, F satisfies the relation $F(x)\gamma F(y) = 0$ for all $x, y \in M$ and $\gamma \in \Gamma$, but neither $F = 0$ nor $d = 0$.

Acknowledgments. The authors are greatly indebted to the referee for his/her valuable comments. The authors would like to thank Professor Mohammad Ashraf for his helpful suggestions and encouragement.

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