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ON FINITE GROUPS WITH SPECIFIC NUMBER OF CENTRALIZERS

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Abstract. For a group G, $\mid Cent(G) \mid$ denotes the number of distinct centralizers of its elements. A group G is called *n-centralizer* if $\mid Cent(G) \mid= n$, and *primitive n-centralizer* if $\mid Cent(G) \mid=\mid Cent(\frac{G}{Z(G)}) \mid= n$. In this paper, among other things, we investigate the structure of finite groups of odd order with $\mid Cent(G) \mid= 9$ and prove that if $|G|$ is odd, then $\mid Cent(G) \mid= 9$ if and only if $\frac{G}{Z(G)} \cong C_7 \rtimes C_3$ or $C_7 \times C_7$.

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1. Introduction

Throughout this paper all groups mentioned are assumed to be finite and we will use usual notation, for example C_n denotes the cyclic group of order n, G' denotes the commutator subgroup of G, D_{2n} denotes the dihedral group of order $2n$, $C_n \rtimes C_p$ denotes the semidirect product of C_n and C_p, where n is a positive integer and p is a prime.

For a group G, $Z(G)$ denotes the center and $Cent(G) = \{C_G(x) \mid x \in G\}$, where $C_G(x)$ is the centralizer of the element x in G i.e., $C_G(x) = \{y \in G \mid xy = yx\}$. A group G is a CA-group if $C_G(x)$ is abelian for every $x \in G \setminus Z(G)$ (see [11]).

Starting with Belcastro and Sherman [6], many authors have studied the influence of $\mid Cent(G) \mid$ on finite group G (see [1-6] and [15-17]). It is clear that a group is 1-centralizer if and only if it is abelian. In [6] Belcastro and Sherman proved that there is no n-centralizer group for $n = 2, 3$. They also proved that G is 4-centralizer if and only if $\frac{G}{Z(G)} \cong C_2 \times C_2$ and G is 5-centralizer if and only if $\frac{G}{Z(G)} \cong C_3 \times C_3$ or S_3. In [3] Ashrafi proved that if G is 6-centralizer, then $\frac{G}{Z(G)} \cong D_8, A_4, C_2 \times C_2 \times C_2$ or $C_2 \times C_2 \times C_2 \times C_2$. In [1] Abdollahi, Amiri and Hassanabadi proved that G is 7-centralizer if and only if $\frac{G}{Z(G)} \cong C_5 \times C_5, D_{10}$ or $< x, y | x^5 = y^4 = 1, y^{-1}xy = x^3 >$. They also proved that if G is 8-centralizer, then $\frac{G}{Z(G)} \cong C_2 \times C_2 \times C_2, A_4$ or D_{12}.

In this paper, we compute $\mid Cent(G) \mid$ under certain conditions on G. We also investigate the structure of odd order 9-centralizer groups and prove that if $|G|$ is odd, then G is 9-centralizer if and only if $\frac{G}{Z(G)} \cong C_7 \rtimes C_3$ or $C_7 \times C_7$.

2. Generalizations of Some Known Results

In this section, we calculate $\mid Cent(G) \mid$ under certain conditions on G. We generalize some of the results obtained in [1], [2], [3], [4], [6] and [8], and obtain some new characterizations of G.

Lemma 2.1. *Let $|G : Z(G)| = pqr$, where p, q, r are primes not necessarily distinct. Then G is a CA-group.*

Proof. Let $x \in G \setminus Z(G)$. Suppose $C_G(x)$ is not abelian. Then $Z(G) \subsetneqq Z(C_G(x)) \subsetneqq C_G(x) \subsetneqq G$. Since $|G : Z(G)| = pqr$, where p, q, r are primes, it follows that $\mid C_G(x) : Z(C_G(x)) \mid$ is a prime, which is a contradiction, since $C_G(x)$ is non-abelian. Therefore, G is a CA-group. $\square$

Let p be a prime. In [6, Theorem 5], Belcastro and Sherman proved that if $|G : Z(G)| = p^2$, then $|Cent(G)| = p + 2$. Now it is natural to ask about $|Cent(G)|$ when $|G : Z(G)| = p^3$. A partial answer is provided by the following proposition. This also generalizes Proposition 3.4 of Abdollahi, Amiri and Hassanabadi in [1], namely, if $\dfrac{G}{Z(G)} \cong C_2 \times C_2 \times C_2$, then $|Cent(G)| = 6$ or 8.

