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DIVISIBILITY PROPERTIES RELATED TO STAR-OPERATIONS ON INTEGRAL DOMAINS

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Abstract. An integral domain R is a GCD-Bezout domain if the Bezout identity holds for any finite set of nonzero elements of R whose gcd exists. Such domains are characterized as the DW-domains having the PSP-property. Using the notion of primitive and superprimitive ideals, we define a (semi)star operation, the q-operation, which is closely related to the w-operation and the p-operation introduced by Anderson. We use q-operation to characterize the GCD-Bezout domains and study various properties of these domains.

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1. Introduction

Throughout R is an integral domain with quotient field K. To avoid triviality, we assume that R is not a field.

For a finite set $\{a_1, \cdots, a_n\}$ of nonzero elements of R, if $(a_1, \cdots, a_n)_v = (z)$ (where I_v is the divisorial closure of an I ideal defined in the following), then z is defined to be a v-gcd of $a_1, \cdots, a_n$. It is easy to check that if z is a v-gcd of $a_1, \cdots, a_n$, then z is a greatest common divisor of $a_1, \cdots, a_n$, but the converse is not always true. For instance, take $R = F[X^2, X^3]$, where F is a field, and consider the ideal (X^2, X^3). Then $(X^2, X^3)_v = (X^2, X^3)$ is not principal, but 1 is a greatest common divisor of X^2, X^3. Thus $\gcd(X^2, X^3)$ exists but v-$\gcd(X^2, X^3)$ does not.

We will see later that the notions of "primitive" and "superprimitive" ideals are closely related to those of "gcd" and "v-gcd", respectively.

The third concept that will be central in our discussion is the "Bezout identity". Given nonzero elements $a_1, \cdots, a_n \in R$, the Bezout identity holds for $a_1, \cdots, a_n$ if $\gcd(a_1, \cdots, a_n)$ exists and it is expressible as a linear combination

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on R of $a_1, \cdots, a_n$. It is evident that the Bezout identity holds for $a_1, \cdots, a_n$ if and only if $(a_1, \cdots, a_n)$ is a principal ideal.

Thus we consider the relations among the following conditions on an integral domain R: given a finite set $\{a_1, \cdots, a_n\}$ of nonzero elements of R,

(i) the Bezout identity holds for $a_1, \cdots, a_n$;
(ii) v-gcd$(a_1, \cdots, a_n)$ exists;
(iii) gcd$(a_1, \cdots, a_n)$ exists.

We observed that (i) implies (ii), (ii) implies (iii), but (iii) does not imply (ii).

On the other hand, it is well-known that if a gcd exists for each finite set of nonzero elements of R (or equivalently, if a gcd exists for any two nonzero elements of R), then a v-gcd exists and it is equal to a gcd. Thus, in a GCD-domain, the two concepts of "gcd" and "v-gcd" are the same. But, even in this case, the Bezout identity is still a stronger property. To show this, consider a Noetherian, GCD-domain which is not a PID: for instance, take $R = F[X, Y]$, where F is a field. It is well-known that R is not a Bezout domain, but it is a GCD-domain and so the v-gcd exists for any two nonzero elements of R.

We denote by $\overline{\mathbf{F}}(R)$ the set of nonzero R-submodules of K, by $\mathbf{f}(R)$ the set of nonzero finitely generated R-submodules of K, and by $\mathbf{F}(R)$ the set of nonzero fractional ideals of R.

Recall (cf. [7] and [15]) that a mapping $E \mapsto E^\star$ of $\overline{\mathbf{F}}(R)$ into $\overline{\mathbf{F}}(R)$ is called a *semistar operation* if the following conditions hold for all $x \in K \setminus \{0\}$ and $E, F \in \overline{\mathbf{F}}(R)$:

($\star_1$) $(xE)^\star = xE^\star$;
($\star_2$) $E \subseteq F$ implies $E^\star \subseteq F^\star$;
($\star_3$) $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

Moreover, if $R^\star = R$, then the semistar operation restricted to the set $\mathbf{F}(R)$ is called a star operation.

Given a semistar operation $\star$, a nonzero ideal I of R such that $I = I^\star$ is called a $\star$*–ideal.*

The most important examples of semistar operations are the d-operation, the v-operation, and the t-operation:

- The d-operation is the identity mapping $E \mapsto E$.
- The v-operation is defined by

$$E \mapsto E_v = \begin{cases} (R : (R : E)) & \text{if } E \in \mathbf{F}(R) \\ K & \text{if } E \in \overline{\mathbf{F}}(R) \setminus \mathbf{F}(R) \end{cases}$$

where $(R : E) = \{x \in K; xE \subseteq R\}$.

