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ON GENERALIZED CODERIVATIONS

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Abstract. Let C (resp. A) be a coalgebra (resp. algebra) over a commutative ring R and M (resp. N) a C-bicomodule (resp. an A-bimodule). We define a dual notion of a generalized derivation from A to N in the sense of the paper *On categorical properties of generalized derivations*, Sci. Math., 2(3) (1999), 345-352, by A. Nakajima, which we call a generalized coderivation from M to C. We give some elementary properties of generalized coderivations and discuss the relations of the set of generalized coderivations $g\mathrm{Coder}(M,\ C)$ between the set of generalized derivations $g\mathrm{Der}(C^*,\ M^*)$ for their dual algebra C^* and module M^*. Using these coderivations, we define a notion of a weakly coseparable coalgebra which is a dual notion of a weakly separable algebra defined in the paper of N. Hamaguchi and A. Nakajima, *Weakly separable polynomials* (in preparation), and give related examples of coseparable coalgebras.

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1. Introduction

Let R be a commutative ring with identity, A an R-algebra and N an R-module. An A-bimodule N means that N is a left and a right A-module such that $a(nb) = (an)b$, $r(an) = a(rn)$, $(nb)r = (nr)b$ and $rn = nr$ for any $a,\ b \in A$, $n \in N$, and $r \in R$. An R-linear map $d : A \to N$ is called a *generalized derivation* if there exists an element $n \in N$ (which depends on d) such that $d(ab) = d(a)b + ad(b) + anb$. We denote it by $(d\ ;\ n)$. For any elements $m,\ n \in N$, an R-linear map $d_{m,n} : A \to N$ defined by $d_{m,n}(a) = ma + an$ is called a *generalized inner derivation* by $m,\ n$. It is easy to see that a generalized inner derivation $d_{m,n}$ is a generalized derivation $(d_{m,n}\ ;\ -m-n)$, $(d\ ;\ 0)$ is a derivation and $d_{m,-m}$ is an inner derivation. A lot of properties of derivations and generalized derivations were obtained till now.

For an R-coalgebra C, we define a dual notion of a generalized derivation in the category of C-bicomodules. Let $\Delta : C \to C \otimes C$ and $\varepsilon : C \to R$ be the coalgebra structure maps of C and M an R-module. A C-bicomodule M means that M is a right and a left C-comodule with comodule structure maps $\rho^+ : M \to M \otimes C$ and $\rho^- : M \to C \otimes M$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\rho^+} & M \otimes C \\ \Big\downarrow{\scriptstyle \rho^-} & & \Big\downarrow{\scriptstyle \rho^- \otimes 1} \\ C \otimes M & \xrightarrow{1 \otimes \rho^+} & C \otimes M \otimes C, \end{array}$$

where 1 is the identity map. Then the relations of ρ^+, ρ^- and Δ are as follows:

$$(\rho^- \otimes 1)\rho^+ = (1 \otimes \rho^+)\rho^-, \ (1 \otimes \Delta)\rho^+ = (\rho^+ \otimes 1)\rho^+, \ (\Delta \otimes 1)\rho^- = (1 \otimes \rho^-)\rho^-. \quad (1.1)$$

An R-linear map $d : M \to C$ is called a *coderivation* if $\Delta d = (d \otimes 1)\rho^+ + (1 \otimes d)\rho^-$ and d is called an *inner coderivation* if there exists $\alpha \in M^* = \mathrm{Hom}_R(M, R)$ such that $d = (\alpha \otimes 1)\rho^+ - (1 \otimes \alpha)\rho^-$. As similar as the case of a commutative ring, if C is cocommutative and the C-bicomodule structure satisfies the relation $t\rho^+ = \rho^- : M \to C \otimes M$, where $t : M \otimes C \to C \otimes M$ is the twisted map, then an inner coderivation is zero. Because, by $(\Delta \otimes 1)t\rho^+ = (1 \otimes t)(t \otimes 1)(1 \otimes \Delta)\rho^+$, we see

$$\begin{aligned} \Delta d &= \Delta\{(\alpha \otimes 1)\rho^+ - (1 \otimes \alpha)\rho^-\} \\ &= (\alpha \otimes 1 \otimes 1)(1 \otimes \Delta)\rho^+ - (1 \otimes 1 \otimes \alpha)(\Delta \otimes 1)t\rho^+ = 0. \end{aligned}$$

We generalize these notions of coderivations as follows. $d : M \to C$ is called a *generalized coderivation* if there exists an R-linear map $\xi : M \to R$ such that

$$\Delta d = (d \otimes 1)\rho^+ + (1 \otimes d)\rho^- + (1 \otimes \xi \otimes 1)(\rho^- \otimes 1)\rho^+, \quad (1.2)$$

and d is called a *generalized inner coderivation* if there exist α, $\beta \in M^*$ such that $d = (\alpha \otimes 1)\rho^+ + (1 \otimes \beta)\rho^-$. We denote the above these coderivations by $(d\ ;\ \xi)$ and $(d\ ;\ \alpha, \beta)$, respectively. Then $(d\ ;\ 0)$ is a coderivation and $(d\ ;\ \alpha, -\alpha)$ is an inner coderivation. The notions of a coderivation and an inner coderivation were defined in [3] and [8]. Some homological properties of coderivations of coalgebras were given in [3], [4] and [8].

In this paper, we give some elementary properties of generalized coderivations and discuss relations to the set of generalized coderivations (resp. coderivations) $g\mathrm{Coder}(M,\ C)$ (resp. $\mathrm{Coder}(M,\ C)$) from M to C and the set of their duals $g\mathrm{Der}(C^*,\ M^*)$ and $\mathrm{Der}(C^*,\ M^*)$.

Throughout the following, R is a commutative ring with identity and we consider all things in the category of R-modules. A is an R-algebra with the multiplication map $\mu : A \otimes A \to A$. C is an R-coalgebra with structure maps $\Delta : C \to C \otimes C$ and

$\varepsilon : C \to R$, M is a C-bicomodule and $M^* = \mathrm{Hom}_R(M,\ R)$. $C^* = \mathrm{Hom}_R(C,\ R)$ is an R-algebra with the convolution product $\circ$:

$$(f \circ g) = (f \otimes g)\Delta \in C^* \quad (f,\ g \in C^*), \tag{1.3}$$

and M^* is a C^*-bimodule via

$$(m^* \leftarrow f) = \sum (m^* \otimes f)\rho^+ \quad \text{and} \quad (g \to m^*) = \sum (g \otimes m^*)\rho^- \tag{1.4}$$

($m^* \in M^*$). $\otimes = \otimes_R$ and 1 means the identity map unless otherwise stated.