Proposition 2.2. *Let p be the smallest prime dividing $|G|$. If $|G : Z(G)| = p^3$, then $|Cent(G)| = p^2 + p + 2$ or $p^2 + 2$.*

Proof. By Lemma 2.1, G is a CA-group, and so by Proposition 3.2 of [11], $C_G(a) = C_G(b)$ for all $a, b \in G \setminus Z(G)$ with $ab = ba$. Therefore, for any $x \in G \setminus Z(G)$, we have $C_G(t) = C_G(x)$ for all $t \in C_G(x) \setminus Z(G)$. Suppose $\mid G : C_G(x) \mid = p^2$ for all $x \in G \setminus Z(G)$. Fix $y \in G \setminus Z(G)$.

We have $G = (G \setminus C_G(y)) \sqcup C_G(y)$. Let k be the number of distinct centralizers produced by the elements of $G \setminus C_G(y)$. Let $C_G(x_1), C_G(x_2), \cdots, C_G(x_k)$ be all of them where $x_i \in G \setminus C_G(y)$, $i \in \{1, 2, \dots, k\}$. Let $A_i = C_G(x_i) \setminus Z(G)$, $i \in \{1, 2, \dots, k\}$.

Clearly, $|A_1| = |A_2| = \cdots = |A_k|$. Again, since $C_G(x_i)$, $i \in \{1, 2, \dots, k\}$ are distinct centralizers, so A_i, $i \in \{1, 2, \dots, k\}$ are disjoint. For if $y' \in A_i \cap A_j$ for some $i \neq j$; $i, j \in \{1, 2, \dots, k\}$, then $y' \in C_G(x_i) \cap C_G(x_j) \setminus Z(G)$. Therefore, by Proposition 3.2 of [11], $C_G(y') = C_G(x_i) = C_G(x_j)$, which is a contradiction. Again $A_i \subset G \setminus C_G(y)$ for all $i \in \{1, 2, \dots, k\}$. For if there exists $y_1 \in A_i$ for some $i \in \{1, 2, \ldots, k\}$ such that $y_1 \in C_G(y)$, then $C_G(x_i) = C_G(y_1) = C_G(y)$ and so $x_i \in C_G(y)$, which is a contradiction. Hence, $G \setminus C_G(y) = \bigsqcup_{i=1}^{k} A_i$. Which implies $|G \setminus C_G(y)| = \sum_{i=1}^{k} |A_i| = k|A_i| = k|C_G(x_i) \setminus Z(G)|$. Moreover we have $|C_G(x) : Z(G)| = p$ for all $x \in G \setminus Z(G)$. Now,

$$k = \frac{| G \setminus C_G(y) |}{| C_G(x_i) \setminus Z(G) |} = \frac{| C_G(y) | (p^2 - 1)}{| Z(G) | (p-1)} = \frac{p(p-1)(p+1)}{(p-1)} = p^2 + p.$$

Therefore, $|Cent(G)| = p^2 + p + 2$.

Next suppose $| G : C_G(x) |= p$ for some $x \in G \setminus Z(G)$. If there exists $y \in G,\ y \neq x$ such that $| G : C_G(y) |= p$, then $C_G(x) = C_G(y)$. Because, if $C_G(x) \neq C_G(y)$, then $C_G(x) \subsetneq C_G(x)C_G(y) \subseteq G$ and so $G = C_G(x)C_G(y)$. Again by Remark 2.1 of [1], $C_G(x) \cap C_G(y) = Z(G)$. Now, $| G |= \frac{| C_G(x) || C_G(y) |}{| C_G(x) \cap C_G(y) |} = \frac{| G || G |}{p^2 | Z(G) |}$. Therefore $| G : Z(G) |= p^2$, which is a contradiction. Hence, G has exactly one centralizer of index p, namely $C_G(x)$, and remaining centralizers are of index p^2. We have $G = (G \setminus C_G(x)) \sqcup C_G(x)$. Let $z \in G \setminus C_G(x)$. Then $| G : C_G(z) |= p^2$. Now, applying the same arguments as above, we get

$$|Cent(G)| = \frac{| G \setminus C_G(x) |}{| C_G(z) \setminus Z(G) |} + 2 = \frac{| C_G(x) | (p-1)}{| Z(G) | (p-1)} + 2 = p^2 + 2.$$

□

In [3, Lemma 2.4], Ashrafi proved that $| Cent(D_{2n}) |= n + 2$ or $\frac{n}{2} + 2$ according to whether n is odd or even. Moreover, in [6, Theorem 5], Belcastro and Sherman proved that if p is a prime and $\frac{G}{Z(G)} \cong C_p \times C_p$, then $| Cent(G) |= p + 2$. The following theorem generalizes these results.

