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# LIE REGULAR GENERATORS OF GENERAL LINEAR GROUPS II

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Abstract. In this paper, it is shown that the linear groups $GL(2, \mathbb{Z}_{3p^n})$ (where $p$ is an odd prime greater than 3) and $GL(2, \mathbb{Z}_{5p^n})$ (where $p$ is an odd prime greater than 5) can be generated by Lie regular matrices. Presentations of linear groups $GL(2, \mathbb{Z}_9)$, $GL(2, \mathbb{Z}_{14})$, $GL(2, \mathbb{Z}_{15})$, $GL(2, \mathbb{Z}_{22})$, $GL(2, \mathbb{Z}_{25})$, $GL(2, \mathbb{Z}_{26})$, $GL(2, \mathbb{Z}_{27})$ and $GL(2, \mathbb{Z}_{34})$ are also given.

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## 1. Introduction

An element $a$ of a ring $R$ is said to be Lie regular if $a = [e, u]$ for some idempotent $e$ and some unit $u$ in $R$. A Lie regular element which is also a unit is called a Lie regular unit. Lie regular elements and Lie regular units were introduced and studied by the authors (cf. [3]) In [3], Lie regular units are used to study generators of linear groups and the presentations of $GL(2, \mathbb{Z}_4)$, $GL(2, \mathbb{Z}_6)$, $GL(2, \mathbb{Z}_8)$, and $GL(2, \mathbb{Z}_{10})$ using these units are given. In this paper we continue this discussion and give Lie regular generators of $GL(2, \mathbb{Z}_{3p^n})$ where $p$ is a prime greater than 3 (Theorem 2.10 and Theorem 2.11) and $GL(2, \mathbb{Z}_{5p^n})$ where $p$ is a prime greater than 5 (Theorem 2.12 and Theorem 2.13). For distinct primes $p$ and $q$ such that $5 < p < q$, Lie regular generators of $GL(2, \mathbb{Z}_{pq^n})$ under some conditions are given (Theorem 2.14). As special cases of Theorem 3.6 and Theorem 3.8 in [3] and the results in this article, presentation of linear groups $GL(2, \mathbb{Z}_9)$, $GL(2, \mathbb{Z}_{14})$, $GL(2, \mathbb{Z}_{15})$, $GL(2, \mathbb{Z}_{22})$, $GL(2, \mathbb{Z}_{25})$, $GL(2, \mathbb{Z}_{26})$, $GL(2, \mathbb{Z}_{27})$ and $GL(2, \mathbb{Z}_{34})$ using Lie regular units are also given.

Throughout this paper, $\phi$ denotes the Euler's totient function and $\mathcal{U}(R)$ will denote the unit group of the ring $R$.

## 2. Generators of Linear Groups

In this section we give Lie regular generators of some linear groups. We first give some results on the order of linear groups $GL(2, \mathbb{Z}_n)$.

**Proposition 2.1.** ([3, Proposition 3.2]) *For any prime $p$, the order of the linear group $GL(2,\mathbb{Z}_{p^n})$ is $p^{2n-1}(p+1)(\phi(p^n))^2$.*

**Corollary 2.2.** *For any $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, where $p_i$'s are distinct primes, the order of $GL(2,\mathbb{Z}_n)$ is $\prod_{i=1}^{k} o(GL(2,\mathbb{Z}_{p_i^{\alpha_i}}))$.*

**Corollary 2.3.** *For any two distinct primes $p$ and $q$, the order of the linear group $GL(2,\mathbb{Z}_{pq})$ is $pq(p+1)(q+1)\phi(pq)^2$.*

Since for each invertible element $\alpha$ in $\mathbb{Z}_n$, the number of matrices in $M_2(\mathbb{Z}_n)$ having determinant $\alpha$ is equal to the order of $SL(2,\mathbb{Z}_n)$, we have the following corollary.

**Corollary 2.4.** *The order of the linear group $SL(2,\mathbb{Z}_n)$ is $\dfrac{o(GL(2,\mathbb{Z}_n))}{\phi(n)}$.*

**Remark 2.5.** *The elements $a = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & k \\ 1 & 0 \end{pmatrix}$, where $k$ is invertible in $\mathbb{Z}_n$, are Lie regular units in $M_2(\mathbb{Z}_n)$. Further, if 2 is invertible in $\mathbb{Z}_n$ then $c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is also a Lie regular unit in $M_2(\mathbb{Z}_n)$. This follows once we observe that*

$$\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} = \left[\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\right].$$

$$\begin{pmatrix} 0 & k \\ 1 & 0 \end{pmatrix} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & k \\ -1 & 0 \end{pmatrix}\right].$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \left[\begin{pmatrix} \frac{1}{2} & 1 \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right].$$

**Lemma 2.6.** *Let $p$ be an odd prime and $\alpha$ be an invertible primitive element modulo $p^n$. Let $a = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $c = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$, $x = (bc)^{\frac{\phi(p^n)}{2}}a$, and $y = c^{-1}b^{-1}cb$. Then the following hold for $a$, $b$, $c$, $x$, and $y$ as elements of $M_2(\mathbb{Z}_{p^n})$.*

(1) *$c^2$ is central element of $M_2(\mathbb{Z}_{p^n})$ and $o(c) = 2\phi(p^n)$.*
(2) *$o(cb) = o(bc) = \phi(p^n)$.*
(3) *$o(y) = o(cb) = \phi(p^n)$.*
(4) *$o(x) = p^n$.*
(5) *$yx = x^{\alpha^{-2}}y$ and $xy = yx^{\alpha^2}$.*

**Proof.** (1) Since $c^2 = \alpha I_2$, $c^2$ is in the center of $M_2(\mathbb{Z}_{p^n})$. Also, since $\alpha$ is a primitive element modulo $p^n$, we have $c^{2\phi(p^n)} = I_2$ and $c^k \neq I_2$ for any $k < \phi(p^n)$.

(2) We have $bc = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$ and $cb = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$. Thus, for any $i$, $(bc)^i = \begin{pmatrix} 1 & 0 \\ 0 & \alpha^i \end{pmatrix}$ and $(cb)^i = \begin{pmatrix} \alpha^i & 0 \\ 0 & 1 \end{pmatrix}$. Since $\alpha$ is a primitive element modulo $p^n$, $o(bc) = o(cb) = \phi(p^n)$.

(3) Since the order of $b$ is 2, we have $c^{-1}b^{-1}cb = c^{-1}bcb = c^{-2}(cb)^2$. Since $c^2$ is central and $o((cb)^2) = \frac{1}{2}o(c^2)$, we have $o(c^{-1}b^{-1}cb) = o(c^2)$.

(4) Since $(bc)^i = \begin{pmatrix} 1 & 0 \\ 0 & \alpha^i \end{pmatrix}$ and $\alpha^{\frac{\phi(p^n)}{2}} \equiv -1(mod\ p^n)$, we get $(bc)^{\frac{\phi(p^n)}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Thus, $x = (bc)^{\frac{\phi(p^n)}{2}} a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

It is now easy to see that $x^n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$ for all $n$. In particular, $x^{p^n} = \begin{pmatrix} 1 & 0 \\ p^n & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Since $x^k \neq I_2$ for any $k < p^n$ we have $o(x) = p^n$.

(5) As in Part 4, $x = (bc)^{\frac{\phi(p^n)}{2}} a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Also, $y = c^{-1}b^{-1}cb = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$.

Thus, $yx = \begin{pmatrix} \alpha & 0 \\ \alpha^{-1} & \alpha^{-1} \end{pmatrix}$ and $xy = \begin{pmatrix} \alpha & 0 \\ \alpha & \alpha^{-1} \end{pmatrix}$.

Also, $x^{\alpha^{-2}}y = \begin{pmatrix} 1 & 0 \\ \alpha^{-2} & 1 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \alpha^{-1} & \alpha^{-1} \end{pmatrix}$ and $yx^{\alpha^2} = \begin{pmatrix} \alpha & 0 \\ \alpha & \alpha^{-1} \end{pmatrix}$. Hence, $yx = x^{\alpha^{-2}}y$ and $xy = yx^{\alpha^2}$. □

The proof of the following lemma is similar to the above proof.