2. Preliminaries

In this section, we give some elementary relations of generalized coderivations and generalized derivations. First, we have the following

Lemma 2.1. *(1) If $(d\ ;\ \xi) : M \to C$ is a generalized coderivation, then $(d^*\ ;\ \xi) : C^* \to M^*$ is a generalized derivation.*

(2) If $(d\ ;\ \alpha, \beta) : M \to C$ is a generalized inner coderivation, then $(d\ ;\ \alpha, \beta) = (d\ ;\ -\alpha - \beta)$ and $(d^*\ ;\ \alpha, \beta) : C^* \to M^*$ is a generalized inner derivation.

Proof. Let $f,\ g \in C^*$ and $m \in M$.

(1) Since C^*-bimodule structure of M^* is given by (1.4), there holds

$$\begin{aligned} d^*(f \circ g) &= (f \otimes g)\Delta d \\ &= (fd \otimes g)\rho^+ + (f \otimes gd)\rho^- + (f \otimes \xi \otimes g)(\rho^- \otimes 1)\rho^+ \\ &= (d^*(f) \leftarrow g) + (f \to d^*(g)) + (f \to \xi \leftarrow g). \end{aligned}$$

Thus $(d^*\ ;\ \xi)$ is a generalized derivation.

(2) If $(d\ ;\ \alpha, \beta)$ is a generalized inner coderivation, then we have

$$\begin{aligned} (d \otimes 1)\rho^+ + (1 \otimes d)\rho^- &= ((\alpha \otimes 1)\rho^+ \otimes 1)\rho^+ + ((1 \otimes \beta)\rho^- \otimes 1)\rho^+ \\ &\quad + (1 \otimes (\alpha \otimes 1)\rho^+)\rho^- + (1 \otimes (1 \otimes \beta)\rho^-)\rho^- \end{aligned}$$

and

$$\Delta d = (\alpha \otimes 1 \otimes 1)(1 \otimes \Delta)\rho^+ + (1 \otimes 1 \otimes \beta)(\Delta \otimes 1)\rho^-.$$

Therefore by (1.1), we see

$$\Delta d = (d \otimes 1)\rho^+ + (1 \otimes d)\rho^- + \{1 \otimes (-\alpha - \beta) \otimes 1\}(\rho^- \otimes 1)\rho^+,$$

and so $(d\ ;\ \alpha, \beta) = (d\ ;\ -\alpha - \beta)$. Moreover, we also have

$$d^*(f) = f((\alpha \otimes 1)\rho^+ + (1 \otimes \beta)\rho^-) = (\alpha \otimes f)\rho^+ + (f \otimes \beta)\rho^- = (\alpha \leftarrow f) + (f \to \beta),$$

and thus $(d^*\ ;\ \alpha, \beta)$ is a generalized inner derivation. □

Lemma 2.2. *(1) If $(d\ ;\ \xi) : M \to C$ is a generalized coderivation, then $\varepsilon d + \xi = 0$. If we set $d_1 = d + (\xi \otimes 1)\rho^+$ (resp. $d_2 = d + (1 \otimes \xi)\rho^-$), then d_1 (resp. d_2) is a coderivation from M to C and d is a right d_1-coderivation and a left d_2-coderivation, that is,*

$$\Delta d = (d \otimes 1)\rho^+ + (1 \otimes d_1)\rho^- = (d_2 \otimes 1)\rho^+ + (1 \otimes d)\rho^-.$$

(2) For any R-linear map $\xi : M \to R$, $((\xi \otimes 1)\rho^+\ ;\ -\xi)$ is a generalized coderivation.

Proof. (1) If $(d\ ;\ \xi)$ is a generalized coderivation, then by

$$\begin{aligned} d &= (1 \otimes \varepsilon)\Delta d = (d \otimes \varepsilon)\rho^+ + (1 \otimes \varepsilon d)\rho^- + (1 \otimes \xi \otimes \varepsilon)(\rho^- \otimes 1)\rho^+ \\ &= d + (1 \otimes \varepsilon d)\rho^- + (1 \otimes \xi)\rho^-, \end{aligned}$$

we have $(1 \otimes \varepsilon d)\rho^- + (1 \otimes \xi)\rho^- = 0$. Therefore $(\varepsilon \otimes 1)((1 \otimes \varepsilon d)\rho^- + (1 \otimes \xi)\rho^-) = (1 \otimes \varepsilon d) + (1 \otimes \xi) = 0$, which means $\varepsilon d + \xi = 0$.

Next, we show that $d_1 = d + (\xi \otimes 1)\rho^+$ is a coderivation. Since M is a C-bicomodule, then by (1.1) and (1.2), we have

$$\begin{aligned} &\Delta d_1 - ((d_1 \otimes 1)\rho^+ + (1 \otimes d_1)\rho^-) \\ &= \Delta d + \Delta(\xi \otimes 1)\rho^+ - (d \otimes 1)\rho^+ - (1 \otimes d)\rho^- \\ &- (\xi \otimes 1 \otimes 1)(\rho^+ \otimes 1)\rho^+ - (1 \otimes \xi \otimes 1)(1 \otimes \rho^+)\rho^- \\ &= \Delta(\xi \otimes 1)\rho^+ - (\xi \otimes 1 \otimes 1)(1 \otimes \Delta)\rho^+ = 0. \end{aligned}$$

Thus d_1 is a coderivation. Moreover, substituting d_1 in the relation $\Delta d_1 = (d_1 \otimes 1)\rho^+ + (1 \otimes d_1)\rho^-$, we have

$$\begin{aligned} \Delta d_1 &= \Delta(d + (\xi \otimes 1)\rho^+) = \Delta d + \Delta(\xi \otimes 1)\rho^+ \\ &= \{(d + (\xi \otimes 1)\rho^+) \otimes 1\}\rho^+ + (1 \otimes d_1)\rho^- \\ &= (d \otimes 1)\rho^+ + ((\xi \otimes 1)\rho^+ \otimes 1)\rho^+ + (1 \otimes d_1)\rho^-. \end{aligned}$$

Thus $\Delta d = (d \otimes 1)\rho^+ + (1 + \otimes d_1)\rho^-$, which shows that d is a right d_1-coderivation. Similarly, d_2 is a coderivation and d is a left d_2-coderivation.