- The t-operation is given by $E \mapsto E_t = \bigcup_{J \in \mathbf{f}(R),\, J \subseteq E} J_v$.

A nonzero ideal J of R is called a *Glaz-Vasconcelos ideal*, in short, a GV-ideal, if J is finitely generated and $J^v = R$ (cfr. [10]). The set of GV-ideals of R is denoted by $\mathrm{GV}(R)$. For each $E \in \overline{\mathbf{F}}(R)$, define E_w to be the R-module

$$E_w := \{x \in K \mid xJ \subseteq E, \text{ for some } J \in \mathrm{GV}(R)\} = \bigcup_{J \in \mathrm{GV}(R)} (E\colon J).$$

Then the operation $E \mapsto E_w$ is a semistar operation, called the w-operation.

A semistar operation $\star$ is of *finite type* if for each $E \in \overline{\mathbf{F}}(R)$:

$$E^\star = \bigcup_{H \subseteq E, H \in \mathbf{f}(R)} H^\star,$$

and $\star$ is *stable* if for each $E, F \in \overline{\mathbf{F}}(R)$:

$$(E \cap F)^\star = E^\star \cap F^\star.$$

For each semistar operation $\star$, $\tilde{\star}$ usually denotes the stable semistar operation of finite type associated to $\star$, which is defined as follows (see [7, p 185]): for $E \in \overline{\mathbf{F}}(R)$,

$$E^{\tilde{\star}} := \bigcup \{(E : F) \mid F \in \mathbf{f}(R),\, F^\star = R^\star\}.$$

According to this notation, $w = \tilde{v}$.

Given a semistar operation $\star$, a $\star$*–maximal* ideal is an ideal that is a maximal element in the set of proper integral $\star$–ideals. If $\star$ is of finite type, then each proper integral $\star$–ideal is contained in a $\star$–maximal ideal and each $\star$–maximal ideal is prime ([7, Lemma 4.20]).

It is well-known that the v-operation is maximal among the star operations on R in the sense that $I^\star \subseteq I_v$ for each $I \in \mathbf{F}(R)$ and each star operation $\star$. Moreover, the t-operation is maximal among the finite-type star operations on R and the w-operation is maximal among the star operations on R which are stable and of finite type.

In [14] and [16], the authors studied the domains in which the star operations w and d coincide, which are called the *DW-domains*. They investigated the multiplicative ideal properties of these domains, such as their behavior with respect to localizations, their integral closure, their relations with Prüfer domains and Noetherian domains.

In Section 2 of this paper, we will see that the DW-domains are exactly the domains in which for any finite set $\{a_1, \cdots, a_n\}$ of nonzero elements, the existence of v-gcd$(a_1, \cdots, a_n)$ implies that the Bezout identity holds for $a_1, \cdots, a_n$ (cfr. Proposition 2.1). Thus we have a connection between the ideal property "$w = d$" and the arithmetic property "Bezout identity $\Leftrightarrow$ v-gcd". By substituting the v-gcd with the gcd, we find that the equivalence "Bezout identity ⇔ gcd" characterizes a class of domains which are very similar to DW-domains (cfr. Proposition 2.6). We call such domains *GCD-Bezout domains* (cfr. Definition 2.4). Bezout domains are of course GCD-Bezout. An interesting example of a GCD-Bezout domain is given by a non-Bezout Prüfer domain in which all maximal ideals are principal (cfr. [1], [5], [13]).

In order to study GCD-Bezout domains as a subclass of DW-domains, we will use the concepts of primitive and superprimitive ideals (Definition 2.2), introduced by J. Arnold & P. Sheldon in a 1975's paper ([1]). This will bring us to study GCD-Bezout domains by means of semistar and star operations (Section 2). In this context we will also investigate the p-operation introduced by D.F. Anderson in 1981 ([2]), which is in a close connection with the gcd concept. We will manipulate the p-operation in order to "make it" a semistar operation suitable to characterize GCD-Bezout domains (cfr. Corollaries 3.6 and 3.15).

In Section 3, we will give several results concerning the GCD-Bezout property in pullback constructions. Lastly, in Section 4, we will briefly consider the Nagata ring to construct examples of nontrivial GCD-Bezout domains.