Theorem 2.3. *Let G be non-abelian and has an abelian normal subgroup of prime index. Then $| Cent(G) |= |G'| + 2$.*

Proof. Let H be an abelian normal subgroup of G of prime index p. Then $H = C_G(x)$ for some $x \in G \setminus Z(G)$. By Lemma 4 (page 303) of [7] , $|G| = p|G'||Z(G)|$, and $|G : C_G(y)| = |G'|$ for $y \in G \setminus C_G(x)$. By Theorem A of [11], G is a CA-group and by Proposition 3.2 of [11], $C_G(a) = C_G(b)$ for all $a, b \in G \setminus Z(G)$ with $ab = ba$. Therefore, $C_G(t) = C_G(y)$ for all $t \in C_G(y) \setminus Z(G)$. We have $G = (G \setminus C_G(x)) \sqcup C_G(x)$. Let k be the number of distinct centralizers produced by the elements of $G \setminus C_G(x)$. Let $C_G(x_1), C_G(x_2), \cdots, C_G(x_k)$ be all of them where $x_i \in G \setminus C_G(x)$, $i \in \{1, 2, \ldots, k\}$. Let $A_i = C_G(x_i) \setminus Z(G)$, $i \in \{1, 2, \ldots, k\}$.

Since $|G : C_G(y)| = |G'|$ for $y \in G \setminus C_G(x)$, $|A_1| = |A_2| = \cdots = |A_k|$. Again, since $C_G(x_i)$, $i \in \{1, 2, \ldots, k\}$ are distinct centralizers, therefore A_i, $i \in \{1, 2, \ldots, k\}$ are disjoint. For if $y' \in A_i \cap A_j$ for some $i \neq j$; $i, j \in \{1, 2, \ldots, k\}$, then $y' \in C_G(x_i) \cap C_G(x_j) \setminus Z(G)$. Therefore, by Proposition 3.2 of [11], $C_G(y') = C_G(x_i) = C_G(x_j)$, which is a contradiction. Again $A_i \subset G \setminus C_G(x)$ for all $i \in \{1, 2, \ldots, k\}$. For if there exists $y_1 \in A_i$ for some $i \in \{1, 2, \ldots, k\}$ such that $y_1 \in C_G(x)$, then $C_G(x_i) = C_G(y_1) = C_G(x)$ and so $x_i \in C_G(x)$, which is a contradiction. Hence, $G \setminus C_G(x) = \bigsqcup\limits_{i=1}^{k} A_i$. This implies $|G \setminus C_G(x)| = \sum\limits_{i=1}^{k} |A_i| = k|A_i| = k|C_G(x_i) \setminus Z(G)|$. Moreover we have $|G : Z(G)| = p|G'|$ and $|G : C_G(x_i)| = |G'|$ for $i \in \{1, 2, \ldots, k\}$. Therefore, $|C_G(x_i) : Z(G)| = p$ for $i \in \{1, 2, \ldots, k\}$. Now,

$$k = \frac{|\ G \setminus C_G(x)\ |}{|\ C_G(x_i) \setminus Z(G)\ |} = \frac{|\ C_G(x)\ |\ (p-1)}{|\ Z(G)\ |\ (p-1)} = |G'|.$$

Hence $|\ Cent(G)\ |= |G'| + 2$. □

Corollary 2.4. *$|\ Cent(D_{2n})\ |= n + 2$ or $\frac{n}{2} + 2$ according to whether n is odd or even.*

Proof. It is easy to see that (see [12, Exercise 9(c), pp 36]), if $G = D_{2n}$, then $|G : G'| = 2|Z(G)|$. Moreover, $|Z(G)| = 1$ or 2 according to whether n is odd or even. Therefore, $|G'| = n$ or $\frac{n}{2}$, according to whether n is odd or even. Hence, the result follows by Theorem 2.3. □

Corollary 2.5. *Let $p \geq q$ be primes. If $|G : Z(G)| = pq$, then $|\ Cent(G)\ |= p + 2$.*

Proof. Suppose $p = q$. Let $x \in G \setminus Z(G)$. Then $C_G(x)$ is an abelian normal subgroup of G of index p. Therefore, by [7, Lemma 4, pp 303], $|G| = p|G'||Z(G)|$ and so $|G'| = p$.