(2) Let $g = (\xi \otimes 1)\rho^+$. Then by (1.1), we have

$$\begin{aligned} &(g \otimes 1)\rho^+ + (1 \otimes g)\rho^- + (1 \otimes (-\xi) \otimes 1)(\rho^- \otimes 1)\rho^+ \\ &= (\xi \otimes 1 \otimes 1)(\rho^+ \otimes 1)\rho^+ + (1 \otimes \xi \otimes 1)(1 \otimes \rho^+)\rho^- - (1 \otimes \xi \otimes 1)(\rho^- \otimes 1)\rho^+ \\ &= (\xi \otimes 1 \otimes 1)(1 \otimes \Delta)\rho^+ = \Delta g. \end{aligned}$$

Thus $(g\ ;\ -\xi)$ is a generalized coderivation. $\square$

In [1], Brešar defined a generalized derivation as follows: an additive map $\delta : A \to A$ is called a *generalized derivation* if there exists a derivation $d : A \to A$ such that $\delta(ab) = \delta(a)b + ad(b)$ for any $a,\ b \in A$. We call that δ is a *right d-derivation.* If d is a right d_1-coderivation, then by $\varDelta d = (d \otimes 1)\rho^+ + (1 \otimes d_1)\rho^-$, we see

$$\begin{aligned} d^*(f \circ g) &= (f \otimes g)\varDelta d = (f \otimes g)((d \otimes 1)\rho^+ + (1 \otimes d_1)\rho^-) \\ &= (fd \otimes g)\rho^+ + (f \otimes gd_1)\rho^- \\ &= (d^*(f) \leftarrow g) + (f \rightarrow d_1^*(g)), \end{aligned}$$

and so d^* is a right d_1^*-derivation. Therefore the notion of a right d_1-coderivation and a left d_2-coderivatoin in Lemma 2.2(1) correspond to the right d-derivation and the left d-derivation in the sense of [1].

Lemma 2.3. *Let M and N be C-bicomodules and $g : M \to N$ a C-bicomodule map. If $(f\ ;\ \xi) : N \to C$ is a generalized coderivation, then $(fg\ ;\ \xi g) : M \to C$ is a generalized coderivation.*

Proof. Let $\rho_X^+ : X \to X \otimes C$ and $\rho_X^- : X \to C \otimes X$ be the C-bicomodule structure maps of X. Since g is a C-bicomodule map, we see $\rho_N^+ g = (g \otimes 1)\rho_M^+$ and $\rho_N^- g = (1 \otimes g)\rho_M^-$. If $(f\ ;\ \xi)$ is a generalized coderivation, then by (1.1), we have

$$\begin{aligned} (\varDelta f)g &= (f \otimes 1)\rho_N^+ g + (1 \otimes f)\rho_N^- g + (1 \otimes \xi \otimes 1)(\rho_N^- \otimes 1)\rho_N^+ g \\ &= (fg \otimes 1)\rho_M^+ + (1 \otimes fg)\rho_M^- + (1 \otimes \xi g \otimes 1)(\rho_M^- \otimes 1)\rho_M^+, \end{aligned}$$

and so $(fg\ ;\ \xi g)$ is a generalized coderivation. □

Two generalized coderivations $(f\ ;\ \xi)$ and $(g\ ;\ \gamma)$ from M to C are *equal* if $f = g$ and $\xi = \gamma$. Then it is easy to see that the set of all generalized coderivations $g\mathrm{Coder}(M,\ C)$ has the following R-module structure:

$$(f\ ;\ \xi) + (g\ ;\ \gamma) = (f + g\ ;\ \xi + \gamma) \quad \text{and} \quad r(f\ ;\ \xi) = (rf\ ;\ r\xi), \ \ (r \in R).$$

Then we have an R-module monomorphism $\psi_M : \mathrm{Coder}(M,\ C) \ni d \mapsto (d\ ;\ 0) \in g\mathrm{Coder}(M,\ C)$, where $\mathrm{Coder}(M,\ C)$ is the set of all coderivations from M to C. We treat these R-modules in the following sections.

3. Elementary properties of generalized coderivations

In this section, we give some elementary properties of $\mathrm{Coder}(M,\ C)$ and $g\mathrm{Coder}(M,\ C)$. First, we have the followings which are easily seen by Lemma 2.2.

Theorem 3.1. *Let M be a C-bicomodule. We set*

$$\begin{aligned}
&\psi_M : Coder(M,\ C) \ni d \mapsto (d\ ;\ 0) \in gCoder(M,\ C),\\
&\psi'_M : gCoder(M,\ C) \ni (d\ ;\ \xi) \mapsto d + (\xi \otimes 1)\rho^+ \in Coder(M,\ C),\\
&\varphi_M : gCoder(M,\ C) \ni (d\ ;\ \xi) \mapsto -\xi \in M^*,\\
&\varphi'_M : M^* \ni \xi \mapsto ((\xi \otimes 1)\rho^+\ ;\ -\xi) \in gCoder(M,\ C).
\end{aligned}$$

Then these four maps are well-defined such that $\psi'_M\psi_M$ and $\varphi_M\varphi'_M$ are identity maps on $Coder(M,\ C)$ and M^*,respectively. Thus the following sequence of R-modules is split exact:

$$0 \longrightarrow Coder(M,\ C) \xrightarrow{\psi_M} gCoder(M,\ C) \xrightarrow{\varphi_M} M^* \longrightarrow 0.$$

Let $f,\ g : C \to C$ be coderivations. Then by

$$\Delta(fg) = (f \otimes 1 + 1 \otimes f)\Delta g = (fg \otimes 1 + f \otimes g + g \otimes f + 1 \otimes fg)\Delta,$$

$[f,\ g] = fg - gf$ is also a coderivation of C and thus $\mathrm{Coder}(C,\ C) = \mathrm{Coder}(C)$ has a Lie algebra structure. We show that $g\mathrm{Coder}(C) = g\mathrm{Coder}(C,\ C)$ has also a Lie algebra structure.

