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PI-RINGS WITH ARTINIAN PROPER CYCLICS ARE NOETHERIAN

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Abstract. Non-Artinian algebras over which proper cyclic right modules are Artinian must be right Ore domains. It is shown that if R is a PI-ring whose proper cyclic right R-modules are Artinian, then R is right Noetherian. In particular, if G is a solvable group and each proper cyclic right $K[G]$-module is Artinian, then the group algebra $K[G]$ is Noetherian. It is also shown that for a group algebra $K[G]$, if every proper cyclic right $K[G]$-module is Artinian and K-finite dimensional, then $K[G]$ is Noetherian.

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1. Introduction

A cyclic right R-module, $M \cong R/A$, is called proper cyclic if $M \cong R/A \ncong R$. An open question of Camilo and Krause [1] asks that if R is any ring whose proper cyclic right R-modules are Artinian, then is it true that R is right Noetherian? In this paper, the question is answered for PI-rings (in the affirmative). In particular, it follows that the group algebra of a solvable group is right Noetherian. An independent proof is provided for this result. Moreover, a group algebra, $K[G]$, over which each proper cyclic right module is both Artinian and finite K-dimensional, is Noetherian. The general problem remains open.

For any unexplained terminology, we refer the reader to [2], [3] and [4].

2. Main Results

The following lemma is folklore. For the sake of completeness, its proof is provided.

Lemma 2.1. *A ring whose proper cyclic right R-modules are Artinian is either right Artinian or a domain.*

Proof. Assume that R is not right Artinian. To show that R is a domain, claim that $\text{u.dim}(R) = 1$. It must be that $\text{u.dim}(R) < \infty$, for if $\text{u.dim}(R) = \infty$, then there exist two right ideals, X and Y, of R such that their sum is direct and they are both infinite direct sums of nonzero right ideals of R. If R/X is not proper cyclic, then it is projective. So, $R = X \oplus K$ for some right ideal K of R. This is a contradiction, because X is an infinite direct sum of right ideals. Thus, R/X is Artinian. Similarly, R/Y is Artinian. This implies R is right Artinian, because R is embeddable in $R/X \times R/Y$; a contradiction. Therefore, $\mathrm{u.dim}(R) < \infty$. Assume $\mathrm{u.dim}(R) = k > 1$, then there exist closed uniform right ideals C_i, $1 \leq i \leq k$, with $\oplus\Sigma_{i=1}^{k} C_i \subseteq R$. Then, for $1 \leq i \neq j \leq k$, $\mathrm{u.dim}(R/C_i) = k - 1 = \mathrm{u.dim}(R/C_j)$. Hence, both R/C_i and R/C_j are Artinian. Therefore, R is Artinian, because R is embeddable in $R/C_i \times R/C_j$, for $1 \leq i \neq j \leq k$; a contradiction. Therefore, $\mathrm{u.dim}(R) = 1$. To prove R is a domain, let x be a nonzero element in R. It is enough to show that $\mathrm{l.ann}(x) = 0$. Assume that $\mathrm{l.ann}(x) \neq 0$, then $R/\mathrm{l.ann}(x)$ is a proper cyclic module since $\mathrm{u.dim}(R) = 1$. Hence, $R/\mathrm{l.ann}(x)$ is Artinian. However, $R/\mathrm{l.ann}(x) \cong xR$, and so xR is Artinian. Moreover, R/xR is Artinian, and therefore R is right Artinian; a contradiction. This completes the proof. □

Theorem 2.2. *If R is an integral domain with PI in which every proper cyclic right R-module is Artinian, then R is right Noetherian.*

Proof. It is known that if R is a PI-domain, then each nonzero right or left ideal contains a nonzero two-sided ideal ([5], Corollary 13.2.9). Now, consider an ascending chain of right ideals

$$I_1 \subset I_2 \subset I_3 \subset \ldots \subset I_k \subset \ldots .$$

Let A be a nonzero two-sided ideal contained in I_1. This leads to an ascending chain of right ideals

$$I_1/A \subset I_2/A \subset I_3/A \subset \ldots \subset I_k/A \subset \ldots$$

in the ring R/A. By hypothesis, R/A is right Artinian ring and, hence, a right Noetherian ring. Thus there exists a positive integer n such that

$$I_n/A = I_{n+1}/A = I_{n+2}/A = \ldots .$$

Thus,

$$I_n = I_{n+1} = I_{n+2} = \ldots .$$

Hence, R is right Noetherian. □

The following theorem is a consequence of Theorem 2.2.

Theorem 2.3. *Let $K[G]$ be a group algebra of a solvable group. If every proper cyclic right $K[G]$-module is Artinian , then $K[G]$ is Noetherian.*

Proof. Since G is solvable, G has a nonidentity normal Abelian subgroup N. However, $K[G/N] \cong K[G]/\omega(N)$. Hence, $K[G/N]$ is an Artinian group algebra and, thus, G/N is finite. By Theorem 5.3.7 and Theorem 5.3.9 in [6], $K[G]$ has PI. Thus, $K[G]$ is Noetherian, by Theorem 2.2.

An Alternative Proof. Recall that every nonidentity solvable group G contains a nonidentity Abelian subgroup A. Also, note that for any group G, $\Delta(G)$ denotes the set of all elements of G that have finitely many conjugates. Since $K[G/A]$ is Artinian, G/A is finite. Hence $A \subset \Delta(G)$. Let N be generated by a nonidentity conjugacy class of $a \in A$ in G. Hence, N is a finitely generated Abelian group and G/N is finite. Thus, G is polycyclic-by-finite and, therefore, $K[G]$ is Noetherian ([6], Corollary 10.2.8). □

Theorem 2.4. *Let $K[G]$ be a group algebra. If every proper cyclic right $K[G]$-module is Artinian and K-finite dimensional, then $K[G]$ is Noetherian.*

Proof. Let $a \in G$ be a nonidentical element, $H = \langle a \rangle$, and $I = (1-a)K[G]$, a right ideal. If elements, g_i, lie in distinct left cosets of H, then they are linearly independent modulo I. Since $K[G]/I \ncong K[G]$, $K[G]/I$ is finite dimensional. Hence H is of finite index in G. Thus $K[G]$ satisfies PI, and the result follows from Theorem 2.2. □

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