**Lemma 2.7.** *Let $p$ be an odd prime and $\alpha$ be an invertible primitive element modulo $p^n$. Let $a = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $c = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$, $x = (bc)^{\frac{\phi(p^n)}{2}} a$, and $y = c^{-1}b^{-1}cb$. Then the following hold for $a$, $b$, $c$, $x$, and $y$ as elements of $M_2(\mathbb{Z}_{2p^n})$.*

(1) *$c^2$ is a central element of $M_2(\mathbb{Z}_{2p^n})$ and $o(c) = 2\phi(p^n)$.*
(2) *$o(cb) = o(bc) = \phi(p^n)$.*
(3) *$o(y) = o(cb) = \phi(p^n)$.*
(4) *$o(x) = 2p^n$.*
(5) *$yx = x^{\alpha^{-2}}y$ and $xy = yx^{\alpha^2}$.*

**Lemma 2.8.** *If $p$ and $q$ are distinct odd primes then the order of any invertible element $x$ in $\mathbb{Z}_{p^m q^n}$ is at most $\frac{1}{2}\phi(p^m q^n)$.*

**Proof.** Let $x$ be an invertible element in $\mathbb{Z}_{p^m q^n}$. Then $gcd(x,\ p^m) = 1$ and $gcd(x,\ q^n) = 1$. Thus, $x^{\phi(p^m)} \equiv 1(mod\ p^m)$ and $x^{\phi(q^n)} \equiv 1(mod\ q^n)$. Consequently, $x^{\frac{1}{2}\phi(p^m q^n)} \equiv 1(mod\ p^m)$ and $x^{\frac{1}{2}\phi(p^m q^n)} \equiv 1(mod\ q^n)$. Thus, $x^{\frac{1}{2}\phi(p^m q^n)} \equiv 1(mod\ p^m q^n)$. □

**Lemma 2.9.** *If $p$ and $q$ are two distinct odd primes such that $q > p$ then for $0 \leq k < p$ there exists $\alpha \in \mathcal{U}(\mathbb{Z}_{pq^n})$ such that $\alpha \equiv\ k(mod\ p)$ and $\alpha$ is primitive element modulo $q^n$.*

**Proof.** The proof follows from the fact that if $\alpha$ is a primitive root modulo $q^n$ then $\alpha + kq^n$ is also a primitive root modulo $q^n$ and the order of $\mathcal{U}(\mathbb{Z}_{pq^n})$ is $(p-1)\phi(q^n)$. □

Let $p$ be a prime greater than 3 and let $\alpha \in \mathcal{U}(\mathbb{Z}_{3p^n})$ be a primitive element modulo $p^n$. Let $A = \begin{pmatrix} 1 & 0 \\ 1-\alpha^{-1} & 1 \end{pmatrix}$. Since $A^m = \begin{pmatrix} 1 & 0 \\ m(1-\alpha^{-1}) & 1 \end{pmatrix}$ for any $m$, $A^k = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ for some $k$ if and only if $1-\alpha^{-1}$ is invertible in $\mathbb{Z}_{3p^n}$. Since $\alpha \in \mathcal{U}(\mathbb{Z}_{3p^n})$, it follows that $\alpha \equiv 2(mod\ 3)$. Moreover, in this case, if $p \equiv\ 3(mod\ 4)$, then $\alpha^{\frac{\phi(p^n)}{2}} \equiv -1(mod\ 3p^n)$, and if $p \equiv\ 1(mod\ 4)$, then $\alpha^k$ is not congruent to $-1$ modulo $3p^n$ for any $k < \phi(p^n)$.

Also, since the order of $\mathcal{U}(\mathbb{Z}_{3p^n}) \simeq \mathcal{U}(\mathbb{Z}_3) \times \mathcal{U}(\mathbb{Z}_{p^n})$, there is an element $\beta$ in $\mathcal{U}(\mathbb{Z}_{3p^n})$, $\beta \neq \alpha^i$, $(1 \leq i \leq \phi(p^n))$ and the order of $\beta$ is 2.

**Theorem 2.10.** *Let $p$ be a prime greater than 3 such that $p \equiv 1\ (mod\ 4)$ and $\alpha \in \mathcal{U}(\mathbb{Z}_{3p^n})$ be a primitive element modulo $p^n$ such that $\alpha \equiv\ 2(mod\ 3)$. Then $GL(2, \mathbb{Z}_{3p^n})$ is generated by Lie regular elements $a = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $c = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$.*

**Proof.** Since $p \equiv 1(mod\ 4)$ and $\alpha \equiv\ 2(mod\ 3)$, as remarked above, $\alpha^i$ is not congruent to $-1$ modulo $3p^n$ for any $i < \phi(p^n)$. Let $G$ be the finite group generated by $a$, $b$, and $c$. Since $\alpha \equiv\ 2(mod\ 3)$, $1-\alpha^{-1}$ is invertible in $\mathbb{Z}_{3p^n}$. Let $m$ be the inverse of $1-\alpha^{-1}$. Let $x = (cabac^{-1}a)^m$ and $y = c^{-1}b^{-1}cb$. Then $x,\ y \in G$. Since $cabac^{-1}a = \begin{pmatrix} 1 & 0 \\ 1-\alpha^{-1} & 1 \end{pmatrix}$, $x = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Also, $y = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$. Thus, the order of $x$ is $3p^n$ and the order of $y$ is $\phi(p^n)$. Also, $yx = x^{\alpha^{-2}}y$. Let $H_1 = \langle x, y \mid x^{3p^n}, y^{\phi(p^n)}, yx = x^{\alpha^{-2}}y\rangle$. Then $H_1$ is a subgroup of $G$. Since $x^i \neq y^j$

for $1 \leq i \leq 3p^n - 1$ and $1 \leq j \leq \phi(p^n) - 1$ and the canonical form of $H_1$ is $\{x^i y^j \mid 0 \leq i \leq 3p^n - 1, 0 \leq j \leq \phi(p^n) - 1\}$, the order of $H_1$ is $3p^n\phi(p^n)$. Moreover, any arbitrary element of $H_1$ is of the form $\begin{pmatrix} \alpha^i & 0 \\ \gamma & \alpha^{-i} \end{pmatrix}$.

Let $r = bxb$ and $s = (b(cabac^{-1}a)^m a)^2$. Then $r,\ s \in G$, the order of $r$ is $3p^n$, and the order of $s$ is 2. Also, $sr = rs$. Let $H_2 = \langle r, s \mid r^{3p^n}, s^2, rs = sr\rangle$. Then $H_2$ is an abelian subgroup of $G$ and any arbitrary element of $H_2$ is of the form $\pm \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$. Thus, the order of $H_2$ is $6p^n$. Since $H_1 \cap H_2 = \{I_2\}$, $o(H_1H_2) = \dfrac{o(H_1)o(H_2)}{o(H_1 \cap H_2)} = 18p^{2n}\phi(p^n)$. Let $H$ be the subgroup of $G$ generated by $x,\ y,\ r$, and $s$. Since $x,\ y,\ r$, and $s$ are all of determinant 1, $H \subset SL(2, \mathbb{Z}_{3p^n})$. Also, $H_1H_2 \subset H$ and by Corollary 2.4, $o(SL(2, \mathbb{Z}_{3p^n})) = 24p^{2n-1}(p+1)\phi(p^n)$. Thus, the order of the subgroup $H$ is greater than equal to $18p^{2n}\phi(p^n)$ and is less than equal to $24p^{2n-1}(p+1)\phi(p^n)$. Since $p > 3$, $18p \geq 12(p+1)$, and hence $o(H_1H_2) > \frac{1}{2}o(SL(2, \mathbb{Z}_{3p^n}))$. Thus, the order of $H$ is $24p^{2n-1}(p+1)\phi(p^n)$.

Let $u = bc = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$ and $v = (cabac^{-1}a)^m a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $u,\ v \in G$, the order of $u$ is $\phi(p^n)$, and the order of $v$ is 2. Let $K = \langle u, v \mid u^{\phi(p^n)}, v^2, uv = vu\rangle$. Since $\alpha^i \neq \pm 1$ for any positive integer $i < \phi(p^n)$, $H \cap K = \{I_2\}$, and hence $o(HK) = \dfrac{o(H)o(K)}{o(H \cap K)} = 48p^{2n-1}(p+1)(\phi(p^n))^2$. Since $HK \subseteq GL(2, \mathbb{Z}_{3p^n})$ and by Proposition 2.1, the order of $GL(2, \mathbb{Z}_{3p^n})$ is $48p^{2n-1}(p+1)(\phi(p^n))^2$, $HK = GL(2, \mathbb{Z}_{3p^n})$. Hence, $GL(2, \mathbb{Z}_{3p^n})$ is generated by $a$, $b$ and $c$. □

**Theorem 2.11.** *Let $p$ be a prime greater than 3 such that $p \equiv 3\ (mod\ 4)$, $\alpha \in \mathcal{U}(\mathbb{Z}_{3p^n})$ be a primitive element modulo $p^n$ such that $\alpha \equiv\ 2(mod\ 3)$, and $\beta$ is an invertible element in $\mathbb{Z}_{3p^n}$ such that the order of $\beta$ is 2 and $\beta \neq \alpha^i$, $(1 \leq i \leq \phi(p^n))$. Then $GL(2, \mathbb{Z}_{3p^n})$ is generated by Lie regular elements $a = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $c = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$, and $d = \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix}$.*

**Proof.** Since $p \equiv 3(mod\ 4)$ and $\alpha \equiv 2(mod\ 3)$, as remarked above $\alpha^{\frac{\phi(p^n)}{2}} \equiv -1\ (mod\ 3p^n)$. Let $G$ be the finite group generated by $a$, $b$, $c$, and $d$.