Lemma 3.2. *Let $(f\ ;\ \xi),\ (g\ ;\ \gamma) : C \to C$ be generalized coderivations. Then $([f,\ g]\ ;\ \xi g - \gamma f) : C \to C$ is a generalized coderivation.*

Proof. We see that

$$\begin{aligned}
&\Delta(fg - gf)\\
&= (f \otimes 1 + 1 \otimes f)\Delta g + (1 \otimes \xi \otimes 1)(\Delta \otimes 1)\Delta g\\
&- (g \otimes 1 + 1 \otimes g)\Delta f - (1 \otimes \gamma \otimes 1)(\Delta \otimes 1)\Delta f\\
&= ([f,\ g] \otimes 1 + 1 \otimes [f,\ g])\Delta\\
&+ (f \otimes 1 \otimes 1 + 1 \otimes 1 \otimes f)(1 \otimes \gamma \otimes 1)(\Delta \otimes 1)\Delta + (1 \otimes \xi \otimes 1)(\Delta \otimes 1)\Delta g\\
&- (g \otimes 1 \otimes 1 + 1 \otimes 1 \otimes g)(1 \otimes \xi \otimes 1)(\Delta \otimes 1)\Delta - (1 \otimes \gamma \otimes 1)(\Delta \otimes 1)\Delta f.
\end{aligned}$$

Since $(g\ ;\ \gamma)$ is a generalized coderivation, we have

$$\begin{aligned}
&(1\otimes\xi\otimes 1)(\Delta\otimes 1)\Delta g\\
&=(1\otimes\xi\otimes 1)(\Delta g\otimes 1+\Delta\otimes g)\Delta+(1\otimes\xi\otimes 1)(\Delta\otimes 1)(1\otimes\gamma\otimes 1)(\Delta\otimes 1)\Delta\\
&=(g\otimes\xi\otimes 1)(\Delta\otimes 1)\Delta+(1\otimes\xi g\otimes 1)(\Delta\otimes 1)\Delta\\
&+(1\otimes\gamma\otimes\xi\otimes 1)(\Delta\otimes 1\otimes 1)(\Delta\otimes 1)\Delta+(1\otimes\xi\otimes g)(\Delta\otimes 1)\Delta\\
&+(1\otimes\xi\otimes 1)(\Delta\otimes 1)(1\otimes\gamma\otimes 1)(\Delta\otimes 1)\Delta\\
&=(g\otimes 1\otimes 1+1\otimes 1\otimes g)(1\otimes\xi\otimes 1)(\Delta\otimes 1)\Delta\\
&+(1\otimes\gamma\otimes\xi\otimes 1)(\Delta\otimes 1\otimes 1)(\Delta\otimes 1)\Delta+(1\otimes\xi g\otimes 1)(\Delta\otimes 1)\Delta\\
&+(1\otimes\xi\otimes\gamma\otimes 1)(\Delta\otimes 1\otimes 1)(\Delta\otimes 1)\Delta
\end{aligned}$$

and similarly

$$\begin{aligned}
&(1\otimes\gamma\otimes 1)(\Delta\otimes 1)\Delta f\\
&=(f\otimes 1\otimes 1+1\otimes 1\otimes f)(1\otimes\gamma\otimes 1)(\Delta\otimes 1)\Delta\\
&+(1\otimes\xi\otimes\gamma\otimes 1)(\Delta\otimes 1\otimes 1)(\Delta\otimes 1)\Delta+(1\otimes\gamma f\otimes 1)(\Delta\otimes 1)\Delta\\
&+(1\otimes\gamma\otimes\xi\otimes 1)(\Delta\otimes 1\otimes 1)(\Delta\otimes 1)\Delta.
\end{aligned}$$

Using these three relations, we can get the following relation:

$$\Delta[f,\ g]=([f,\ g]\otimes 1+1\otimes[f,\ g])\Delta+(1\otimes(\xi g-\gamma f)\otimes 1)(\Delta\otimes 1)\Delta.$$

This shows that $([f,\ g]\ ;\ \xi g-\gamma f)$ is a generalized coderivation. □

For generalized coderivations $(f\ ;\ \xi)$ and $(g\ ;\ \gamma)$, we can define a bracket operation by Lemma 3.2:

$$[(f\ ;\ \xi),\ (g\ ;\ \gamma)]=([f,\ g]\ ;\ \xi g-\gamma f).$$

By this bracket operation $g\mathrm{Coder}(C)$ has a Lie algebra structure. Since C^* is an R-algebra via convolution product $\circ$ given by (1.3), it is also Lie algebra by the following operation:

$$\{\xi,\ \gamma\}=\gamma\circ\xi-\xi\circ\gamma=(\gamma\otimes\xi-\xi\otimes\gamma)\Delta,\ (\xi,\ \gamma\in C^*).$$

Then in the exact sequence

$$0\longrightarrow\mathrm{Coder}(C)\xrightarrow{\psi_C}g\mathrm{Coder}(C)\xrightarrow{\varphi_C}C^*\longrightarrow 0 \tag{3.1}$$

given by Theorem 3.1, the map ψ_C is a Lie algebra map, but φ_C is not a Lie algebra map in the above Lie algebra stuctures. But we have the following

Theorem 3.3. *For any R-coalgebra C,*

$$g\,Coder(C) \cong Coder(C) \oplus C^*$$

as Lie algebras.

