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A FINITE DIMENSIONAL L_∞ MODULE

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Abstract. We construct a concrete example of an L_∞ module defined over a finite dimensional L_∞ algebra. We then explore a canonical L_∞ algebra defined over the direct sum of these structures.

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1. Introduction

L_∞ algebras (or sh Lie algebras) are topics of current research. Concrete examples of these structures facilitate deeper insight into their properties. However these examples still remain elusive. In [1], Daily and Lada constructed an example of a finite dimensional L_∞ algebra that consists of a Lie algebra in conjunction with with a non-Lie action on a different vector space. Furthermore, this small structure was proved by Kadeishvili and Lada in [3] to be an example of an open-closed homotopy algebra (OCHA), as defined by Kajiura and Stasheff in [4].

L_∞ modules provide an alternative but equivalent convention to representations of L_∞ algebras in a manner that generalizes the relationship between classical Lie algebra representations and Lie modules. Concrete examples of L_∞ modules are highly nontrivial. Such examples will prove essential for expanding the representation theory of L_∞ algebras, as evident by the robust nature of the example in [1]. In this article we construct one such example and subsequently explore a new L_∞ algebra structure that it induces.

2. L_∞ Algebras and Modules

We will utilize the Koszul sign convention that is common in graded settings. That is, whenever two symbols (objects or maps) of degree p and q are transposed, a factor of $(-1)^{pq}$ is introduced. In general, we denote the total Koszul sign of a permutation σ by $\epsilon(\sigma)$. For brevity, we will also use commas in lieu of tensor or direct sum symbols when convenient.

We first recall the definition of an L_∞ algebra (see [2], [6], [7]).

Definition 2.1. Let V be a graded vector space. An L_∞ *algebra* structure on V is a collection of linear maps $\{l_k : V^{\otimes k} \to V\}$ of degree $k-2$ which are skew-symmetric in the sense that

$$l_k(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)}) = \chi(\sigma) l_k(x_1, x_2, \ldots, x_k)$$

for all $\sigma \in S_k,\ x_i \in V$, with $\chi(\sigma) = (-1)^\sigma \epsilon(\sigma)$, and are also required to satisfy the generalized form of the Jacobi identity:

$$\sum_{i+j=n+1} \sum_\sigma \chi(\sigma)(-1)^{i(j-1)} l_j(l_i(v_{\sigma(1)}, \ldots, v_{\sigma(i)}), v_{\sigma(i+1)}, \ldots, v_{\sigma(n)}) = 0$$

where the inner summation is taken over all $(i, n-i)$-unshuffles, $i \geq 1$, $n \geq 1$.

This utilizes the chain complex convention. One may alternatively use the cochain complex convention by requiring each map l_k to have degree $2-k$.

It is worth noting that L_∞ algebras are generalizations of Lie algebras from a homotopy theoretic point of view with l_2 acting as a (graded) Lie bracket. Remarkably, many classical relationships involving Lie algebras generalize to this context. In [6], Lada and Markl defined L_∞ algebra representations and, equivalently, L_∞ modules. We now recall the definition of an L_∞ module.

Definition 2.2. Let (L, l_k) be an L_∞ algebra and M a differential graded vector space with graded differential k_1. A *(left) L-module* on M is a collection of skew-symmetric linear maps $\{k_n : L^{\otimes n-1} \otimes M \to M | 1 \leq n < \infty\}$ of degree $n-2$ such that the following identity holds:

$$\sum_{i+j=n+1} \sum_\sigma \chi(\sigma)(-1)^{i(j-1)} k_j(k_i(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(i)}), \xi_{\sigma(i+1)}, \ldots, \xi_{\sigma(n)}) = 0$$

where σ ranges over all $(i, n-i)$-unshuffles, $i \geq 1$, with $n \geq 1$, $\xi_1, \cdots, \xi_{n-1} \in L$, and $\xi_n \in M$.