Let $x = (bc)^{\frac{\phi(p^n)}{2}}a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $y = c^{-1}b^{-1}cb = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$. Then $x,\ y \in G$. As in Lemma 2.6, the order of $x$ is $3p^n$, and the order of $y$ is $\phi(p^n)$. Also, $yx = x^{\alpha^{-2}}y$. Let $H_1 = \langle x, y \mid x^{3p^n}, y^{\phi(p^n)}, yx = x^{\alpha^{-2}}y\rangle$. Then $H_1$ is a subgroup of $G$. Since $x^i \neq y^j$ for $1 \leq i \leq 3p^n - 1$ and $1 \leq j \leq \phi(p^n) - 1$ and the canonical form of $H_1$ is $\{x^i y^j \mid 0 \leq i \leq 3p^n - 1, 0 \leq j \leq \phi(p^n) - 1\}$, the order of $H_1$ is $3p^n\phi(p^n)$. Moreover, any element of $H_1$ is of the form $\begin{pmatrix} \alpha^i & 0 \\ \gamma & \alpha^{-i} \end{pmatrix}$.

Let $r = bxb = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $s = d^2 = \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}$. Then $r,\ s \in G$. Since $\beta$ is of order 2, order of $r$ is $3p^n$, and order of $s$ is 2. Also, $sr = rs$. Let $H_2 = \langle r, s \mid r^{3p^n}, s^2, rs = sr\rangle$. Then $H_2$ is an abelian subgroup of $G$ and any arbitrary element of $H_2$ is of the form $\begin{pmatrix} \beta^i & k \\ 0 & \beta^i \end{pmatrix}$, where $i = 0$ or 1. Thus, the order of $H_2$ is $6p^n$. Since $H_1 \cap H_2 = \{I_2\}$, $o(H_1H_2) = \dfrac{o(H_1)o(H_2)}{o(H_1 \cap H_2)} = 18p^{2n}\phi(p^n)$. Let $H$ be the subgroup of $G$ generated by $x,\ y,\ r$, and $s$. Since $x,\ y,\ r$, and $s$ are all of determinant 1, $H \subset SL(2, \mathbb{Z}_{3p^n})$. Also, $H_1H_2 \subset H$ and by Corollary 2.4, $o(SL(2, \mathbb{Z}_{3p^n})) = 24p^{2n-1}(p+1)\phi(p^n)$, the order of the subgroup $H$ is greater than equal to $18p^{2n}\phi(p^n)$ and is less than equal to $24p^{2n-1}(p + 1)\phi(p^n)$. Since $p > 3$, $18p \geq 12(p + 1)$, and hence $o(H_1H_2) > \frac{1}{2}o(SL(2, \mathbb{Z}_{3p^n}))$. Thus, the order of $H$ is $24p^{2n-1}(p + 1)\phi(p^n)$.

Let $u = bc = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$ and $v = db = \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}$. Then $u,\ v \in G$, the order of $u$ is $\phi(p^n)$, the order of $v$ is 2, and $uv = vu$. Let $K = \langle u, v \mid u^{\phi(p^n)}, v^2, uv = vu\rangle$. Since $\alpha^i \neq \beta$ for any $i \leq \phi(p^n)$, $\det(u^iv^j) \neq 1$ for $0 \leq i < \phi(p^n)$ and $j = 0$ or 1, $H \cap K = \{I_2\}$. Thus, $o(HK) = \dfrac{o(H)o(K)}{o(H \cap K)} = 48p^{2n-1}(p + 1)(\phi(p^n))^2$. Since $HK \subset GL(2, \mathbb{Z}_{3p^n})$ and by Proposition 2.1, the order of $GL(2, \mathbb{Z}_{3p^n})$ is $48p^{2n-1}(p+1)(\phi(p^n))^2$, $HK = GL(2, \mathbb{Z}_{3p^n})$. Hence, $GL(2, \mathbb{Z}_{3p^n})$ is generated by $a,\ b,\ c$ and $d$ in this case. □

Next we give generators of $GL(2, \mathbb{Z}_{5p^n})$, where $p$ is a prime greater than 5. First observe that, in this case, if $\alpha$ is a primitive element modulo $p^n$ such that $\alpha \equiv 2(mod\ 5)$ then $\alpha^{-1} \equiv 3(mod\ 5)$, and thus $1 - \alpha^{-1} \in \mathcal{U}(\mathbb{Z}_5)$. Also, if $1 - \alpha^{-1} \notin \mathcal{U}(\mathbb{Z}_{p^n})$ then $1 - \alpha \notin \mathcal{U}(\mathbb{Z}_{p^n})$. Thus, $\alpha = p^mk + 1$ for some nonnegative integer $k$ and positive integer $m < n$. But then $\alpha^{p^{n-m}} \equiv 1(mod\, p^n)$, a contradiction as $p^{n-m} < \phi(p^n)$ and $\alpha$ is a primitive element modulo $p^n$. Thus, $1-\alpha^{-1} \in \mathcal{U}(\mathbb{Z}_{p^n})$. Moreover, if $\alpha \equiv 2(mod\ 5)$ then the order of $\alpha$ modulo 5 is 4. It follows that if $\alpha$ is a primitive element modulo $p^n$ such that $\alpha \equiv 2(mod\ 5)$, then the order of $\alpha$ modulo $5p^n$ is $\phi(p^n)$ if $p \equiv 1(mod\ 4)$ and the order of $\alpha$ modulo $5p^n$ is $2\phi(p^n)$ if $p \equiv 3(mod\ 4)$ .

**Theorem 2.12.** *Let $p$ be a prime greater than 5 such that $p \equiv 3(mod\ 4)$ and $\alpha \in \mathcal{U}(\mathbb{Z}_{5p^n})$ be a primitive element modulo $p^n$ such that $\alpha \equiv 2(mod\ 5)$. Then*

*$GL(2,\mathbb{Z}_{5p^n})$ is generated by Lie regular elements $a = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $c = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$.*

**Proof.** Since $p \equiv 3(mod\ 4)$ and $\alpha \equiv 2(mod\ 5)$, the order of $\alpha$ modulo $5p^n$ is $2\phi(p^n)$. Let $G$ be the finite group generated by $a$, $b$, $c$, and $d$. Let $m$ be the inverse of $1-\alpha^{-1}$ in $\mathbb{Z}_{5p^n}$. Let $x = (cabac^{-1}a)^m$ and $y = c^{-1}b^{-1}cb$. Then $x,\ y \in G$. Since $cabac^{-1}a = \begin{pmatrix} 1 & 0 \\ 1-\alpha^{-1} & 1 \end{pmatrix}$, $x = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Also, $y = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$. As in Lemma 2.6, the order of $x$ is $5p^n$ and the order of $y$ is $2\phi(p^n)$. Also, $yx = x^{\alpha^{-2}}y$. Let

$$H_1 = \langle x, y \mid x^{5p^n}, y^{2\phi(p^n)}, yx = x^{\alpha^{-2}}y\rangle.$$

Since $x^i \neq y^j$ for $0 \leq i \leq 5p^n-1, 0 \leq j \leq 2\phi(p^n)-1$ and the canonical form of $H_1$ is $\{x^iy^j \mid 0 \leq i \leq 5p^n-1, 0 \leq j \leq 2\phi(p^n)-1\}$, the order of $H_1$ is $10p^n\phi(p^n)$. Moreover, any element of $H_1$ is of the form $\begin{pmatrix} \alpha^i & 0 \\ \gamma & \alpha^{-i} \end{pmatrix}$.

Let $r = bxb = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $s = (bxa)^2 = -I_2$. Then $r,\ s \in G$, the order of $r$ is $5p^n$ and the order of $s$ is 2. Also, $rs = sr$. Let $H_2 = \langle r, s \mid r^{5p^n}, s^2, rs = sr\rangle$. Then $H_2$ is an abelian subgroup of $G$ and the order of $H_2$ is $10p^n$. Clearly, $H_1 \cap H_2 = \{I_2\}$. Thus, $o(H_1H_2) = \dfrac{o(H_1)o(H_2)}{o(H_1 \cap H_2)} = 100p^{2n}\phi(p^n)$. Let $H$ be the subgroup of $G$ generated by $x,\ y,\ r$, and $s$. Since $x,\ y,\ r$, and $s$ are all of determinant 1, $H \subset SL(2,\mathbb{Z}_{5p^n})$. Also, $H_1H_2 \subset H$ and by Corollary 2.4, $o(SL(2,\mathbb{Z}_{5p^n})) = 120p^{2n-1}(p+1)\phi(p^n)$, the order of the subgroup $H$ generated by $x,\ y,\ r$ and $s$ is greater than equal to $100p^{2n}\phi(p^n)$ and is less than equal to $120p^{2n-1}(p+1)\phi(p^n)$. Since $p > 5$, $100p \geq 60(p+1)$, and hence the order of $H$ is $120p^{2n-1}(p+1)\phi(p^n)$. Thus, $H = SL(2,\mathbb{Z}_{5p^n})$.