Proof. In the exact sequence

$$0 \longrightarrow C^* \xrightarrow{\varphi'_C} g\mathrm{Coder}(C) \xrightarrow{\psi'_C} \mathrm{Coder}(C) \longrightarrow 0, \tag{3.2}$$

we show that φ'_C and ψ'_C are Lie algebra maps. Firstly, we note that the following four relations hold for any $\xi,\ \gamma \in C^*$ and $f,\ g \in \mathrm{Hom}(C,\ C)$:

$$\begin{aligned}
&(\xi \otimes 1)\Delta(\gamma \otimes 1)\Delta = (\gamma \otimes \xi \otimes 1)(\Delta \otimes 1)\Delta,\\
&\xi(\gamma \otimes 1)\Delta = (\gamma \otimes \xi)\Delta,\\
&f(\gamma \otimes 1)\Delta = (\gamma \otimes f)\Delta,\\
&(\xi \otimes 1)\Delta g = (\xi g \otimes 1)\Delta + (\xi \otimes g)\Delta + (\xi \otimes \gamma \otimes 1)(\Delta \otimes 1)\Delta.
\end{aligned}$$

Then we have

$$\begin{aligned}
[\varphi'_C(\xi),\ \varphi'_C(\gamma)] &= [((\xi \otimes 1)\Delta\ ;\ -\xi),\ ((\gamma \otimes 1)\Delta\ ;\ -\gamma)]\\
&= ([(\xi \otimes 1)\Delta,\ (\gamma \otimes 1)\Delta]\ ;\ -\xi(\gamma \otimes 1)\Delta + \gamma(\xi \otimes 1)\Delta)\\
&= (\{(\gamma \otimes \xi - \xi \otimes \gamma) \otimes 1\}(\Delta \otimes 1)\Delta\ ;\ (\xi \otimes \gamma - \gamma \otimes \xi)\Delta)\\
&= \varphi'_C(\{\xi,\ \gamma\})
\end{aligned}$$

and so φ'_C is a Lie algebra map. If $(g\ ;\ \gamma) : C \to C$ is a generalized coderivation, then, we see

$$\begin{aligned}
&[\psi'_C(f\ ;\ \xi),\ \psi'_C(g\ ;\ \gamma)]\\
&= [f + (\xi \otimes 1)\Delta,\ g + (\gamma \otimes 1)\Delta]\\
&= [f,\ g] + f(\gamma \otimes 1)\Delta + (\xi \otimes 1)\Delta g + (\xi \otimes 1)\Delta(\gamma \otimes 1)\Delta\\
&- g(\xi \otimes 1)\Delta - (\gamma \otimes 1)\Delta f - (\gamma \otimes 1)\Delta(\xi \otimes 1)\Delta\\
&= [f,\ g] + ((\xi g - \gamma f) \otimes 1)\Delta\\
&= \psi'_C([(f\ ;\ \xi),\ (g\ ;\ \gamma)]).
\end{aligned}$$

Thus ψ'_C is also a Lie algebra map. Since ψ_C is a Lie algebra map such that $\psi'_C\psi_C$ is the identity on $\mathrm{Coder}(C)$, we have $g\mathrm{Coder}(C) \cong \mathrm{Coder}(C) \oplus \mathrm{Ker}\ \psi'_C$. Moreover, by the exact sequence (3.2), we see $\mathrm{Ker}\ \psi'_C \cong C^*$. This shows the theorem. □

By Theorem 3.1, we have the functors $\mathrm{Coder}(-,\ C)$, $g\mathrm{Coder}(-,\ C)$ and $\mathrm{Hom}_R(-,\ R)$ from the category of C-bicomodules to the category of R-modules and

$$g\mathrm{Coder}(-,\ C) \cong \mathrm{Coder}(-,\ C) \oplus \mathrm{Hom}_R(-,\ R)$$

as functors.

Now let L be the cokernel of the map $\Delta : C \to C \otimes C$. Then the sequence of C-bicomodules

$$0 \longrightarrow C \xrightarrow{\Delta} C \otimes C \xrightarrow{\omega} L \longrightarrow 0$$

is exact. Define a map

$$\lambda : L \ni \omega(c \otimes c') \mapsto c\varepsilon(c') - \varepsilon(c)c' \in C.$$

Then λ is a coderivation and λ is a universal coderivation in the following sense: For any C-bicomodules M, the map

$$\mathrm{Com}(M, L) \ni \sigma \mapsto \lambda\sigma \in \mathrm{Coder}(M, C)$$

is an R-module isomorphism, where $\mathrm{Com}(M, L)$ is the set of all C-bicomodule maps from M to L (cf. [3, Proposition 13]). Using this isomorphism and Theorem 3.1, the following is easily seen.

Corollary 3.4. *For any C-bicomodule M, the map*

$$\Phi : \mathit{Com}(M, L) \oplus M^* \ni (\sigma,\ \xi) \mapsto (\lambda\sigma + (\xi \otimes 1)\rho^+ \ ;\ -\xi) \in g\mathit{Coder}(M, C)$$

is an R-module isomorphism.

4. A relation of gCoder(C) and gDer$(C^*,\ M^*)$

Since C^* is an R-algebra by the convolution product $\circ$, M^* is a C^*-bimodule by (1.4) for any C-bicomodule M. Define two R-module maps

$$\psi_M^* : \mathrm{Der}(C^*,\ M^*) \ni \alpha \mapsto (\alpha\ ;\ 0) \in g\mathrm{Der}(C^*,\ M^*),$$
$$\varphi_M^* : g\mathrm{Der}(C^*,\ M^*) \ni (\beta\ ;\ \ m^*) \mapsto m^* \in M^*.$$

Then we see that

$$0 \longrightarrow \mathrm{Der}(C^*,\ M^*) \xrightarrow{\psi_M^*} g\mathrm{Der}(C^*,\ M^*) \xrightarrow{\varphi_M^*} M^* \longrightarrow 0$$

is an exact sequence of R-modules (cf. [9, Theorem 2.4.]) By Lemma 2.1, we show that a generalized coderivation $(f\ ;\ \xi) : M \to C$ induces a generalized derivation $(f^*\ ;\ \xi) : C^* \to M^*$ and therefore we have an R-linear map

$$\theta : g\mathrm{Coder}(C,\ M) \ni (f\ ;\ \xi) \mapsto (f^*\ ;\ \xi) \in g\mathrm{Der}(C^*,\ M^*).$$

In this section, we give the relations of an exact sequence of generalized coderivations in Theorem 3.1 and the above exact sequence of derivations. First we have the following

Theorem 4.1. *The following diagram of R-modules is commutative and each rows are split exact:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Coder(M,\ C) & \xrightarrow{\psi_M} & gCoder(M,\ C) & \xrightarrow{\varphi_M} & M^* & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \theta_0} & & \Big\downarrow{\scriptstyle \theta} & & \Big\downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & Der(C^*,\ M^*) & \xrightarrow{\psi^*_M} & gDer(C^*,\ M^*) & \xrightarrow{\varphi^*_M} & M^* & \longrightarrow & 0, \end{array}$$

where θ_0 is the restriction map of θ and 1 is the identity map on M^*. Moreover, if R is a field, then θ_0 is a monomorphism and thus θ is a monomorphism.