Again, suppose $p > q$. Let $xZ(G)$ be an element of order p in $\frac{G}{Z(G)}$. Since $\frac{C_G(x)}{Z(G)} \leq C_{\frac{G}{Z(G)}}(xZ(G))$, $|\frac{C_G(x)}{Z(G)}| = p$ and by Correspondence Theorem, $C_G(x)$ is an abelian normal subgroup of G of index q. Therefore, by [7, Lemma 4, pp 303], $|G| = q|G'||Z(G)|$ and so $|G'| = p$. Hence, the result follows by Theorem 2.3. □

Let $I(G)$ be the set of all solutions of the equation $x^2 = 1$ in G. Define $\alpha(G) = \frac{|I(G)|}{|G|}$. Then we have the following immediate corollary.

Corollary 2.6. *Let $|G| = 2^n m$, where n is any integer and $m > 1$ is an odd integer. If $\alpha(G) > \frac{1}{2}$, then $|\ Cent(G)\ |= |G'| + 2$.*

Proof. By [7, Exercise 2, pp 315], G contains an abelian subgroup of index 2. Therefore, by Theorem 2.3, $|\ Cent(G)\ |= |G'| + 2$. □

In the following simple proposition, we obtain a characterization of primitive n-centralizer groups.

Proposition 2.7. *Let G be non-abelian and both G and $\frac{G}{Z(G)}$ have an abelian normal subgroup of prime index. Then G is primitive n-centralizer if and only if $G' \cap Z(G) = \{1\}$.*

Proof. Suppose G is primitive n-centralizer. Then $\mid Cent(G) \mid = \mid Cent(\frac{G}{Z(G)}) \mid$. Therefore, by Theorem 2.3, $|G'| + 2 = |(\frac{G}{Z(G)})'| + 2 = |\frac{G'Z(G)}{Z(G)}| + 2 = |\frac{G'}{G' \cap Z(G)}| + 2$. Hence, $G' \cap Z(G) = \{1\}$.

Conversely, if $G' \cap Z(G) = \{1\}$, then G is primitive n-centralizer by [2, Lemma 3.1]. □

Proposition 2.8. *Let $\frac{G}{Z(G)}$ be non-abelian, n be an integer and p be a prime. If $\frac{G}{Z(G)} \cong C_n \rtimes C_p$, then G has an abelian normal subgroup of index p and $|G'| = n$.*

Proof. $\frac{G}{Z(G)}$ has a cyclic normal subgroup of order n, say H. Then $H = < xZ(G) >$, for some $xZ(G) \in \frac{G}{Z(G)}$. Clearly, $xZ(G) \notin Z(\frac{G}{Z(G)})$. For if $xZ(G) \in Z(\frac{G}{Z(G)})$, then $H = < xZ(G) > \subseteq Z(\frac{G}{Z(G)}) \subsetneq \frac{G}{Z(G)}$, and so $H = Z(\frac{G}{Z(G)})$. Therefore, $|\frac{G}{Z(G)} : Z(\frac{G}{Z(G)})| = p$, which is a contradiction since $\frac{G}{Z(G)}$ is non-abelian. Now, consider $C_{\frac{G}{Z(G)}}(xZ(G))$. Then $< xZ(G) > \subseteq C_{\frac{G}{Z(G)}}(xZ(G)) \subsetneq \frac{G}{Z(G)}$. Since $< xZ(G) >$ is of prime index, $< xZ(G) > = C_{\frac{G}{Z(G)}}(xZ(G))$. Again it is easy to see that

$$\frac{C_G(x)}{Z(G)} \leq C_{\frac{G}{Z(G)}}(xZ(G)) = < xZ(G) > \trianglelefteq \frac{G}{Z(G)}.$$