2. GCD-Bezout domains

In this section we introduce the notion of GCD-Bezout domain and study some properties of this class of domains (among others, connections with Noetherian and Prüfer domains). We start with a result concerning DW-domains, from which the definition of GCD-Bezout domain arises naturally.

Proposition 2.1. *Let R be an integral domain. The following conditions are equivalent:*

(i) *R is a DW-domain.*

(ii) *There do not exist proper finitely generated ideals I such that $I_v = R$.*

(iii) *For any finite set $\{a_1, \cdots, a_n\}$ of nonzero elements of R such that v-gcd$(a_1, \cdots, a_n)$ exists, the Bezout identity holds for $a_1, \cdots, a_n$.*

Proof. (i) ⇔ (ii) It follows from [16, Corollary 2.6].

(ii) ⇒ (iii) If d is a v-gcd of $a_1, \cdots, a_n$, then $(a_1, \cdots, a_n)_v = (d)$ and hence $(\frac{a_1}{d}, \cdots, \frac{a_n}{d})_v = R$. By (ii), $(\frac{a_1}{d}, \cdots, \frac{a_n}{d}) = (1)$, i.e., $(a_1, \cdots, a_n) = (d)$. Therefore, the Bezout identity holds for $a_1, \cdots, a_n$.

(iii) ⇒ (ii) Let $I := (a_1, \cdots, a_n)$ be a finitely generated ideal such that $I_v = R$. This implies that 1 is a v-gcd$(a_1, \cdots, a_n)$, and hence by (iii), $(a_1, \cdots, a_n) = (1)$. Thus we have $I = R$. □

J. Arnold and P. Sheldon in [1] give the following definition:

Definition 2.2. Let I be a nonzero finitely generated ideal in R. Then I is called *primitive* if $I \nsubseteq aR$ for any nonunit element $a \in R$, and it is called *superprimitive* if $I_v = R$.

Proposition 2.3. *For a nonzero finitely generated ideal $I = (a_1, \cdots, a_n)$ in R, the following equivalences hold:*

(1) *I is primitive if and only if* $\gcd(a_1, \cdots, a_n) = 1$*;*

(2) *I is superprimitive if and only if* v-$\gcd(a_1, \cdots, a_n) = 1$ *if and only if I is a GV-ideal.*

Proof. The proof follows directly from the definitions. □

Thus a DW-domain is an integral domain in which there do not exist proper superprimitive ideals. In Proposition 2.1 we have seen that these domains may be characterized in terms of some divisibility properties.

Now it is natural to ask whether the integral domains in which there do not exist proper primitive ideals also satisfy some relevant divisibility properties. For this purpose we give the following definition.

Definition 2.4. An integral domain R is a *GCD-Bezout domain* if given nonzero elements $a_1, \cdots, a_n \in R$, the existence of a $\gcd(a_1, \cdots, a_n)$ implies that the Bezout identity holds for $a_1, \cdots, a_n$.

We compare the Bezout domains and the GCD-Bezout domains. In a Bezout domain R, the Bezout identity holds for any finite set of elements (and hence R is a GCD-domain).

In a GCD-Bezout domain R, the Bezout identity holds only for the finite sets of elements having a gcd (and hence R is not necessarily a GCD-domain).

By replacing gcd with v-gcd in Definition 2.4, we can also give the following definition:

Definition 2.5. An integral domain R is a *v-GCD-Bezout domain* if given nonzero elements $a_1, \cdots, a_n \in R$, the existence of a v-$\gcd(a_1, \cdots, a_n)$ implies that the Bezout identity holds for $a_1, \cdots, a_n$.

By Proposition 2.1, we have that the v-GCD-Bezout domains are exactly the DW-domains. Therefore, the class of GCD-Bezout domains is a subclass of DW-domains.

Proposition 2.6. *Let R be an integral domain. The following conditions are equivalent:*

(i) *R is a GCD-Bezout domain.*
(ii) *There do not exist proper primitive ideals in R.*
(iii) *For any finite set* $\{a_1, \cdots, a_n\}$ *of nonzero elements of* R *such that* $\gcd(a_1, \cdots, a_n)$ *exists, the Bezout identity holds for* $a_1, \cdots, a_n$.

Proof. (i) $\Leftrightarrow$ (iii) It follows from Definition 2.4.