Proof. The commutativity of the diagram is easily seen by the definitions of each maps, and the split exactness is obtained by Theorem 3.1. Assume that R is a field and $0 \neq f \in \mathrm{Coder}(M, C)$ such that $\theta_0(f) = 0$. Then there exists $m \in M$ such that $f(m) \neq 0$. Since R is a field, $C = Rf(m) \oplus C_1$ for some R-subspace C_1 of C and so we have an R-linear map $\lambda : C \to R$ such that $\lambda(f(m)) = 1$ and $\lambda(C_1) = 0$. This contradicts to $\theta_0(f)(\lambda)(m) = \lambda f(m) = 0$ for all $\lambda \in C^*$. Therefore θ_1 is a monomorphism and thus θ is a monomorphism by five lemma. $\square$

Although C^* is an algebra, but A^* is not a coalgebra in general, and thus a generalized derivation $(d\ ;\ \xi) : A \to M$ does not necessarily induce a generalized coderivation from M^* to A^*. If A is a finitely generated projective R-module, then it is well known that A^* has a coalgebra structure with the comultiplication

$$\varDelta_{A^*} : A^* \ni \alpha \mapsto \sum_{i=1}^{n} \alpha\mu(- \otimes a_i) \otimes f_i \in (A \otimes A)^* \cong A^* \otimes A^*,$$

where $\{a_i, f_i\}$ $(i = 1, 2, \cdots, n)$ is an R-projective coordinate system of A and the map $\alpha\mu(- \otimes a_i) : A \to A$ is defined by $\alpha\mu(- \otimes a_i)(a) = \alpha(aa_i)$. Under these notations, we prove the following

Lemma 4.2. *Assume that A is finitely generated projective R-module and $(f\ ;\ x) : A \to A$ is a generalized derivation. Then $(f^*\ ;\ x) : A^* \to A^*$ is a generalized coderivation. Especially, if f is a derivation, then f^* is a coderivation.*

Proof. Let a, $b \in A$ and $\alpha \in A^*$. Then by $a = \sum_{i=1}^n f_i(a)a_i$, we have

$$\begin{aligned}(\varDelta_{A^*} f^*)(\alpha)(a \otimes b) &= \sum_{i=1}^{n} (\alpha f\mu(- \otimes a_i) \otimes f_i)(a \otimes b) = \sum_{i=1}^{n} \alpha(f(aa_i))f_i(b) = \alpha(f(ab)) \\ &= \alpha(f(a)b + af(b) + axb).\end{aligned}$$

Moreover, by

$$\begin{aligned}
&(f^* \otimes 1 + 1 \otimes f^*)\Delta_{A^*}(\alpha)(a \otimes b) \\
&= ((f^* \otimes 1 + 1 \otimes f^*)(\sum_{i=1}^{n} \alpha\mu(- \otimes a_i) \otimes f_i))(a \otimes b) \\
&= \sum_{i=1}^{n}(\alpha\mu(- \otimes a_i)(f(a)) \otimes f_i(b) + \alpha(aa_i)f_i(f(b))) \\
&= \sum_{i=1}^{n}(\alpha(f(a)a_i f_i(b) + aa_i f_i(f(b))) = \alpha(f(a)b + af(b))
\end{aligned}$$

and

$$\begin{aligned}
&(1 \otimes x \otimes 1)(\Delta_{A^*} \otimes 1)\Delta_{A^*}(\alpha)(a \otimes b) \\
&= (1 \otimes x \otimes 1)(\Delta_{A^*} \otimes 1)(\sum_{i=1}^{n} \alpha\mu(- \otimes a_i) \otimes f_i)(a \otimes b) \\
&= (1 \otimes x \otimes 1)\{\sum_{i,j=1}^{n} (\alpha\mu(- \otimes a_i))(\mu(- \otimes a_j) \otimes f_j) \otimes f_i\}(a \otimes b) \\
&= \sum_{i,j=1}^{n} \alpha(aa_j a_i)f_j(x)f_i(b) = \alpha(axb).
\end{aligned}$$

Combining these three relations, we see that $(f^* \,;\, x)$ is a generalized coderivation of A^*. □

By (3.1), Theorem 4.1 and Lemma 4.2, we have the following

Theorem 4.3. *In the commutative diagram in Theorem 4.1, if C is finitely generated projective R-module, then the maps θ_0 and θ are isomorphisms. Especially, the following diagram of Lie algebras is commutative and each rows are split exact:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathit{Coder}(C) & \xrightarrow{\psi_C} & g\mathit{Coder}(C) & \xrightarrow{\varphi_C} & C^* & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \theta_0} & & \Big\downarrow{\scriptstyle \theta} & & \Big\downarrow{\scriptstyle 1} & & \\
0 & \longrightarrow & \mathit{Der}(C^*) & \xrightarrow{\psi_C^*} & g\mathit{Der}(C^*) & \xrightarrow{\varphi_C^*} & C^* & \longrightarrow & 0.
\end{array}$$

Let $gInn\mathrm{Coder}(M,\ C)$ be the set of generalized inner coderivations from M to C. Then $gInn\mathrm{Coder}(M,\ C)$ is an R-submodule of $g\mathrm{Coder}(M,\ C)$ and we have an exact sequence of R-modules

$$0 \longrightarrow gInn\mathrm{Coder}(M,\ C) \xrightarrow{\iota_M} g\mathrm{Coder}(M,\ C) \xrightarrow{\pi_M} P_M \longrightarrow 0,$$

where ι_M is the inclusion map, $P_M = g\mathrm{Coder}(M,\ C)/gInn\mathrm{Coder}(M,\ C)$ is the quotient R-module and π_M is the natural projection. Then by Lemma 2.1(2),

$$\theta_{-1} : gInn\mathrm{Coder}(M,\ C) \ni (d\ ;\ \alpha, \beta) \mapsto (d^*\ ;\ \alpha, \beta) \in gInn\mathrm{Der}(C^*,\ M^*) \qquad (4.1)$$