Since $xZ(G) \in \frac{C_G(x)}{Z(G)}$, $< xZ(G) > \subseteq \frac{C_G(x)}{Z(G)}$. That is $C_{\frac{G}{Z(G)}}(xZ(G)) \subseteq \frac{C_G(x)}{Z(G)}$ and hence $\frac{C_G(x)}{Z(G)} = C_{\frac{G}{Z(G)}}(xZ(G))$. Therefore, $\frac{C_G(x)}{Z(G)}$ is a cyclic normal subgroup of $\frac{G}{Z(G)}$ and hence $C_G(x)$ is abelian. Moreover, by Correspondence Theorem, $C_G(x) \trianglelefteq G$. Since $|\frac{C_G(x)}{Z(G)}| = n$, $|\frac{G}{C_G(x)}| = p$. Thus, we have $C_G(x)$ is an abelian normal subgroup of G of prime index p.

For the second part, by [7, Lemma 4, pp 303], we have $|G| = p|G'||Z(G)|$ and hence $|G'| = n$. □

In [6, Theorem 5], Belcastro and Sherman proved that if $\frac{G}{Z(G)} \cong D_{2p}$, p is an odd prime, then $\mid Cent(G) \mid = p + 2$. In [1, Proposition 2.2], Abdollahi, Amiri and Hassanabadi and in [4, Lemma 2], Ashrafi and Taeri generalized this result to the case where p is an arbitrary positive integer and proved that if $\frac{G}{Z(G)} \cong D_{2n}$, $n \geq 2$ is any integer, then $\mid Cent(G) \mid = n + 2$. In the following proposition, we generalize this result which will be used in characterising odd order 9-centralizer groups.

Proposition 2.9. *Let $\frac{G}{Z(G)}$ be non-abelian, n be an integer and p be a prime. If $\frac{G}{Z(G)} \cong C_n \rtimes C_p$, then $\mid Cent(G) \mid = n + 2$.*

Proof. By Proposition 2.8, G has an abelian normal subgroup of prime index and $|G'| = n$. Now using Proposition 2.3, we get $|\ Cent(G)\ | = n + 2$. $\square$

Lemma 2.10. *Let $\frac{G}{Z(G)}$ be non-abelian, n be an integer and p be a prime. If $\frac{G}{Z(G)} \cong C_n \rtimes C_p$, then G is a CA-group.*

Proof. Using Proposition 2.8 and Theorem A of [11], we get the result. $\square$

Proposition 2.11. *If $\frac{G}{Z(G)} \cong D_{2n}$, where n is an odd integer, then $G' \cap Z(G) = \{1\}$.*

Proof. By Proposition 2.8, both G and $\frac{G}{Z(G)}$ have an abelian normal subgroup of prime index. Again, by [1, Corollary 2.3], $|\ Cent(G)\ | = |\ Cent(\frac{G}{Z(G)})\ |$. Therefore, by Proposition 2.7, $G' \cap Z(G) = \{1\}$. $\square$

Proposition 2.12. *If $\frac{G}{Z(G)} \cong C_p \rtimes C_q$, where p and q are primes, $q \mid p - 1$, then $G' \cap Z(G) = \{1\}$.*

Proof. By Corollary 2.5, it follows that both G and $\frac{G}{Z(G)}$ have an abelian normal subgroup of prime index and $|\ Cent(G)\ | = |\ Cent(\frac{G}{Z(G)})\ |$. Therefore, by Proposition 2.7, $G' \cap Z(G) = \{1\}$. $\square$

For a finite group G, $Pr(G)$ denotes the probability that any two group elements commute. In [8, Proposition 5.2.16], it is proved that if $\frac{G}{Z(G)} \cong D_{2n}$, then $Pr(G) = \frac{n+3}{4n}$. In the following proposition we generalize this result.