Let $u = bc = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$ and $v = xa = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $u,\ v \in G$, the order of $u$ is $2\phi(p^n)$ and the order of $v$ is 2. Also, $uv = vu$. Let $K = \langle u, v \mid u^{2\phi(p^n)}, v^2, uv = vu\rangle$. Since the determinant of any nonidentity element in $K$ is different from 1, $H \cap K = \{I_2\}$, and hence $o(HK) = \dfrac{o(H)o(K)}{o(H \cap K)} = 480p^{(2n-1)}(p+1)(\phi(p^n))^2$. As $HK \subset GL(2,\mathbb{Z}_{5p^n})$ and by Proposition 2.1, the order of $GL(2,\mathbb{Z}_{5p^n})$ is $480p^{(2n-1)}(p+1)(\phi(p^n))^2$, $HK = GL(2,\mathbb{Z}_{5p^n})$. Hence, $GL(2,\mathbb{Z}_{5p^n})$ is generated by $a$, $b$, $c$, and $d$. □

**Theorem 2.13.** *Let $p$ be a prime greater than 5 such that $p \equiv 1\ (mod\ 4)$ and $\alpha \in \mathcal{U}(\mathbb{Z}_{5p^n})$ be a primitive element modulo $p^n$ such that $\alpha \equiv 2(mod\ 5)$ and*

*$\beta$ is an invertible element in $\mathbb{Z}_{5p^n}$ such that the order of $\beta$ is 4 and $\beta \neq \alpha^i$, ($1 \leq i \leq \phi(p^n)$). Then $GL(2, \mathbb{Z}_{5p^n})$ is generated by Lie regular elements $a = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $c = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$, and $d = \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix}$.*

**Proof.** Since $p \equiv 1(mod\ 4)$ and $\alpha \equiv\ 2(mod\ 5)$, the order of $\alpha$ modulo $5p^n$ is $\phi(p^n)$. Let $G$ be the finite group generated by $a$, $b$, $c$, and $d$. Let $m$ be the inverse of $1-\alpha^{-1}$ in $\mathbb{Z}_{5p^n}$. Let $x = (cabac^{-1}a)^m$ and $y = c^{-1}b^{-1}cb$. Then $x,\ y \in G$. Since $cabac^{-1}a = \begin{pmatrix} 1 & 0 \\ 1-\alpha^{-1} & 1 \end{pmatrix}$, $x = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Also, $y = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$. As in Lemma 2.6, the order of $x$ is $5p^n$ and the order of $y$ is $\phi(p^n)$. Also, $yx = x^{\alpha^{-2}}y$. Let

$$H_1 = \langle x, y \mid x^{5p^n}, y^{\phi(p^n)}, yx = x^{\alpha^{-2}}y\rangle.$$

Since $x^i \neq y^j$ for $0 \leq i \leq 5p^n - 1, 0 \leq j \leq \phi(p^n) - 1$ and the canonical form of $H_1$ is $\{x^iy^j \mid 0 \leq i \leq 5p^n - 1, 0 \leq j \leq \phi(p^n) - 1\}$, the order of $H_1$ is $5p^n\phi(p^n)$. Also, elements of $H_1$ are of the form $\begin{pmatrix} \alpha^i & 0 \\ \gamma & \alpha^{-i} \end{pmatrix}$.

Let $r = bxb = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $s = d^{-1}b^{-1}db = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$. Then $r,\ s \in G$, the order of $r$ is $5p^n$ and the order of $s$ is 4. Also, $rs^2 = sr$. Let $H_2 = \langle r, s \mid r^{5p^n}, s^4, rs^2 = sr\rangle$. Since $r^i \neq s^j$ for $0 \leq i \leq 5p^n - 1, 0 \leq j \leq 3$, and the canonical form of $H_2$ is $\{r^is^j \mid 0 \leq i \leq 5p^n - 1, 0 \leq j \leq 3\}$, the order of $H_2$ is $20p^n$. Also, any element of $H_2$ is of the form $\begin{pmatrix} \beta^i & \delta \\ 0 & \beta^{-i} \end{pmatrix}$. Since $\beta \neq \alpha^i$, ($1 \leq i \leq \phi(p^n)$), $H_1 \cap H_2 = \{I_2\}$. Therefore, $o(H_1H_2) = \dfrac{o(H_1)o(H_2)}{o(H_1 \cap H_2)} = 100p^{2n}\phi(p^n)$. Let $H$ be the subgroup of $G$ generated by $x$, $y$, $r$, and $s$. Since $x$, $y$, $r$, and $s$ are all of determinant 1, $H \subset SL(2, \mathbb{Z}_{5p^n})$. Also, $H_1H_2 \subset H$ and by Corollary 2.4, $o(SL(2, \mathbb{Z}_{5p^n})) = 120p^{2n-1}(p+1)\phi(p^n)$, the order of the subgroup $H$ generated by $x$, $y$, $r$ and $s$ is greater than equal to $100p^{2n}\phi(p^n)$ and is less than equal to $120p^{2n-1}(p+1)\phi(p^n)$. Since $p > 5$, $100p \geq 60(p+1)$, and hence the order of $H$ is $120p^{2n-1}(p+1)\phi(p^n)$. Thus, $H = SL(2, \mathbb{Z}_{5p^n})$.

Let $u = bc = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$ and $v = d^2 = \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}$. Then $u, v \in G$, the order of $u$ is $\phi(p^n)$, and the order of $v$ is 2. Let $K = \langle u, v \mid u^{2\phi(p^n)}, v^2, uv = vu\rangle$. Then the order of $K$ is $4\phi(p^n)$. Since determinant of nonidentity elements of $K$ is different from 1, we have $H \cap K = \{I_2\}$, and hence $o(HK) = \dfrac{o(H)o(K)}{o(H \cap K)} = 480p^{(2n-1)}(p+1)(\phi(p^n))^2$. As $HK \subset GL(2, \mathbb{Z}_{5p^n})$ and by Proposition 2.1, the order of $GL(2,\mathbb{Z}_{5p^n})$ is $480p^{(2n-1)}(p+1)(\phi(p^n))^2$ , $HK = GL(2,\mathbb{Z}_{5p^n})$. Hence, $GL(2,\mathbb{Z}_{5p^n})$ is generated by $a$, $b$, $c$ and $d$. □

For the case when $p$ and $q$ are distinct primes greater than 5, we have the following result.

**Theorem 2.14.** *Let $p$ and $q$ be distinct primes such that $5 < p < q$. Suppose there exists an invertible element $\alpha$ of order $\frac{(p-1)\phi(q^n)}{2}$ in $\mathcal{U}(\mathbb{Z}_{pq^n})$ such that $\alpha$ is a primitive element modulo $q^n$. Let $\beta$ be an invertible element in $\mathbb{Z}_{pq^n}$ such that the order of $\beta$ is $p-1$ and $\beta \neq \alpha^i$ for any $i$, $1 \leq i \leq \phi(q^n)$. Then $GL(2,\mathbb{Z}_{pq^n})$ is generated by Lie regular elements $a = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $c = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$, and $d = \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix}$.*

**Proof.** Let $G$ be a finite group generated by $a$, $b$, $c$, and $d$. First observe that $1-\alpha^{-1}$ is invertible in $\mathbb{Z}_{pq^n}$. Let $m$ be the inverse of $1-\alpha^{-1}$ in $\mathbb{Z}_{pq^n}$. Let $x = (cabac^{-1}a)^m$, and $y = c^{-1}b^{-1}cb$. Then $x,\ y \in G$. Since $cabac^{-1}a = \begin{pmatrix} 1 & 0 \\ 1-\alpha^{-1} & 1 \end{pmatrix}$, $x = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Also, $y = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$. As in Lemma 2.6, the order of $x$ is $pq^n$ and the order of $y$ is $\frac{(p-1)\phi(q^n)}{2}$. Also, $yx = x^{\alpha^{-2}}y$. Let

$$H_1 = \langle x, y \mid x^{pq^n}, y^{\frac{(p-1)\phi(q^n)}{2}}, yx = x^{\alpha^{-2}}y\rangle.$$

Since $x^i \neq y^j$ for $0 \leq i \leq pq^n-1, 0 \leq j \leq \phi(q^n)-1$ and the canonical form of $H_1$ is $\{x^iy^j \mid 0 \leq i \leq pq^n-1, 0 \leq j \leq \phi(q^n)-1\}$, the order of $H_1$ is $\dfrac{pq^n\phi(pq^n)}{2}$. Moreover, any element of $H_1$ is of the form $\begin{pmatrix} \alpha^i & 0 \\ \gamma & \alpha^{-i} \end{pmatrix}$.