A few remarks are in order:

- If $\xi_1, \cdots, \xi_n \in L$, then we define $k_n(\xi_1, \cdots, \xi_n) = l_n(\xi_1, \cdots, \xi_n)$.
- By definition of an unshuffle, either $\xi_{\sigma(i)} = \xi_n$ or $\xi_{\sigma(n)} = \xi_n$.
- Since we have $k_n : L^{\otimes n-1} \otimes M \to M$, we must utilize the skew-symmetry of k_n in the case where $\xi_{\sigma(i)} = \xi_n$ as follows:

$$k_j(\underbrace{k_i(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(i)})}_{\in M}, \xi_{\sigma(i+1)} \ldots, \xi_{\sigma(n)}) = \alpha k_j(\xi_{\sigma(i+1)} \ldots \xi_{\sigma(n)}, \underbrace{k_i(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(i)})}_{\in M})$$

With $\alpha = (-1)^{j-1}(-1)^{(i+\sum_{k=1}^{i} |\xi_{\sigma(k)}|)(\sum_{k=i+1}^{n} |\xi_{\sigma(k)}|)}$.

One may easily verify that L_∞ modules can be viewed as generalizations of Lie modules from a homotopy theoretic point of view by interpreting k_2 as a module action.

3. A Finite Dimensional Example

Let L denote the graded vector space given by $L = \bigoplus_{i\in\mathbb{Z}} L_i$ where L_0 has basis $< v_1, v_2 >$, L_{-1} has basis $< w >$, and $L_i = 0$ for $i \neq 0, -1$ with skew-symmetric linear maps $l_n : L^{\otimes n} \to L$ defined by the following:

$$\begin{aligned} l_1(v_1) = l_1(v_2) &= w \\ l_2(v_1 \otimes v_2) &= v_1 \\ l_2(v_1 \otimes w) &= w \\ For\ n \geq 3:\ l_n(v_2 \otimes w^{\otimes n-1}) &= C_n w \end{aligned}$$

where $C_3 = 1$ and $C_n = (-1)^{n-1}(n-3)C_{n-1}$ and $l_n = 0$ when evaluated on any element of $L^{\otimes n}$ that is not listed above. In [1], Daily and Lada proved that this structure forms an L_∞ algebra.

Remark 3.1. *When convenient, the recursive definition of C_n may be recognized more explicitly as*

$$C_n = (-1)^{\frac{(n-2)(n-3)}{2}}(n-3)!.$$

Now let M denote the graded vector space given by $M = \bigoplus_{i\in\mathbb{Z}} M_i$ where M_0 is a one dimensional vector space with basis $< m >$, M_{-1} is a one dimensional vector space with basis $< u >$ and $M_i = 0$ for $i \neq 0, -1$. Define a structure on M by the following linear maps $k_n : L^{\otimes n-1} \otimes M \to M$:

$$\begin{aligned} k_1(m) &= u \\ k_2(v_1 \otimes m) &= m \\ k_2(v_1 \otimes u) &= u \\ For\ n \geq 3:\ k_n(v_2 \otimes w^{\otimes n-2} \otimes m) &= C_n m \\ k_n(v_2 \otimes w^{\otimes n-2} \otimes u) &= C_n u \end{aligned}$$

Extend these maps to be skew-symmetric and define $k_n = 0$ when evaluated on any element of $L^{\otimes n-1} \otimes M$ that is not listed above.

Theorem 3.2. *The maps k_n on M, in conjunction with the L_∞ algebra structure on L defined above, define an L_∞ module.*

Proof. We aim to prove the following:

$$\sum_{i+j=n+1} \sum_{\sigma} \chi(\sigma)(-1)^{i(j-1)} k_j(k_i(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(i)}), \xi_{\sigma(i+1)}, \ldots, \xi_{\sigma(n)}) = 0$$

where σ ranges over all $(i, n-i)$-unshuffles, $i \geq 1$, with $n \geq 1$, $\xi_1, \cdots, \xi_{n-1} \in L$, and $\xi_n \in M$.

In shorthand notation, this is equivalent to showing that

$$\sum_{s=1}^{n}(-1)^{s(n-s)}k_{n-s+1}k_s(\xi_1,\xi_2,\cdots,\xi_n)=0 \tag{1}$$

where it is understood that k_s will be extended on $n>s$ elements over $(s,n-s)$-unshuffles.