Proposition 2.13. *Let $\frac{G}{Z(G)}$ be non-abelian, n be an integer and p be a prime. If $\frac{G}{Z(G)} \cong C_n \rtimes C_p$, then $Pr(G) = \frac{1}{p^2} + \frac{p^2-1}{p^2 n}$.*

Proof. By Proposition 2.8, G has an abelian normal subgroup of prime index p and $|G'| = n$. Therefore, by [7, Lemma 5, pp 303], $Pr(G) = \frac{1}{p^2} + \frac{p^2-1}{p^2 n}$. $\square$

Proposition 2.14. *If $\frac{G}{Z(G)} \cong D_{2n}$, where n is an odd integer, then $Pr(G) = Pr(\frac{G}{Z(G)})$.*

Proof. By Proposition 2.11, we have $G' \cap Z(G) = \{1\}$. Therefore, $Pr(G) = Pr(\frac{G}{Z(G)})$ by [13, Proposition 3]. $\square$

Proposition 2.15. *If $\frac{G}{Z(G)} \cong C_p \rtimes C_q$, where p and q are primes satisfying $q \mid p-1$, then $Pr(G) = Pr(\frac{G}{Z(G)})$.*

Proof. By Proposition 2.12, we have $G' \cap Z(G) = \{1\}$. Therefore, $Pr(G) = Pr(\frac{G}{Z(G)})$ by [13, Proposition 3]. $\square$

3. Odd Order Groups with Nine Centralizers

In this section, we study the structure of odd order groups having nine centralizers. The following lemma and proposition will be used in proving the main result of this section.

Lemma 3.1. *Let G be a CA-group. Then $\frac{C_G(x)}{Z(G)} = \frac{C_G(y)}{Z(G)}$ if and only if $C_G(x) = C_G(y)$ for any $x, y \in G \setminus Z(G)$.*

Proof. Suppose $\frac{C_G(x)}{Z(G)} = \frac{C_G(y)}{Z(G)}$. Then $xZ(G) \in \frac{C_G(x)}{Z(G)} = \frac{C_G(y)}{Z(G)}$. Therefore, $x \in C_G(x) \cap C_G(y) \setminus Z(G)$ and by Remark 2.1 of [1], $C_G(x) = C_G(y)$. The reverse implication is trivial. □

Proposition 3.2. *Let $|G|$ be odd. If $|\ Cent(G)\ | = 9$, then G cannot have a centralizer of index 5.*

Proof. Suppose that G has a centralizer of index 5. Let $\{x_1, x_2, \ldots, x_r\}$ be a set of pairwise non-commuting elements of G having maximal size. Suppose $X_i = C_G(x_i)$, $1 \leq i \leq r$ and $|G : X_1| \leq |G : X_2| \leq \cdots \leq |G : X_r|$. By Lemma 2.4 of [1], $5 \leq r \leq 8$. Suppose that $r = 5$. Then by Remark 2.1 of [1], $|\frac{G}{Z(G)}| \leq 16$. Clearly $|\frac{G}{Z(G)}| \neq 16, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4$. Hence $r \neq 5$ and so $|\ Cent(G)\ | < r + 4$.

Suppose $r = 6$. By Remark 2.1 of [1], $G = X_1 \cup X_2 \cup \cdots \cup X_6$ and by Lemma 3.3 of [14], $|G : X_2| \leq 5$. If $|G : X_2| = 5$, then by Lemma 3.3 of [14], $|G : X_2| = |G : X_3| = \cdots = |G : X_6| = 5$. Therefore, $|G| = \sum_{i=2}^{6} |X_i|$ and by [9, Theorem 1], $G = X_1X_2$. Again by Proposition 2.5 of [1], $X_1 \cap X_2 = Z(G)$. Since $|G|$ is odd, $|G : X_1| = 5$ or 3. If $|G : X_1| = 5$, then $|\frac{G}{Z(G)}| = 25$ and by Theorem 5 of [6], $|\ Cent(G)\ | = 7$. Again if $|G : X_1| = 3$, then $|\frac{G}{Z(G)}| = 15$, which is not possible since G is non-abelian. Hence, $r \neq 6$.