Let $r = bxb$ and $s = d^{-1}b^{-1}db$. Then $r,\ s \in G$, the order of $r$ is $pq^n$, and the order of $s$ is $p-1$. Also, $sr = r^{\beta^{-2}}s$. Let $H_2 = \langle r, s \mid r^{pq^n}, s^{p-1}, sr = r^{\beta^{-2}}s\rangle$. Further, $r^i \neq s^j$ for $0 \leq i \leq pq^n-1, 0 \leq j \leq p-2$ and the canonical form of $H_2$ is $\{r^is^j \mid 0 \leq i \leq pq^n-1, 0 \leq j \leq p-2\}$, the order of $H_2$ is $pq^n(p-1)$. Also, any element of $H_2$ is of the form $\begin{pmatrix} \beta^i & 0 \\ \delta & \beta^{-i} \end{pmatrix}$. Since $\beta \neq \alpha^i$, $(1 \leq i \leq \phi(q^n))$, $H_1 \cap H_2 = \{I_2\}$, and therefore $o(H_1H_2) = \dfrac{o(H_1)o(H_2)}{o(H_1 \cap H_2)} = \dfrac{p^2q^{2n}(p-1)^2\phi(q^n)}{2}$. Let $H$ be the subgroup of $G$ generated by $x$, $y$, $r$, and $s$. Since $x$, $y$, $r$, and $s$ are all of determinant 1, $H \subset SL(2,\mathbb{Z}_{pq^n})$. Also, $H_1H_2 \subset H$ and by Corollary 2.4, $o(SL(2,\mathbb{Z}_{5p^n})) = p(p-1)(p+1)(q+1)q^{2n-1}\phi(q^n)$, the order of the subgroup $H$ generated by $x$, $y$, $r$ and $s$ is greater than equal to $\dfrac{p^2q^{2n}(p-1)^2\phi(q^n)}{2}$ and is less than equal to $p(p-1)(p+1)(q+1)q^{2n-1}\phi(q^n)$. Since for $3 < p < q$, $p(p-1) > 2(p+1)$ and $q > \frac{q+1}{2}$, we have $pq(p-1) > (p+1)(q+1)$. Thus, the order of $H$ is $p(p-1)(p+1)(q+1)q^{2n-1}\phi(q^n)$, and hence $H = SL(2, \mathbb{Z}_{pq^n})$.

Let $u = bc$ and $v = d^{\frac{p-1}{2}}$. Then $u \in G$ and the order of $u$ is $\dfrac{(p-1)\phi(q^n)}{2}$. $K = \langle u, v \mid u^{2\phi(p^n)}, v^2, uv = vu\rangle$. Thus, $H \cap K = \{I_2\}$, and hence $o(HK) = \dfrac{o(H)o(K)}{o(H\cap K)} = p(p-1)^2(p+1)(q+1)q^{2n-1}(\phi(q^n))^2$. As $HK \subset GL(2, \mathbb{Z}_{pq^n})$ and by Proposition 2.1, the order of $GL(2, \mathbb{Z}_{pq^n})$ is $p(p-1)^2(p+1)(q+1)q^{2n-1}(\phi(q^n))^2$, $HK = GL(2, \mathbb{Z}_{pq^n})$. Hence, $GL(2, \mathbb{Z}_{pq^n})$ is generated by $a$, $b$, $c$ and $d$. $\square$

## 3. Presentations of Some Linear Groups

In this section, we use Lie regular units to give presentation of linear groups over some finite rings.

**Theorem 3.1.** *Let* $a = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *and* $c = \begin{pmatrix} 0 & 5 \\ 1 & 0 \end{pmatrix}$. *Then*

$$\begin{aligned} GL(2,\mathbb{Z}_9) = \langle a, b, c \mid a^2,\ b^2,\ c^{12},\ c^2a = ac^2, c^2b = bc^2,\ (ab)^3,\ (bc)^6,\\ (ac)^{12} = (ca)^{12},\ ((bc)^3a)^9,\ c^4bca = ((bc)^3a)^4(cb)^2\rangle. \end{aligned}$$

**Proof.** Let $G$ be the subgroup of $GL(2, \mathbb{Z}_9)$ generated by $a$, $b$, $c$ and having the presentation

$$\begin{aligned} \langle a, b, c \mid a^2,\ b^2,\ c^{12},\ c^2a = ac^2, c^2b = bc^2,\ (ab)^3,\ (bc)^6,\\ (ac)^{12} = (ca)^{12},\ ((bc)^3a)^9,\ c^4bca = ((bc)^3a)^4(cb)^2\rangle. \end{aligned}$$

Let $x = (bc)^3a$. and $y = c^{-1}b^{-1}cb$. Then $x$, $y \in G$. Using the relators in $G$ we get the order of $x$ is 9, the order of $y$ is 6 and $yx = x^4y$. Let $H_1 = \langle x, y \mid x^9,\ y^6,\ yx = x^4y\rangle$. $H_1$ is a subgroup of $G$ and the canonical form of $H_1$ is $\{x^iy^j \mid 0 \le i \le 8, 0 \le j \le 5\}$. Since no power of $x$ is same as any power of $y$, the order of $H_1$ is 54 and any element of $H_1$ is of the form $\begin{pmatrix} 5^i & 0 \\ \beta & 2^i \end{pmatrix}$.

Let $r = bxb$. Then $r \in G$. Also, since the order of $x$ is 9 and $r$ is a conjugate of $x$, the order of $r$ is 9. Let $H_2 = \langle r\rangle$, the cyclic group generated by $r$. Then $H_2$ is also a subgroup of $G$ and the order of $H_2$ is 9. Also, elements of $H_2$ are of the form $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$. Clearly, $H_1 \cap H_2 = \{I_2\}$. Thus, $o(H_1H_2) = \dfrac{o(H_1)o(H_2)}{o(H_1 \cap H_2)} = 486$. Let $H$ be the subgroup of $G$ generated by $x$, $y$, and $r$. Since $x$, $y$, and $r$ are all of determinant 1, $H \subset SL(2, \mathbb{Z}_9)$. Also, $H_1H_2 \subset H$ and by Corollary 2.4, the order of $SL(2, \mathbb{Z}_9)$ is 648, the order of the subgroup $H$ is greater than equal to 486 and less than equal to 648. Now, since $486 > \frac{1}{2}o(SL(2, \mathbb{Z}_9))$, the order of $H$ is 648, and hence $H = SL(2, \mathbb{Z}_9)$.

Now $bc \in G$, being of determinant 5, does not belong to $H$ $(= SL(2,\mathbb{Z}_9))$. Let $K = \langle bc\rangle$. Since the order of $bc$ is 6, the order of $K$ is 6. Also, $K \cap H = \{I_2\}$. Therefore, $o(HK) = \dfrac{o(H)o(K)}{o(H\cap K}= 3888$. Since $HK \subset G \leq GL(2,\mathbb{Z}_9)$ and by Proposition 2.1, $o(GL(2,\mathbb{Z}_9)) = 3888$, we get $HK = G = GL(2,\mathbb{Z}_9)$. Hence, the theorem follows. □

**Theorem 3.2.** *Let* $a = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *and* $c = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}$. *Then*

$$\begin{aligned} GL(2,\mathbb{Z}_{14}) = \langle a,\ b,\ c \mid a^2,\ b^2,\ c^{12}, c^2a = ac^2,\ c^2b = bc^2,\ (ab)^3,\ (bc)^6,\\ (ac)^6,\ ((bc)^3a)^{14}, c^4bca = ((bc)^3a)^{11}(cb)^2\rangle. \end{aligned}$$

**Proof.** Let $a,\ b,\ c$ be as above and let $G$ be the subgroup of $GL(2,\mathbb{Z}_{14})$ having the presentation

$$\begin{aligned} \langle a,\ b,\ c \mid a^2,\ b^2,\ c^{12}, c^2a = ac^2,\ c^2b = bc^2,\ (ab)^3,\ (bc)^6,\ (ac)^6,\\ ((bc)^3a)^{14}, c^4bca = ((bc)^3a)^{11}(cb)^2\rangle. \end{aligned}$$

Let $x = (bc)^3a$ and $y = c^{-1}b^{-1}cb$. Then $x,\ y \in G$. Using the relators in $G$, it is clear that the order of $x$ is 14, the order of $y$ is 6, and $yx = x^{11}y$. Let $H_1 = \langle x, y \mid x^{14},\ y^6,\ yx = x^{11}y\rangle$. Then $H_1$ is a subgroup of $G$. Since $x^i \neq y^j$ for $0 \leq i \leq 13, 0 \leq j \leq 5$ and the canonical form of $H_1$ is $\{x^iy^j \mid 0 \leq i \leq 13, 0 \leq j \leq 5\}$, we get that the order of $H_1$ is 84. Since $x = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$, it can be seen that elements of $H_1$ are of the form $\begin{pmatrix} 3^i & 0 \\ \beta & 5^i \end{pmatrix}$.