(ii) $\Rightarrow$ (iii) Let d be a gcd of $a_1, \cdots, a_n$. Then 1 is a gcd of $\frac{a_1}{d}, \cdots, \frac{a_n}{d}$, and hence the ideal $(\frac{a_1}{d}, \cdots, \frac{a_n}{d})$ is primitive. By the assumption (ii), we have that $(\frac{a_1}{d}, \cdots, \frac{a_n}{d}) = (1)$, whence $(a_1, \cdots, a_n) = (d)$. Thus, the Bezout identity holds for $a_1, \cdots, a_n$.

(iii) $\Rightarrow$ (ii) Let $I := (a_1, \cdots, a_n)$ be a primitive ideal of R. Then 1 is a gcd of $a_1, \cdots, a_n$. By the assumption (iii), $(a_1, \cdots, a_n) = (1) = R$. Thus, I is not a proper ideal. $\square$

Since the DW-domains are a very natural generalization of the domains in which $t = d$ (in the integrally closed case, these are exactly the Prüfer domains), we ask whether there is any relationship between the GCD-Bezout domains and the domains in which $t = d$.

The following examples show that these two concepts are not related (even in the integrally closed case). Recall first that an integral domain R is called a pseudo-valuation domain if there is a valuation overring V such that $\operatorname{Spec}(R) = \operatorname{Spec}(V)$. In this case, V is uniquely determined and called the associated valuation domain of R.

Example 2.7. *Let R be an integrally closed pseudo-valuation (not valuation) domain with associated valuation domain V and idempotent maximal ideal M. (We will construct such a domain specifically using a pullback diagram in Example 4.3.) Then $t \neq d$ on R, otherwise R would be a valuation domain against the assumption.*

Now let I be a proper finitely generated ideal of R. Then IV is a principal ideal of V, say aV for some $a \in M$. Since $M = M^2$, $a \in bM$ for some $b \in M$. Then we have $I \subseteq aV \subseteq bMV = bM \subseteq bR \subsetneq R$. It follows that R does not have proper primitive ideals. Thus, by Proposition 2.6, R is a GCD-Bezout domain in which $t \neq d$.

Example 2.8. *Take a Dedekind domain R with a nonprincipal maximal ideal M. (It is enough to consider a Dedekind domain which is not a PID; for instance, $\mathbb{Z}[\sqrt{-5}]$.) Then, M is a proper primitive ideal and by Proposition 2.6 whence R is not a GCD-Bezout domain. But $t = d$, because R is a Prüfer domain.*

The above example suggests the following:

Lemma 2.9. *Let R be a GCD-Bezout domain. A maximal ideal of R is finitely generated if and only if it is principal.*

Proof. Let M be a finitely generated maximal ideal of R. Since R is a GCD-Bezout domain, M cannot be a primitive ideal, and hence $M \subsetneq (a)$ for some nonunit element $a \in R$. By maximality of M, M must be equal to (a). □

Corollary 2.10. *Let R be a Noetherian domain. Then R is a GCD-Bezout domain if and only if R is a PID.*

Proof. By Lemma 2.9, if R is a Noetherian GCD-Bezout domain, then every maximal ideal of R is principal, and hence R is a PID. The other direction is obvious. □

Thus, for a Noetherian domain, R being GCD-Bezout is equivalent to R being Bezout, whereas there exist Noetherian GCD-domains which are not Bezout; for example, consider $R = \mathbb{Z}[X]$.

Following [1], we will say that an integral domain R satisfies the *PSP-property*, or that R is a *PSP-domain*, if each primitive ideal of R is superprimitive (that a superprimitive ideal is primitive does always hold).

Proposition 2.11. *An integral domain R is a PSP-domain if and only if for any finite set $\{a_1, \cdots, a_n\}$ of nonzero elements of R, the existence of* $\gcd(a_1, \cdots, a_n)$ *implies the existence of* v-$\gcd(a_1, \cdots, a_n)$.

Proof. ($\Rightarrow$) Let d be a gcd of $a_1, \cdots, a_n$. Then 1 is a gcd of $\frac{a_1}{d}, \cdots, \frac{a_n}{d}$. This implies that the ideal $(\frac{a_1}{d}, \cdots, \frac{a_n}{d})$ is primitive. Since R is a PSP-domain, $(\frac{a_1}{d}, \cdots, \frac{a_n}{d})$ is superprimitive, i.e., $(\frac{a_1}{d}, \cdots, \frac{a_n}{d})_v = R$. Therefore, $(a_1, \cdots, a_n)_v = (d)$, whence d is a v-gcd of $a_1, \cdots, a_n$.