Since k_n is linear and skew-symmetric, it suffices to show this equation holds when evaluated only on basis elements and in any string order. Each element of the left hand side of Equation 1 also has degree $(n-s-1-2)+(s-2)+|\xi_1|+|\xi_2|+\cdots+|\xi_n|=n-3+|\xi_1|+|\xi_2|+\cdots+|\xi_n|$, which must equal 0 or -1 in order for the elements to be nonzero. So $(\xi_1\otimes\xi_2\otimes\cdots\otimes\xi_n)\in L^{\otimes n-1}\otimes M$ must contain either 2 or 3 elements in degree 0. If this tensor product contains v_1 or v_2 twice, then Equation 1 holds trivially since $|v_1|=|v_2|=0$ and k_n is skew-symmetric. For example,

$$\begin{aligned}
k_{n-s+1}k_s(v_1\otimes v_1\otimes w^{\otimes n-3}\otimes u) &= -(-1)^{|v_1||v_1|}k_{n-s+1}k_s(v_1\otimes v_1\otimes w^{\otimes n-3}\otimes u)\\
&= -k_{n-s+1}k_s(v_1\otimes v_1\otimes w^{\otimes n-3}\otimes u)
\end{aligned}$$

by permuting the first two elements. So $k_{n-s+1}k_s(v_1\otimes v_1\otimes w\otimes^{n-3}\otimes u)=0$. Hence it suffices to prove that Equation 1 holds on the following string choices for $(\xi_1\otimes\xi_2\otimes\cdots\otimes\xi_n)$:

$$(v_1\otimes v_2\otimes w^{\otimes n-3}\otimes u) \tag{2}$$
$$(v_1\otimes w^{\otimes n-2}\otimes m) \tag{3}$$
$$(v_2\otimes w^{\otimes n-2}\otimes m) \tag{4}$$
$$(v_1\otimes v_2\otimes w^{\otimes n-3}\otimes m) \tag{5}$$

For String 2, in regards to the summands of Equation 1 we observe the following:

$$\begin{aligned}
k_1k_n(v_1\otimes v_2\otimes w^{\otimes n-3}\otimes u)) &= k_1(0)=0\\
k_2k_{n-1}(v_1\otimes v_2\otimes w^{\otimes n-3}\otimes u) &= (-1)^{n-1}k_2\big(k_{n-1}(v_2\otimes w^{\otimes n-2}\otimes u)\otimes v_1\big)\\
&= (-1)^nC_{n-1}k_2(v_1\otimes u)\\
&= (-1)^nC_{n-1}u
\end{aligned}$$

$k_r k_{n-r+1}(v_1 \otimes v_2 \otimes w^{\otimes n-3} \otimes u) = 0$ for $3 \leq r \leq n-2$ since k_r and k_{n-r+1} are only nonzero when they are evaluated on a tensor product containing v_2.

$$\begin{aligned}
k_{n-1}k_2(v_1 \otimes v_2 \otimes w^{\otimes n-3} \otimes u) &= -(n-3)k_{n-1}\big(l_2(v_1 \otimes w) \otimes v_2 \otimes w^{\otimes n-4} \otimes u\big) \\
&\quad - k_{n-1}\big(k_2(v_1 \otimes u) \otimes v_2 \otimes w^{\otimes n-3}\big) \\
&= (n-3)k_{n-1}(v_2 \otimes w^{\otimes n-3} \otimes u) \\
&\quad + k_{n-1}(v_2 \otimes w^{\otimes n-3} \otimes u) \\
&= (n-2)k_{n-1}(v_2 \otimes w^{\otimes n-3} \otimes u) \\
&= (n-2)C_{n-1}u \\
k_n k_1(v_1 \otimes v_2 \otimes w^{\otimes n-3} \otimes u) &= k_n\big(k_1(v_1) \otimes v_2 \otimes w^{\otimes n-3} \otimes u\big) \\
&= k_n(w \otimes v_2 \otimes w^{\otimes n-3} \otimes u) \\
&= -k_n(v_2 \otimes w^{\otimes n-2} \otimes u) \\
&= -C_n u \\
&= -(-1)^{n-1}(n-3)C_{n-1}u \\
&= (-1)^n(n-3)C_{n-1}u
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{s=1}^{n}(-1)^{s(n-s)}k_{n-s+1}k_s(v_1 \otimes v_2 \otimes w^{\otimes n-3} \otimes u) &= (-1)^{1(n-1)}(-1)^n(n-3)C_{n-1}u \\
&\quad + (-1)^{2(n-2)}(n-2)C_{n-1}u \\
&\quad + (-1)^{(n-1)(n-(n-1))}(-1)^n C_{n-1}u \\
&= -(n-3)C_{n-1}u \\
&\quad + (n-2)C_{n-1}u \\
&\quad - C_{n-1}u \\
&= 0
\end{aligned}$$

The case where Equation 1 is evaluated on String 3 is trivial by definition of k_n and l_n.