Let $r = bxb$. Then $r \in G$. Since the order of $x$ is 14 and $r$ is a conjugate of $x$, the order of $r$ is 14. Let $H_2 = \langle r\rangle$, the cyclic subgroup of $G$ generated by $r$. Then the order of $H_2$ is 14 and it can be shown that the elements of $H_2$ are of the form $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$. Clearly, $H_1 \cap H_2 = \{I_2\}$. Thus, $o(H_1H_2) = \dfrac{o(H_1)o(H_2)}{o(H_1\cap H_2)} = 1176$. Let $H$ be the subgroup of $G$ generated by $x,\ y$, and $r$. Since $x,\ y$, and $r$ are all of determinant 1, $H \subset SL(2,\mathbb{Z}_{14})$. Also, $H_1H_2 \subset H$. Thus, the order of $H$ is greater than equal to 1176. Since, by Corollary 2.4, the order of $SL(2,\mathbb{Z}_{14})$ is 2016 and $1176 > \frac{1}{2}\cdot o(SL(2,\mathbb{Z}_{14}))$, it follows that the order of $H$ is 2016. Hence, $H = SL(2,\mathbb{Z}_{14})$.

Let $w = bc \in G$. Then $w$, being of determinant 3, does not belong to $H$. Let $K = \langle w\rangle$, the cyclic group generated by $w$. Then $K$ is a subgroup of $G$, and since the order of $w$ is 6, $o(K) = 6$. Since $H \cap K = \{I_2\}$, $o(HK) = \dfrac{o(H)o(K)}{o(H\cap K} = 12096$. Now $HK \subset G \leq GL(2,\mathbb{Z}_{14})$ and by Proposition 2.1, $o(GL(2,\mathbb{Z}_{14})) = 12096$. Thus, $HK = G = GL(2,\mathbb{Z}_{14})$. Hence, the theorem follows . □

**Theorem 3.3.** *Let* $a = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *and* $c = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$. *Then*

$$GL(2,\mathbb{Z}_{15}) = \langle a,\ b,\ c \mid a^2,\ b^2,\ c^8,\ c^2a = ac^2,\ c^2b = bc^2,\ (ab)^3,\ (bc)^4, (ac)^8,\\ b(ac)^4 = (ac)^4b,\ (cabac^7a)^{30}, (cb)^2(cabac^7a)^2 = (cabac^7a)^8(cb)^2,\\ (b(cabac^7a)^2)^5ca = ca(b(cabac^7a)^2)^5\rangle.$$

**Proof.** Let $G$ be the subgroup of $GL(2,\mathbb{Z}_{15})$ generated by $a$, $b$, $c$ and having the presentation

$$\langle a,\ b,\ c \mid a^2,\ b^2,\ c^8,\ c^2a = ac^2,\ c^2b = bc^2,\ (ab)^3,\ (bc)^4, (ac)^8,\\ b(ac)^4 = (ac)^4b,\ (cabac^7a)^{30}, (cb)^2(cabac^7a)^2 = (cabac^7a)^8(cb)^2,\\ (b(cabac^7a)^2)^5ca = ca(b(cabac^7a)^2)^5\rangle.$$

Let $x = (cabac^7a)^2$ and $y = c^{-1}b^{-1}cb$. Then $x,\ y \in G$. Also, using the relators in $G$, it follows that the order of $x$ is 15, the order of $y$ is 4, and $yx = x^4y$. Let $H_1 = \langle x, y \mid x^{15},\ y^4,\ yx = x^4y\rangle$. Then $H_1$ is a subgroup of $G$. Since $x^i \neq y^j$ for $0 \le i \le 14, 0 \le j \le 3$, and the canonical form of elements of $H_1$ is $\{x^iy^j \mid 0 \le i \le 14, 0 \le j \le 3\}$, the order of $H_1$ is 60. Since $x = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}$, any element of $H_1$ is of the form $\begin{pmatrix} 2^i & 0 \\ \beta & 8^i \end{pmatrix}$.

Let $r = bxb$ and $s = c^4(ac)^4$. Clearly, $r,\ s \in G$. Since the order of $x$ is 15 and $r$ is a conjugate of $x$, the order of $r$ is 15. Since both $c^4$ and $(ac)^4$ are of order 2 and are commuting, the order of $s$ is 2. Also, as $c^4$ and $(ac)^4$ commute with $a$, $b$, $c$, it follows that $rs = sr$. Let $H_2 = \langle r, s \mid r^{15}, s^2, rs = sr\rangle$. Then $H_2$ is an abelian subgroup of $G$. Since $r^i \neq s^j$ for $0 \le i \le 19, 0 \le j \le 3$, it follows that the order of $H_2$ is 30. Moreover, any element of $H_2$ is of the form $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$. Clearly, $H_1 \cap H_2 = \{I_2\}$. Thus, $o(H_1H_2) = \dfrac{o(H_1)o(H_2)}{o(H_1 \cap H_2)} = 1800$. Let $H$ be the subgroup of $G$ generated by $x, y$ and $r$. Since $x,\ y$, and $r$ are all of determinant 1, $H \subset SL(2,\mathbb{Z}_{14})$. Also, $H_1H_2 \subset H$. Thus, the order of $H$ is greater than equal to 1800. Since, by Corollary 2.4, the order of $SL(2,\mathbb{Z}_{15})$ is 2880 and $1800 > \frac{1}{2} \cdot o(SL(2,\mathbb{Z}_{15}))$, it follows that the order of $H$ is 2880. Hence, $H = SL(2,\mathbb{Z}_{15})$.

Now, let $w = bc$ and $z = sb$. Then $w, z \in G$. Since determinants of $w$ and $z$ are being of determinant 2 and $-1$ respectively, $w$ and $z$ do not belong to $H$. Also, the order of $w$ is 4, the order of $z$ is 2, and $wz = zw$. Let $K = \langle w, z \mid w^4, z^2, wz = zw\rangle$. Then $K$ is an abelian subgroup of $G$. Since the order of $w$ is 4 and the order of $z$ is 2, $o(K) = 8$. Since $H \cap K = \{I_2\}$, $o(HK) = \dfrac{o(H)o(K)}{o(H \cap K)} = 23040$. As $HK \subset G \le GL(2,\mathbb{Z}_{15})$ and by Proposition 2.1, $o(GL(2,\mathbb{Z}_{15})) = 23040$, we get $HK = G = GL(2,\mathbb{Z}_{15})$. Hence, the theorem follows. □

**Theorem 3.4.** *Let* $a = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *and* $c = \begin{pmatrix} 0 & 7 \\ 1 & 0 \end{pmatrix}$. *Then*

$$\begin{aligned} GL(2,\mathbb{Z}_{22}) = \langle a,\ b,\ c \mid a^2,\ b^2,\ c^{20},\ c^2a = ac^2,\ c^2b = bc^2,\ (ab)^3,\ (bc)^{10}, \\ (ac)^6 = c^{10}(ca)^6,\ ((bc)^5a)^{22},\ c^4(bc)^3a = ((bc)^5a)^9(cb)^2\rangle. \end{aligned}$$

**Proof.** Let $G$ be the subgroup of $GL(2,\mathbb{Z}_{22})$ generated by $a,\ b,\ c$ and having presentation

$$\begin{aligned} \langle a,\ b,\ c \mid a^2,\ b^2,\ c^{20},\ c^2a = ac^2,\ c^2b = bc^2,\ (ab)^3,\ (bc)^{10}, \\ (ac)^6 = c^{10}(ca)^6,\ ((bc)^5a)^{22},\ c^4(bc)^3a = ((bc)^5a)^9(cb)^2\rangle. \end{aligned}$$

Let $x = (bc)^5a$ and $y = c^{-1}b^{-1}cb$. Then $x,\ y \in G$. Using the relators in $G$, it is clear that the order of $x$ is 22, the order of $y$ is 10, and $yx = x^9y$. Let $H_1 = \langle x, y \mid x^{22}, y^{10}, yx = x^9y\rangle$. Then $H_1$ is a subgroup of $G$. Since $x^i \neq y^j$ for $0 \le i \le 21, 0 \le j \le 9$ and the canonical form of elements of $H_1$ is $\{x^iy^j \mid 0 \le i \le 21, 0 \le j \le 9\}$, it follows that the order of $H_1$ is 220. Since $x = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 7 & 0 \\ 0 & 19 \end{pmatrix}$, it follows that elements of $H_1$ are of the form $\begin{pmatrix} 7^i & 0 \\ \beta & 19^i \end{pmatrix}$.