Lastly, suppose that $r = 7$. By Remark 2.1 of [1], $G = X_1 \cup X_2 \cup \cdots \cup X_7$ and by Lemma 3.3 of [14], $|G : X_2| \leq 6$. Since $|\ Cent(G)\ | = r + 2$, it follows from [1, Proposition 2.5] that there exists a proper non-abelian centralizer $C_G(x)$ which contains $C_G(x_{i_1}), C_G(x_{i_2})$ and $C_G(x_{i_3})$ for three distinct $i_1, i_2, i_3 \in \{1, 2, \ldots, r\}$. Therefore, $G = C_G(x) \cup X_{j_1} \cup X_{j_2} \cup X_{j_3} \cup X_{j_4}$ with $|G : X_{j_1}| \leq |G : X_{j_2}| \leq |G : X_{j_3}| \leq |G : X_{j_4}|$, where $j_1, j_2, j_3, j_4 \in \{1, 2, \ldots, r\} \setminus \{i_1, i_2, i_3\}$. By Lemma 3.3 of [14], $|G : X_{j_1}| \leq 4$. Therefore, we have seen that $|G : X_i| = 3$ for some $i \in \{1, 2, \ldots, r\}$. Now if $|G : X_2| = 5$, then $|G : X_1| = 3$ and so $G = X_1X_2$. Again by Proposition 2.5 of [1], $X_1 \cap X_2 = Z(G)$. Hence, $|\frac{G}{Z(G)}| = 15$, which is not possible since G is non-abelian. If $|G : X_2| = 3$, then by Proposition 2.5 of [1], $|\ Cent(G)\ | = 5$, which is again a contradiction.

Thus, $r = 8$. By Lemma 2.6 of [1], G is a CA-group and by Remark 2.1 of [1], $X_i \cap X_j = Z(G)$ for all distinct $i, j \in \{1, 2, \ldots, r\}$. Now $G = X_1 \cup X_2 \cup \cdots \cup X_8$ and by Lemma 3.3 of [14], $|G : X_2| \leq 7$. If $|G : X_2| = 7$, then $|G : X_1| = 5$ and by Exercise 1.8 of [10], $G = X_1X_2$. Since $X_1 \cap X_2 = Z(G)$, therefore $|\frac{G}{Z(G)}| = 35$, which is not possible since G is non-abelian. If $|G : X_2| = 5$, then $|G : X_1| = 3$ or 5. If $|G : X_1| = 3$, then $|\frac{G}{Z(G)}| = 15$, which is also not possible since G is non-abelian. Hence, $|G : X_1| = 5$.

We have $G =< X_1, X_2 >$. Therefore, by Theorem 4.2 of [14], $|\frac{G}{Z(G)}| \leq 49$. Since $|G|$ is odd and G is non-abelian, $|\frac{G}{Z(G)}| = 45$ or 25. If $|\frac{G}{Z(G)}| = 25$, then by Theorem 5 of [6], $| \, Cent(G) \, |= 7$. Therefore, $|\frac{G}{Z(G)}| = 45$. If X_1 or X_2 is normal in G, then $G = X_1X_2$ and since $X_1 \cap X_2 = Z(G)$, $|\frac{G}{Z(G)}| = 25$, and by Theorem 5 of [6], $| \, Cent(G) \, |= 7$. Hence $\frac{G}{Z(G)}$ is non-abelian. Again it is easy to see that $\frac{C_G(g)}{Z(G)} \leq C_{\frac{G}{Z(G)}}(gZ(G))$ for any $g \in G \setminus Z(G)$. Therefore, $|G : X_i| = 5, 9$ or 15 for $i \in \{1, 2, \ldots 8\}$. Let $xZ(G)$ be an element of order 5 in $\frac{G}{Z(G)}$. Then $C_{\frac{G}{Z(G)}}(xZ(G))$ is the normal sylow 5-subgroup of $\frac{G}{Z(G)}$. Since $\frac{C_G(x)}{Z(G)} \leq C_{\frac{G}{Z(G)}}(xZ(G))$, $\frac{C_G(x)}{Z(G)} = C_{\frac{G}{Z(G)}}(xZ(G))$. Hence, by Lemma 3.1, G has exactly one centralizer of index 9. Suppose $|G : X_3| = 9$. Then $|G : X_4| = |G : X_5| = |G : X_6| = |G : X_7| = |G : X_8| = 15$ and $|G| > \sum_{i=2}^{8} |X_i|$, which is a contradiction by [9, Theorem 1]. Therefore, $|G : X_3| = 5$. Similarly, we can show that $|G : X_4| = |G : X_5| = 5$. Thus, we have seen that $|G : X_1| = |G : X_2| = |G : X_3| = |G : X_4| = |G : X_5| = 5$.