Regarding String 4, we observe the following:

$$k_1 k_n(v_2 \otimes w^{\otimes n-2} \otimes m) = k_1(C_n m) = C_n u$$

$k_r k_{n-r+1}(v_2 \otimes w^{\otimes n-2} \otimes m) = 0$ for $2 \le r \le n-1$ for similar reasons encountered above.

$$\begin{aligned} k_n k_1(v_2 \otimes w^{\otimes n-2} \otimes m) &= (-1)^{n-1} k_n\big(k_1(m) \otimes v_2 \otimes w^{\otimes n-2}\big) \\ &= -(-1)^{n-1} k_n(v_2 \otimes w^{\otimes n-2} \otimes u) \\ &= (-1)^n C_n u \end{aligned}$$

Hence,

$$\sum_{s=1}^{n} (-1)^{s(n-s)} k_{n-s+1} k_s(v_2 \otimes w^{\otimes n-2} \otimes m) = (-1)^{1(n-1)}(-1)^n C_n u + (-1)^{n(n-n)} C_n u \tag{6}$$

$$= -C_n u + C_n u \tag{7}$$

$$= 0 \tag{8}$$

For String 5, we observe the following:

$$\begin{aligned} k_1 k_n(v_1 \otimes v_2 \otimes w^{\otimes n-3} \otimes m) &= k_1(0) = 0 \\ k_2 k_{n-1}(v_1 \otimes v_2 \otimes w^{\otimes n-3} \otimes m) &= (-1)^{n-1} k_2\big(k_{n-1}(v_2 \otimes w^{\otimes n-2} \otimes m) \otimes v_1\big) \\ &= (-1)^n C_{n-1} k_2(v_1 \otimes m) \\ &= (-1)^n C_{n-1} m \end{aligned}$$

$k_r k_{n-r+1}(v_1 \otimes v_2 \otimes w^{\otimes n-3} \otimes m) = 0$ for $3 \le r \le n-2$ for similar reasons encountered above.

$$\begin{aligned} k_{n-1} k_2(v_1 \otimes v_2 \otimes w^{\otimes n-3} \otimes m) &= -(n-3) k_{n-1}\big(l_2(v_1 \otimes w) \otimes v_2 \otimes w^{\otimes n-4} \otimes m\big) \\ &\quad + (-1)^{n-2} k_{n-1}\big(k_2(v_1 \otimes m) \otimes v_2 \otimes w^{\otimes n-3}\big) \\ &= -(n-3) k_{n-1}(w \otimes v_2 \otimes w^{\otimes n-4} \otimes m) \\ &\quad + (-1)^{n-2} k_{n-1}(m \otimes v_2 \otimes w^{\otimes n-3}) \\ &= (n-3) k_{n-1}(v_2 \otimes w^{\otimes n-3} \otimes m) \\ &\quad + (-1)^{(n-2)+(n-2)} k_{n-1}(v_2 \otimes w^{\otimes n-3} \otimes m) \\ &= (n-3) C_{n-1} m \\ &\quad + C_{n-1} m \\ &= (n-2) C_{n-1} m \\ k_n k_1(v_1 \otimes v_2 \otimes w^{\otimes n-3} \otimes m) &= k_n\big(k_1(v_1) \otimes v_2 \otimes w^{\otimes n-3} \otimes m\big) \\ &= -k_n(v_2 \otimes w^{\otimes n-2} \otimes m) \\ &= -C_n m \\ &= -(-1)^{n-1}(n-3) C_{n-1} m \\ &= (-1)^n (n-3) C_{n-1} m \end{aligned}$$