Let $r = bxb$. Then $r \in G$. Since the order of $x$ is 22 and $r$ is a conjugate of $s$, the order of $r$ is 22. Let $H_2 = \langle r\rangle$, the cyclic subgroup of $G$ generated by $r$. Then the order of $H_2$ is 22. It can be seen that the elements of $H_2$ are of the form $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$. Clearly $H_1 \cap H_2 = \{I_2\}$. Thus, $o(H_1H_2) = \dfrac{o(H_1)o(H_2)}{o(H_1 \cap H_2)} = 4840$. Let $H$ be the subgroup of $G$ generated by $x, y$ and $r$. Since $x,\ y$, and $r$ are all of determinant 1, $H \subset SL(2,\mathbb{Z}_{22})$. Also, $H_1H_2 \subset H$. Thus, the order of $H$ is greater than equal to 4840. Since, by Corollary 2.4, the order of $SL(2,\mathbb{Z}_{22})$ is 7920 and $4840 > \frac{1}{2} \cdot o(SL(2,\mathbb{Z}_{22}))$, it follows that the order of $H$ is 7920. Hence, $H = SL(2,\mathbb{Z}_{22})$.

Further, $w = bc \in G$, being of determinant 7, does not belong to $H$. Let $K = \langle w\rangle$, the cyclic group generated by $w$. Then $K$ is a subgroup of $G$ and since the order of $w$ is 10, $o(K) = 10$. Since $H \cap K = \{I_2\}$, $o(HK) = \dfrac{o(H)o(K)}{o(H \cap K)} = 79200$. As $HK \subset G \le GL(2,\mathbb{Z}_{22})$ and by Proposition 2.1, $o(GL(2,\mathbb{Z}_{22})) = 79200$, we get $HK = G = GL(2,\mathbb{Z}_{22})$. Hence, the theorem follows. □

**Theorem 3.5.** *Let* $a = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *and* $c = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$. *Then*

$$\begin{aligned} GL(2,\mathbb{Z}_{25}) = \langle a,\ b,\ c \mid a^2,\ b^2,\ c^{40},\ c^2a = ac^2,\ c^2b = bc^2,\ (ab)^3,\ (bc)^{20}, \\ (ac)^4 = (ca)^4,\ (ac)^{20},\ ((bc)^{10}a)^{25}, abacab = cb(c(bc)^9ab)^{13}, \\ (cb)^2(bc)^{10}a = ((bc)^{10}a)^{19}(cb)^2\rangle. \end{aligned}$$

**Proof.** Let $G$ be the subgroup of $GL(2,\mathbb{Z}_{25})$ generated by $a,\ b,\ c$ having the presentation

$$\begin{aligned}\langle a,\ b,\ c \mid a^2,\ b^2,\ c^{40},\ c^2a = ac^2,\ c^2b = bc^2,\ (ab)^3,\ (bc)^{20},\\ (ac)^4 = (ca)^4,\ (ac)^{20},\ ((bc)^{10}a)^{25}, abacab = cb(c(bc)^9ab)^{13},\\ (cb)^2(bc)^{10}a = ((bc)^{10}a)^{19}(cb)^2\rangle.\end{aligned}$$

Let $x = (bc)^{10}a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $y = c^{-1}b^{-1}cb = \begin{pmatrix} 2 & 0 \\ 0 & 13 \end{pmatrix}$. Then $x,\ y \in G$. Also, using the relators in $G$, it is clear that the order of $x$ is 25, the order of $y$ is 20, and $yx = x^{19}y$. Let $H_1 = \langle x, y \mid x^{25},\ y^{20},\ yx = x^{19}y\rangle$. Then $H_1$ is a subgroup of $G$. Since $x^i \neq y^j$ for $0 \leq i \leq 24, 0 \leq j \leq 19$ and the canonical form of $H_1$ is $\{x^iy^j \mid 0 \leq i \leq 24, 0 \leq j \leq 19\}$, the order of $H_1$ is 500. Since $x = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 2 & 0 \\ 0 & 13 \end{pmatrix}$, it follows that the elements of $H_1$ are of the form $\begin{pmatrix} 2^i & 0 \\ \beta & 13^i \end{pmatrix}$.

Let $r = bxb$. Then $r \in G$. Since the order of $x$ is 25 and $x$ is a conjugate of $b$, the order of $r$ is 25. Let $H_2 = \langle r\rangle$, the cyclic subgroup of $G$ generated by $r$. Then the order of $H_2$ is 25 and any arbitrary element of $H_2$ is of the form $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$. Clearly $H_1 \cap H_2 = \{I_2\}$. Thus, $o(H_1H_2) = \dfrac{o(H_1)o(H_2)}{o(H_1 \cap H_2)} = 12500$. Let $H$ be the subgroup of $G$ generated by $x, y$ and $r$. Since $x,\ y$, and $r$ are all of determinant 1, $H \subset SL(2,\mathbb{Z}_{25})$. Also, $H_1H_2 \subset H$. Thus, the order of $H$ is greater than equal to 12500. Since, by Corollary 2.4, the order of $SL(2,\mathbb{Z}_{25})$ is 15000 and $12500 > \frac{1}{2}\cdot o(SL(2,\mathbb{Z}_{25}))$, it follows that the order of $H$ is 15000. Hence, $H = SL(2,\mathbb{Z}_{25})$.

Now $w = bc \in G$, being of determinant 2, does not belong to $H$. Let $K = \langle w\rangle$, the cyclic group generated by $w$. Then $K$ is a subgroup of $G$ and since the order of $w$ is 20, $o(K) = 20$. Further, $H \cap K = \{I_2\}$. Hence, $o(HK) = \dfrac{o(H)o(K)}{o(H \cap K)} = 300000$. Since $HK \subset G \leq GL(2,\mathbb{Z}_{25})$ and by Proposition 2.1, $o(GL(2,\mathbb{Z}_{25})) = 300000$, we get $HK = G = GL(2,\mathbb{Z}_{25})$. Hence, the theorem follows. $\square$

**Theorem 3.6.** *Let* $a = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *and* $c = \begin{pmatrix} 0 & 7 \\ 1 & 0 \end{pmatrix}$. *Then*

$$\begin{aligned}GL(2,\mathbb{Z}_{26}) = \langle a,\ b,\ c \mid a^2,\ b^2,\ c^{24},\ c^2a = ac^2,\ c^2b = bc^2,\ (ab)^3,\ (bc)^{12},\ (ac)^{168},\\ (ac)^{42} = (ca)^{42},\ ((bc)^6a)^{26},\ abacab = cb(c(bc)^5ab)^{12},\\ (cb)^2(bc)^6a = ((bc)^6a)^{17}(cb)^2\rangle.\end{aligned}$$

**Proof.** Let $G$ be the subgroup of $GL(2,\mathbb{Z}_{26})$ generated by $a,\ b,\ c$ having presentation as:

$$\begin{aligned}\langle a,\ b,\ c \mid a^2,\ b^2,\ c^{24},\ c^2a = ac^2,\ c^2b = bc^2,\ (ab)^3,\ (bc)^{12},\ (ac)^{168},\\ (ac)^{42} = (ca)^{42},\ ((bc)^6a)^{26},\ abacab = cb(c(bc)^5ab)^{12},\\ (cb)^2(bc)^6a = ((bc)^6a)^{17}(cb)^2\rangle.\end{aligned}$$

Let $x = (bc)^6a$ and $y = c^{-1}b^{-1}cb$. Then $x,\ y \in G$. Using the relators in $G$ it follows that the order of $x$ is 26, the order of $y$ is 12, and $yx = x^{17}y$. Let $H_1 = \langle x, y \mid x^{26},\ y^{12},\ yx = x^{17}y\rangle$. Then $H_1$ is a subgroup of $G$. Since $x^i \neq y^j$ for $0 \le i \le 25, 0 \le j \le 11$ and the canonical form of $H_1$ is $\{x^iy^j \mid 0 \le i \le 25, 0 \le j \le 11\}$, we get that the order of $H_1$ is 312. Since $x = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 7 & 0 \\ 0 & 15 \end{pmatrix}$, it follows that any element of $H_1$ is of the form $\begin{pmatrix} 7^i & 0 \\ \beta & 15^i \end{pmatrix}$.

Let $r = bxb$. Then $r \in G$. Since the order of $x$ is 26 and $r$ is a conjugate of $x$, the order of $r$ is 26. Let $H_2 = \langle r\rangle$, the cyclic subgroup of $G$ generated by $r$. Then the order of $H_2$ is 26 and any arbitrary element of $H_2$ is of the form $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$. Clearly, $H_1 \cap H_2 = \{I_2\}$. Thus, $o(H_1H_2) = \dfrac{o(H_1)o(H_2)}{o(H_1 \cap H_2)} = 8112$. Let $H$ be the subgroup of $G$ generated by $x, y$ and $r$. Since $x,\ y$, and $r$ are all of determinant 1, $H \subset SL(2,\mathbb{Z}_{26})$. Also, $H_1H_2 \subset H$. Thus, the order of $H$ is greater than equal to 8112. Since by Corollary 2.4, the order of $SL(2,\mathbb{Z}_{26})$ is 13104 and $8112 > \frac{1}{2}\cdot o(SL(2,\mathbb{Z}_{26}))$, it follows that the order of $H$ is 13104. Hence, $H = SL(2,\mathbb{Z}_{26})$.