is an R-module homomorphism and so we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & gInn\mathrm{Coder}(M,\ C) & \xrightarrow{\iota_M} & g\mathrm{Coder}(M,\ C) & \xrightarrow{\pi_M} & P_M & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \theta_{-1}} & & \Big\downarrow{\scriptstyle \theta} & & \Big\downarrow{\scriptstyle p} & & \\
0 & \longrightarrow & gInn\mathrm{Der}(C^*,\ M^*) & \xrightarrow{\iota_{M^*}} & g\mathrm{Der}(C^*,\ M^*) & \xrightarrow{\pi_{M^*}} & P_{M^*} & \longrightarrow & 0,
\end{array}$$

where $P_{M^*} = g\mathrm{Der}(C^*,\ M^*)/gInn\mathrm{Der}(C^*,\ M^*)$ and p is the natural map. In the next section, we define a notion of weakly coseparable coalgebras which relates to generalized inner coderivations.

5. Weakly coseparable coalgebras.

A coalgebra C is called *coseparable* if there exists a C-bicomodule map $\pi : C \otimes C \to C$ such that $\pi\Delta = 1$. It was defined in [6], and several properties of coseparable coalgebras were given in [3], [4], [6], [7] and [8]. A coseparable coalgebra is a dual notion of a separable algebra, and it is known that an algebra A is separable if and only if any derivation in $\mathrm{Der}(A,\ N)$ is inner for every A-bimodule N (cf. [2, pp.75-76]). The corresponding result for coalgebras was also proved in [3, Theorem 3] and [8, Theorem 1.2], that is, C is coseparable if and only if any coderivation in $\mathrm{Coder}(M,\ C)$ is inner for every C-bicomodule M. Recently, the notion of separable algebras is generalized as follows. A is called *weakly separable* if any derivation in $\mathrm{Der}(A)$ is inner, and characterize weakly separable polynomials in $R[X]$ (cf. [5]). From these point of view, we define that C is *weakly coseparable* if any coderivation in $\mathrm{Coder}(C)$ is inner.

In this section, we treat weakly coseparable coalgebras and give such an example. First, we have the following

Lemma 5.1. *Consider the following commutative diagram*

$$\begin{array}{ccccc}
0 & \longrightarrow & InnCoder(M,\ C) & \xrightarrow{\iota'_M} & Coder(M,\ C) \\
 & & \Big\downarrow{\scriptstyle \psi_{M0}} & & \Big\downarrow{\scriptstyle \psi_M} \\
0 & \longrightarrow & gInnCoder(M,\ C) & \xrightarrow{\iota_M} & gCoder(M,\ C),
\end{array}$$

where ι'_M is the canonical inclusion and ψ_{M0} is the restriction of ψ_M. Then ι'_M is an isomorphism if and only if ι_M is an isomorphism. Therefore, a coalgebra

C is weakly coseparable if and only if any generalized coderivation in Coder(C) is generalized inner.

Proof. Assume that $Inn\mathrm{Coder}(M,\ C) = \mathrm{Coder}(M,\ C)$. If $(d\ ;\ \alpha) : M \to C$ is a generalized coderivation, then by Lemma 2.2(1), $d_1 = d + (\alpha \otimes 1)\rho^+$ is a coderivation, and so there exists $\beta \in M^*$ such that $d_1 = (\beta \otimes 1)\rho^+ - (1 \otimes \beta)\rho^-$. Therefore, $d = \{(\beta - \alpha) \otimes 1)\}\rho^+ - (1 \otimes \beta)\rho^-$ is a generalized inner coderivation. Conversely, assume that $gInn\mathrm{Coder}(M,\ C) = g\mathrm{Coder}(M,\ C)$. If $d : M \to C$ is a coderivation, then d is generalized inner by assumption, and so $d = (\alpha \otimes 1)\rho^+ + (1 \otimes \beta)\rho^-$ for some $\alpha,\ \beta \in M^*$. Substituting d in the relation $\Delta d = (d \otimes 1)\rho^+ + (1 \otimes d)\rho^-$, we see

$$\begin{aligned}
0 &= ((d \otimes 1)\rho^+ + (1 \otimes d)\rho^-) - \Delta d \\
&= (((\alpha \otimes 1)\rho^+ + (1 \otimes \beta)\rho^-) \otimes 1)\rho^+ + (1 \otimes ((\alpha \otimes 1)\rho^+ + (1 \otimes \beta)\rho^-))\rho^- \\
&\quad - \Delta((\alpha \otimes 1)\rho^+ + (1 \otimes \beta)\rho^-) \\
&= (1 \otimes \beta \otimes 1)(\rho^- \otimes 1)\rho^+ + (1 \otimes \alpha \otimes 1)(1 \otimes \rho^+)\rho^-.
\end{aligned}$$

Using $(\varepsilon \otimes 1 \otimes \varepsilon)((\rho^- \otimes 1)\rho^+) = (\varepsilon \otimes 1 \otimes \varepsilon)((1 \otimes \rho^+)\rho^-) = 1$, we have $\alpha = -\beta$, which shows that d is an inner coderivation. □

Note that for an algebra A, there holds $Inn\mathrm{Der}(A, N) = \mathrm{Der}(A, N)$ if and only if $gInn\mathrm{Der}(A, N) = g\mathrm{Der}(A, N)$.

Theorem 5.2. *Assume that C is finitely generated projective R-module. Then C is coseparable if and only if C^* is separable. Especially, C is weakly coseparable if and only if C^* is weakly coseparable.*

Proof. Assume that C is a finitely generated projective R-module. Then by Theorem 4.3, $\theta_{-1} : gInn\mathrm{Coder}(M,\ C) \to gInn\mathrm{Der}(C^*,\ M^*)$ and $\theta : g\mathrm{Coder}(M,\ C) \to g\mathrm{Der}(C^*,\ M^*)$ are isomorphisms. By Lemma 5.1, $Inn\mathrm{Coder}(C) = \mathrm{Coder}(C)$ if and only if $gInn\mathrm{Coder}(C) = g\mathrm{Coder}(C)$. Thus in the commutative diagram

$$\begin{array}{ccc}
gInn\mathrm{Coder}(M,\ C) & \xrightarrow{\iota_M} & g\mathrm{Coder}(M,\ C) \\
\Big\downarrow{\scriptstyle \theta_{-1}} & & \Big\downarrow{\scriptstyle \theta} \\
gInn\mathrm{Der}(C^*,\ M^*) & \xrightarrow{\iota_{M^*}} & g\mathrm{Der}(C^*,\ M^*),
\end{array}$$

ι_M is an isomorphism if and only if ι_{M^*} is an isomorphim. □

Finally, we give a simple example of a weakly coseparable coalgebra.