Hence,

$$\begin{aligned}\sum_{s=1}^{n}(-1)^{s(n-s)}k_{n-s+1}k_s(v_1\otimes v_2\otimes w^{\otimes n-3}\otimes m) &= (-1)^{1(n-1)}(-1)^n(n-3)C_{n-1}m\\ &\quad + (-1)^{2(n-2)}(n-2)C_{n-1}m\\ &\quad + (-1)^{(n-1)(n-(n-1))}(-1)^nC_{n-1}m\\ &= -(n-3)C_{n-1}m\\ &\quad + (n-2)C_{n-1}m\\ &\quad - C_{n-1}m\\ &= 0\end{aligned}$$

In all 4 cases, Equation 1 holds. Hence, M is an L-module. □

A concrete L_∞ module will induce another interesting L_∞ algebra structure in a manner that generalizes the relationship between a classical Lie module and a canonical Lie algebra. This will be addressed in the next section.

4. A Canonical L_∞ Structure

One of the most fundamental results in the study of classical Lie algebras is that given a Lie algebra L and an $L-$module M, the vector space $L\oplus M$ forms a Lie algebra via the bracket

$$\big[(x_1,m_1),(x_2,m_2)\big] = \Big([x_1,x_2], x_1\cdot m_2 - x_2\cdot m_1\Big)$$

where '$\cdot$' denotes the module action in M. It is not surprising that a homotopy theoretic version of the classical Lie algebra $L\oplus M$ exists. Given an L_∞ structure, L, and an L-module, M, we may construct a new graded vector space $L\oplus M$ that can be endowed with its own L_∞ structure as follows.

Theorem 4.1. ([5]) *Let (L,l_k) be an L_∞ algebra and (M,k_n) be an L-module. Then the graded vector space $L\oplus M$ inherits a canonical L_∞ structure under the collection of maps $\{j_n : (L\oplus M)^{\otimes n}\to L\oplus M\}$ defined by*

$$\begin{aligned}&j_n\big((x_1,m_1)\otimes\cdots\otimes(x_n,m_n)\big)\\ &=\Big(l_n(x_1\otimes\cdots\otimes x_n),\sum_{i=1}^{n}(-1)^{n-i}(-1)^{m_i\sum_{k=i+1}^{n}x_k}k_n(x_1\otimes\cdots\otimes\hat{x_i}\otimes\cdots\otimes x_n\otimes m_i)\Big)\end{aligned}$$

where $\hat{x_i}$ means omit x_i.

Given the newly constructed concrete example of an L_∞ module in Theorem 3.2, it is natural to investigate the type of L_∞ algebra structure it induces. Let L

and M denote the L_∞ algebra and L-module in Theorem 3.2. That is, $L = \bigoplus_{i\in\mathbb{Z}} L_i$, where $L_i = 0$ if $i \neq 0, -1$ and

$$L_0 =< v_1, v_2 >, L_{-1} =< w >$$

and $M = \bigoplus_{i\in\mathbb{Z}} M_i$, where $M_i = 0$ if $i \neq 0, -1$ and

$$M_0 =< m >, M_{-1} =< u > .$$

As a graded vector space, elements $(x, m) \in L \oplus M$ must satisfy $|x| = |m|$. Hence $L \oplus M = \bigoplus_{i\in\mathbb{Z}} (L \oplus M)_i$, where $(L \oplus M)_i = 0$ if $i \neq 0, -1$ and

$$(L \oplus M)_0 =< (v_1, m), (v_2, m) >, (L \oplus M)_{-1} =< (w, u) > .$$

Using the definition given in Theorem 4.1, we may explicitly define the structure maps $\{j_n\}$ on $L \oplus M$. As a result of linearity and skew-symmetry, it suffices to define these maps by how they act on any number of basis elements.

Since $deg(j_1) = -1$, $j_1(w, u) = 0$. Furthermore,

$$\begin{aligned} j_1(v_1, m) &= (l_1(v_1), k_1(m)) = (w, u) \\ j_1(v_2, m) &= (l_1(v_2), k_1(m)) = (w, u). \end{aligned}$$