Now $w = bc \in G$, being of determinant 7, does not belong to $H$. Let $K = \langle w\rangle$, the cyclic group generated by $w$. Then $K$ is a subgroup of $G$ and since the order of $w$ is 12, $o(K) = 12$. Since $H \cap K = \{I_2\}$, $o(HK) = \dfrac{o(H)o(K)}{o(H \cap K)} = 157248$. Since $HK \subset G \le GL(2,\mathbb{Z}_{26})$ and by Proposition 2.1, $o(GL(2,\mathbb{Z}_{26})) = 157248$, we get $HK = G = GL(2,\mathbb{Z}_{26})$. Hence, the theorem follows. □

**Theorem 3.7.** *Let* $a = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *and* $c = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$. *Then*

$$\begin{aligned}GL(2,\mathbb{Z}_{27}) = \langle a,\ b,\ c \mid a^2,\ b^2,\ c^{36},\ c^2a = ac^2,\ c^2b = bc^2,\ (ab)^3,\ (bc)^{18},\ (ac)^{72},\\ (ac)^4 = (ca)^4,\ ((bc)^9a)^{27},\ abacab = cb(c(bc)^8ab)^{14},\\ (cb)^2(bc)^9a = ((bc)^9a)^7(cb)^2\rangle.\end{aligned}$$

**Proof.** Let $G$ be the subgroup of $GL(2,\mathbb{Z}_{27})$ generated by $a,\ b,\ c$ having the presentation

$$\begin{aligned}\langle a,\ b,\ c \mid\ & a^2,\ b^2,\ c^{36},\ c^2a = ac^2,\ c^2b = bc^2,\ (ab)^3,\ (bc)^{18},\ (ac)^{72},\\ & (ac)^4 = (ca)^4,\ ((bc)^9a)^{27},\ abacab = cb(c(bc)^8ab)^{14},\\ & (cb)^2(bc)^9a = ((bc)^9a)^7(cb)^2\rangle.\end{aligned}$$

Let $x = (bc)^9a$ and $y = c^{-1}b^{-1}cb$. Then $x,\ y \in G$. Also, using the relators in $G$, it is clear that the order of $x$ is 27, the order of $y$ is 18, and $yx = x^7y$. Let $H_1 = \langle x, y \mid x^{27}, y^{18}, yx = x^7y\rangle$. Then $H_1$ is a subgroup of $G$. Since $x^i \neq y^j$ for $0 \leq i \leq 26, 0 \leq j \leq 17$ and the canonical form of $H_1$ is $\{x^iy^j \mid 0 \leq i \leq 26, 0 \leq j \leq 17\}$, the order of $H_1$ is 486. Since $x = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 2 & 0 \\ 0 & 14 \end{pmatrix}$, it follows that any element of $H_1$ is of the form $\begin{pmatrix} 2^i & 0 \\ \beta & 14^i \end{pmatrix}$.

Let $r = bxb$. Then $r \in G$. Since the order of $x$ is 27 and $r$ is a conjugate of $x$, the order of $r$ is 27. Let $H_2 = \langle r\rangle$. Then the order of $H_2$ is 25 and any element of $H_2$ is of the form $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$. Clearly $H_1 \cap H_2 = \{I_2\}$. Thus, $o(H_1H_2) = \dfrac{o(H_1)o(H_2)}{o(H_1 \cap H_2)} =$ 13122. Let $H$ be the subgroup of $G$ generated by $x, y$ and $r$. Since $x,\ y$, and $r$ are all of determinant 1, $H \subset SL(2,\mathbb{Z}_{27})$. Also, $H_1H_2 \subset H$. Thus, the order of $H$ is greater than equal to 13122. Since by Corollary 2.4, the order of $SL(2,\mathbb{Z}_{27})$ is 17496 and $13122 > \frac{1}{2} \cdot o(SL(2,\mathbb{Z}_{27}))$, it follows that the order of $H$ is 17496. Hence, $H = SL(2,\mathbb{Z}_{27})$.

Now $w = bc \in G$, being of determinant 2, does not belong to $H$. Let $K = \langle w\rangle$, the cyclic group generated by $w$. Then $K$ is a subgroup of $G$ and since the order of $w$ is 18, $o(K) = 18$. Since $H \cap K = \{I_2\}$, $o(HK) = \dfrac{o(H)o(K)}{o(H \cap K)} = 314928$. As $HK \subset G \leq GL(2,\mathbb{Z}_{27})$ and by Proposition 2.1, $o(GL(2,\mathbb{Z}_{27})) = 314928$, we get $HK = G = GL(2,\mathbb{Z}_{27})$. Hence, the theorem follows. $\square$

**Theorem 3.8.** *Let* $a = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *and* $c = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}$. *Then*

$$\begin{aligned}GL(2,\mathbb{Z}_{34}) = \langle a,\ b,\ c \mid\ & a^2,\ b^2,\ c^{32},\ c^2a = ac^2,\ c^2b = bc^2,\ (ab)^3,\ (bc)^{16},\ (ac)^{96},\\ & (ac)^6 = (ca)^6,\ ((bc)^8a)^{34},\ abacab = cb(c(bc)^7ab)^{12},\\ & (cb)^2(bc)^8a = ((bc)^8a)^{19}(cb)^2\rangle.\end{aligned}$$

**Proof.** Let $G$ be the subgroup of $GL(2,\mathbb{Z}_{34})$ generated by $a,\ b,\ c$ having presentation as:

$$\begin{aligned}\langle a,\ b,\ c \mid\ & a^2,\ b^2,\ c^{32},\ c^2a = ac^2,\ c^2b = bc^2,\ (ab)^3,\ (bc)^{16},\ (ac)^{96},\\ & (ac)^6 = (ca)^6,\ ((bc)^8a)^{34},\ abacab = cb(c(bc)^7ab)^{12},\\ & (cb)^2(bc)^8a = ((bc)^8a)^{19}(cb)^2\rangle.\end{aligned}$$

Let $x = (bc)^8a$ and $y = c^{-1}b^{-1}cb$. Then $x,\ y \in G$. Also, it follows from the relators in $G$ that the order of $x$ is 34, the order of $y$ is 16, and $yx = x^{19}y$. Let $H_1 = \langle x, y \mid x^{34}, y^{16}, yx = x^{19}y\rangle$. Then $H_1$ is a subgroup of $G$. Since $x^i \neq y^j$ for $0 \leq i \leq 33, 0 \leq j \leq 15$ and the canonical form of $H_1$ is $\{x^iy^j \mid 0 \leq i \leq 33, 0 \leq j \leq 15\}$, we get that the order of $H_1$ is 544. Since $x = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 3 & 0 \\ 0 & 23 \end{pmatrix}$, any element of $H_1$ is of the form $\begin{pmatrix} 3^i & 0 \\ \beta & 23^i \end{pmatrix}$.

Let $r = bxb$. Then $r \in G$. Since the order of $x$ is 34 and $r$ is a conjugate of $x$, the order of $r$ is 34. Let $H_2 = \langle r\rangle$, the cyclic subgroup of $G$ generated by $r$. Then the order of $H_2$ is 34 and it can be seen that the elements of $H_2$ are of the form $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$. Clearly, $H_1 \cap H_2 = \{I_2\}$. Thus, $o(H_1H_2) = \dfrac{o(H_1)o(H_2)}{o(H_1 \cap H_2)} = 18496$.
Let $H$ be the subgroup of $G$ generated by $x, y$ and $r$. Since $x$, $y$, and $r$ are all of determinant 1, $H \subset SL(2, \mathbb{Z}_{34})$. Also, $H_1H_2 \subset H$. Thus, the order of $H$ is greater than equal to 18496. Since by Corollary 2.4, the order of $SL(2, \mathbb{Z}_{34})$ is 29376 and $18496 > \frac{1}{2} \cdot o(SL(2, \mathbb{Z}_{34}))$, it follows that the order of $H$ is 29376. Hence, $H = SL(2, \mathbb{Z}_{34})$.
Now $w = bc \in G$, being of determinant 3, does not belong to $H$. Let $K = \langle w\rangle$, the cyclic group generated by $w$. Then $K$ is a subgroup of $G$ and since the order of $w$ is 16, $o(K) = 16$. Further, $H \cap K = \{I_2\}$, $o(HK) = \dfrac{o(H)o(K)}{o(H \cap K)} = 470016$. Since $HK \subset G \leq GL(2, \mathbb{Z}_{34})$ and by Proposition 2.1, $o(GL(2, \mathbb{Z}_{34})) = 470016$, we get $HK = G = GL(2, \mathbb{Z}_{34})$. Hence, the theorem follows. □

We may remark that none of the relators in the presentation given here is redundant. However, the relators may not be defining and there may be another presentation with less number of relators. We also would like to remark that although the proofs are theoretical and self contained, we have used the MAGMA software for verification purposes.

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