Example 5.3. *Let $R[X]$ be a free R-module with free basis $\{1,\ X,\ \cdots,\ X^k, \cdots\}$. We denote the dual basis of $R[X]$ by $\{x_0,\ x_1,\ \cdots,\ x_k, \cdots\}$.*

(1) Let $C_1 = R[X]$ and define a coalgebra structure maps $\varDelta_1 : C_1 \to C_1 \otimes C_1$ and $\varepsilon_1 : C_1 \to R$ as follows:

$$\varDelta_1(X^n) = (X \otimes X)^n, \quad \varepsilon_1(X^n) = 1 \quad \text{for all} \quad n = 0,\ 1,\ 2,\ \cdots$$

Assume that

$$(f_1\ ;\ \xi) : C_1 \ni X \mapsto f_1(X) = \sum_{i=0}^{m} a_i X^i \in C_1 = R[X]$$

is a generalized coderivation. Then by $\varDelta_1 f_1(X) = ((f_1 \otimes 1 + 1 \otimes f_1))\varDelta_1 + (1 \otimes \xi \otimes 1)(\varDelta_1 \otimes 1)\varDelta_1(X)$, we see $f_1(X) = a_1 X$ and $\xi(X) = -a_1$. By induction, we can easily prove that any generalized coderivation of C_1 is the following form

$$f_1(X^n) = a_n X^n \quad \text{and} \quad \xi(X^n) = -a_n.$$

Therefore any generalized coderivation is generalized inner and thus C_1 is weakly coseparable.

Next, we show that if 2 is not a zero divisor in R, the dual algebra C_1^* is also coseparable. Since C_1 is cocommutative, C_1^* is commutative and so it is enough to show that $Der(C_1^*) = 0$. In the dual basis $\{x_0,\ x_1,\ \cdots,\ x_k,\ \cdots\}$ of $\{1,\ X,\ \cdots,\ X^k,\ \cdots\}$, the convolution product $\circ$ in C_1^* is given by

$$(x_i \circ x_j)(X^n) = (x_i \otimes x_j)(X^n \otimes X^n) = \delta_{in}\delta_{jn},$$

where δ_{ij} is the Kronecker's δ. Then the set of dual basis $\{x_0,\ x_1,\ \cdots,\ x_k,\ \cdots\}$ is the system of orthogonal idempotents in C_1^*. Assume that

$$d : C_1^* \ni x_k \mapsto \sum_{j=0}^{m} a_{ij} x_j \in C_1^* = R[X]^*$$

is a derivation. Since d is a derivation, we have

$$\begin{aligned} d(x_i x_k) = d(x_i)x_k + x_i d(x_k) &= \sum_{j=0}^{m} a_{ij} x_j x_k + x_i \sum_{j=0}^{m} a_{kj} x_j \\ &= a_{ik} x_k + a_{ki} x_i. \end{aligned}$$

Thus if $i \neq k$, then $a_{ik}x_k + a_{ki}x_i = 0$ and so $a_{ik} = a_{ki} = 0$ for all $i \neq k$. Moreover, by $d(x_i^2) = d(x_i) = \sum_{j=0}^{m} a_{ij}x_j = 2a_{ii}x_i$, we have $a_{ii} = 0$. Thus $d(x_i) = 0$ for all i, which show that C_1^* is a weakly separable algebra.

(2) Let $C_2 = R[X]$ and define a coalgebra structure maps $\varDelta_2 : C_2 \to C_2 \otimes C_2$ and $\varepsilon_2 : C_2 \to R$ as follows:

$$\varDelta_2(X^n) = (X \otimes 1 + 1 \otimes X)^n, \quad \varepsilon_2(1) = 1,\ \varepsilon_2(X^n) = 0 \quad \text{for all} \quad n = 1,\ 2,\ \cdots.$$

Define an R-linear map $f_2 : C_2 \to C_2$ as follows:

$$f_2(1) = 0,\ f_2(X) = a_1 X,\ f_2(X^2) = a_2 X + 2a_1 X^2, \cdots ,$$
$$f_2(X^n) = a_n X + \binom{n}{1} a_{n-1} X^2 + \cdots + \binom{n}{i} a_{n-i} X^{i+1} + \cdots + a_1 \binom{n}{n-1} X^n,$$

$(a_i \in R)$. Then we can check that f_2 is a coderivation. Therefore there are many coderivations in $Coder(C_2)$. Take $a_1 = 1$. If f_2 is an inner coderivation, then there exists a non-zero R-linear map $\alpha : C_2 \to R$ such that $f_2 = (\alpha \otimes 1 - 1 \otimes \alpha)\Delta_2$ and so we have $f_2(X) = X = (\alpha \otimes 1 - 1 \otimes \alpha)\Delta_2(X) = 0$, a contradiction. Thus there exists a coderivation which is not inner. This shows that the coalgebra C_2 is not weakly coseparable. In this case, the dual algebra C_2^* is not weakly separable. Because, noting the relation

$$\Delta(X^n) = \sum_{i=0}^{n} \binom{n}{i} X^{n-i} \otimes X^i,$$

we see $x_i \circ x_j = \binom{i+j}{i} x_{i+j}$. Define an R-linear map

$$d : C_2^* \ni x_n \mapsto n x_n \in C_2^*, \quad \textit{for all} \quad n = 1,\ 2,\ \cdots .$$

Then we can check that d is a non-zero derivation. Since C_2^* is commutative algebra, d is not inner. Therefore C_2^* is not weakly separable.

In our paper [5], we only treat a weakly separable polynomial $f(X)$, that is, $R[X]/(f(X))$ is a finitely generated projective R-module. The above example shows that there exists a non-finitely generated weakly coseparable algebra and a weakly separable algebra.

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