Hence, j_1 is very simply defined by

$$j_1(v_1, m) = j_1(v_2, m) = (w, u).$$

Examining all combinations of basis elements, we find j_2 to be defined as follows:

$$\begin{aligned} j_2\big((v_1, m), (v_1, m)\big) &= \Big(l_2(v_1, v_1), k_2(v_1, m) - (-1)^{v_1 m} k_2(v_1, m)\Big) = (0, 0) \\ j_2\big((v_2, m), (v_2, m)\big) &= \Big(l_2(v_2, v_2), k_2(v_2, m) - (-1)^{v_2 m} k_2(v_2, m)\Big) = (0, 0) \\ j_2\big((v_1, m), (v_2, m)\big) &= \Big(l_2(v_1, v_2), k_2(v_1, m) - (-1)^{v_2 m} k_2(v_2, m)\Big) = (v_1, m) \\ j_2\big((v_1, m), (w, u)\big) &= \Big(l_2(v_1, w), k_2(v_1, u) - (-1)^{wm} k_2(w, m)\Big) = (w, u) \\ j_2\big((v_2, m), (w, u)\big) &= \Big(l_2(v_2, w), k_2(v_2, u) - (-1)^{wm} k_2(w, m)\Big) = (0, 0) \\ j_2\big((w, u), (w, u)\big) &= (0, 0) \text{ since } deg(j_2) = 0. \end{aligned}$$

Hence,

$$\begin{aligned} j_2\big((v_1, m), (v_2, m)\big) &= (v_1, m) \\ j_2\big((v_1, m), (w, u)\big) &= (w, u) \end{aligned}$$

with $j_2 = 0$ when evaluated on any other element.

It is apparent that the graded differential j_1 and bracket j_2 are acting in precisely the same manner as their counterparts in L. The homotopies, however, do not. Let $n \geq 3$ and consider $j_n : (L \oplus M)^{\otimes n} \to L \oplus M$. By definition of j_n in Theorem 4.1, the only nonzero action of j_n on basis elements occurs on $(v_2, m)\otimes(w, u)^{\otimes n-1}$ since both l_n and k_n are only nonzero when evaluated on a string containing v_2 when $n \geq 3$. Hence, for $n \geq 3$,

$$\begin{aligned} j_n\big((v_2, m)\otimes(w, u)^{\otimes n-1}\big) &= \Big(l_n(v_2 \otimes w^{\otimes n-1}), \\ &\quad 0 + \underbrace{k_n(v_2 \otimes w^{\otimes n-2} \otimes u) + k_n(v_2 \otimes w^{\otimes n-2} \otimes u) + \cdots + k_n(v_2 \otimes w^{\otimes n-2} \otimes u)}_{(n-1)k_n(v_2\otimes w^{\otimes n-2}\otimes u)}\Big) \\ &= (C_n w, (n-1)C_n u) \\ &= C_n(w, (n-1)u). \end{aligned}$$

These structure maps form a new L_∞ algebra as follows.

Theorem 4.2. *Let $L \oplus M = \bigoplus_{i\in\mathbb{Z}}(L \oplus M)_i$ where $(L \oplus M)_0$ is two dimensional with basis $< (v_1, m), (v_2, m) >$ and $(L \oplus M)_{-1}$ is one dimensional with basis $< (w, u) >$ and $(L \oplus M)_i = 0$ for $i \neq 0, -1$. Define a structure on $(L \oplus M)$ by the following linear maps $\{j_n : (L \oplus M)^{\otimes n} \to L \oplus M\}$:*

$$\begin{aligned} j_1(v_1, m) = j_1(v_2, m) &= (w, u) \\ j_2\big((v_1, m) \otimes (v_2, m)\big) &= (v_1, m) \\ j_2\big((v_1, m) \otimes (w, u)\big) &= (w, u) \\ j_n\big((v_2, m) \otimes (w, u)^{\otimes n-1}\big) &= C_n(w, (n-1)u), \end{aligned}$$

where $C_3 = 1$, $C_n = (-1)^{n-1}(n-3)C_{n-1}$, and $j_n = 0$ when evaluated on any element of $(L \oplus M)^{\otimes n}$ that is not listed above. Then $(L \oplus M, j_n)$ is an L_∞ algebra.

The proof is an immediate consequence of the previous computations and Theorem 4.1.

It is worth noting that this is not isomorphic to the L_∞ structure given in [1] due to the extra coefficient, $n-1$, attached to the higher homotopies j_n. It is also worth noting that this is another example of an L_∞ structure that is a strict Lie algebra in degree 0. Finding an example that is not strictly Lie in degree 0 remains an interesting question for further investigation.

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