Again we have $G = Z(G) \sqcup g_1Z(G) \sqcup g_2Z(G) \sqcup \cdots \sqcup g_{44}Z(G)$. Since $X_i \cap X_j = Z(G)$ for all distinct $i, j \in \{1, 2, \ldots, 8\}$, $X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5$ will contain 41 distinct cosets of $Z(G)$ in G, say $Z(G), g_1Z(G), \ldots, g_{40}Z(G)$. Therefore $X_6 \cup X_7 \cup X_8$ will contain $Z(G), g_{41}Z(G), \ldots, g_{44}Z(G)$. Hence, $|X_i : Z(G)| = 2$ for some $i \in \{6, 7, 8\}$, which is a contradiction since $|\frac{G}{Z(G)}| = 45$. Hence, $|G : X_2| \neq 5$. Therefore, $|G : X_2| = 3$ and by Proposition 2.5 of [1], $| \, Cent(G) \, |= 5$, which is again a contradiction. Therefore, G cannot have a centralizer of index 5. □

Now we are ready to state the main result of this section.

Theorem 3.3. *Let $|G|$ be odd. Then $| \, Cent(G) \, |= 9$ if and only if $\frac{G}{Z(G)} \cong C_7 \rtimes C_3$ or $C_7 \times C_7$.*

Proof. Let $\{x_1, x_2, \ldots, x_r\}$ be a set of pairwise non-commuting elements of G having maximal size. Suppose $X_i = C_G(x_i)$, $1 \leq i \leq r$ and $|G : X_1| \leq |G : X_2| \leq \cdots \leq |G : X_r|$. By [1, Lemma 2.4], we have $5 \leq r \leq 8$.

Suppose $r = 5$. By [1, Remark 2.1], $|\frac{G}{Z(G)}| \leq 16$. If $|\frac{G}{Z(G)}| = 9$, then $| Cent(G) |= 5$ by [6, Theorem 5]. Therefore, $|\frac{G}{Z(G)}| \neq 16, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4$. Hence, $r \neq 5$ and so $| Cent(G) |< r + 4$.

Suppose $r = 6$. Then by Remark 2.1 of [1], $G = X_1 \cup X_2 \cup \cdots \cup X_6$ and by Lemma 3.3 of [14], $|G : X_2| \leq 5$. By Proposition 3.2, we have $|G : X_2| \neq 5$. Therefore, $|G : X_1| \leq |G : X_2| = 3$ and by Proposition 2.5 of [1], $| Cent(G) |= 5$. Hence, $r \neq 6$.

Lastly suppose $r = 7$. Then by [1, Remark 2.1], $G = X_1 \cup X_2 \cup \cdots \cup X_7$, and by [14, Lemma 3.3], $|G : X_2| \leq 6$. By Proposition 3.2, we have $|G : X_2| \neq 5$. Therefore, $|G : X_2| = 3$ and by Proposition 2.5 of [1], $| Cent(G) |= 5$. Hence, $r \neq 7$.

Therefore, $r = 8$. By Lemma 2.6 of [1], G is a CA-group and by Remark 2.1 of [1], $X_i \cap X_j = Z(G)$ for all distinct $i, j \in \{1, 2, \ldots, r\}$.

Now, $G = X_1 \cup X_2 \cup \cdots \cup X_8$ and by Lemma 3.3 of [14], $|G : X_2| \leq 7$. If $|G : X_2| < 7$, then $|G : X_2| = 3$ and by Proposition 2.5 of [1], $| Cent(G) |= 5$. Hence, $|G : X_2| = 7$ and so by Lemma 3.3 of [14], $|G : X_2| = |G : X_3| = \cdots = |G : X_8| = 7$ and by Theorem 1 of [9], $G = X_1X_2$.

Since $|G : X_2| = 7$, we get $|G : X_1| = 3$ or 7. If $|G : X_1| = 3$, then $|\frac{G}{Z(G)}| = 21$ since $G = X_1X_2$ and $X_1 \cap X_2 = Z(G)$, and so $\dfrac{G}{Z(G)} \cong C_7 \rtimes C_3$. If $|G : X_1| = 7$, then since $G = X_1X_2$ and $X_1 \cap X_2 = Z(G)$, we obtain $|\frac{G}{Z(G)}| = 49$ and so $\dfrac{G}{Z(G)} \cong C_7 \times C_7$.

Conversely, if $\frac{G}{Z(G)} \cong C_7 \rtimes C_3$, then by Proposition 2.9 and if $\frac{G}{Z(G)} \cong C_7 \times C_7$, then by [6, Theorem 5], $| Cent(G) |= 9$. □

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