($\Leftarrow$) Let $(a_1, \cdots, a_n)$ be a primitive ideal of R. Then 1 is a gcd of $a_1, \cdots, a_n$. By the assumption, a v-gcd of $a_1, \cdots, a_n$ exists. It is obvious that 1 is a v-$\gcd(a_1, \cdots, a_n)$ and hence that $(a_1, \cdots, a_n)$ is a superprimitive ideal. Thus R is a PSP-domain. □

Corollary 2.12. *An integral domain R is a GCD-Bezout domain if and only if R is a DW-domain satisfying the PSP-property.*

Proof. It follows directly from Propositions 2.1, 2.6, and 2.11. □

Remark 2.13. *Examples 2.7 and 2.8 show that there is no relation between integrally closed GCD-Bezout domains and Prüfer domains. One of our interests is to give characterizations of Prüfer domains with the GCD-Bezout property. By Corollary 2.12, we know that a Prüfer domain R is a GCD-Bezout domain if and only* *if R is a PSP-domain (because a Prüfer domain is always a DW-domain). Therefore, our question is equivalent to looking for characterizations of Prüfer domains with the PSP-property. This is a quite old problem considered, at first, in* [1], *and then for some particular cases, in* [5] *and* [13]. *We leave the question open so far, but the following argument shows that the investigation of such domains could be interesting.*

In [1], *the authors ask whether a Prüfer domain with the PSP-property is a Bezout domain. (They are motivated by some results linking Bezout-, PSP-, and GL-property.) In order to answer this question, they suggest to consider the Prüfer domains in which all maximal ideals are principal, defined in the next section. Even though A. Loper showed later that such Prüfer domains are not necessarily Bezout domains (*[13]*), this case is also very interesting to us in the "GCD-Bezout context", because a domain whose maximal ideals are all principal is a GCD-Bezout domain, by Proposition 2.6. Thus, in a Prüfer domain, the GCD-Bezout property does not necessarily imply that the given domain is a Bezout domain, unlike in the Noetherian case (see Corollary 2.10).*

In [14] and in [16], the authors investigate some questions about the localization of DW-domains. They show that if R_M is a DW-domain for each maximal ideal M of R, then R is a DW-domain, and that the converse also holds when R is v-coherent [14, Theorem 2.9]. (Recall that an integral domain R is *v-coherent* if for each nonzero finitely generated ideal J of R, $J_v = H^{-1} = (R\colon H)$ for some finitely generated ideal H in R.) Moreover, in [16, Proposition 3.11], the last statement is proven for a class of domains larger than the v-coherent domains.

But, for an arbitrary domain R, it is not known yet whether the DW-property is preserved under the localization at each maximal ideal. In the following, we give a positive answer in the case where the representation $R = \bigcap_{M\in \mathrm{Max}(R)} R_M$ is locally finite (i.e., each nonzero element of R is contained in only finitely many maximal ideals of R). Nextly, we will give some results about the localization of GCD-Bezout domains.

Proposition 2.14. *Let R be an integral domain with the representation $R = \bigcap_{M\in \mathrm{Max}(R)} R_M$ being locally finite. Then R is a DW-domain if and only if R_M is a DW-domain for each $M \in \mathrm{Max}(R)$.*

Proof. If R_M is a DW-domain for each $M \in \mathrm{Max}(R)$, then R is a DW-domain by [14, Theorem 2.9].

Conversely, assume that R is a DW-domain and let M be a maximal ideal of R. We will show that R_M is a DW-domain, i.e., $\mathrm{GV}(R_M) = \{R_M\}$. Suppose, on the contrary, that there exists a proper GV-ideal J of R_M. Then $J = IR_M$ for some finitely generated ideal I of R contained in M. Let $S = \{M_1, \cdots, M_r\}$ be the set of maximal ideals of R containing I and let $M = M_1$. Choose an element $a \in M \setminus \bigcup_{i=2}^{r} M_i$ and put $J' = (I, a)R_M$. Then $J \subseteq J'$, whence $R_M = J^{-1} \supseteq J'^{-1} = (I, a)^{-1}R_M \supseteq R_M$. Thus we have $(I, a)^{-1}R_M = R_M$. Moreover, by construction, $(I, a)R_N = R_N$, so $(I, a)^{-1}R_N = ((I, a)R_N)^{-1} = R_N$ for each $N \in \mathrm{Max}(R) \setminus \{M\}$. Therefore, $(I, a)^{-1} = R$, i.e., $(I, a) \in \mathrm{GV}(R)$. But $(I, a) \subseteq M \subsetneq R$ and this contradicts that R is a DW-domain. □

Proposition 2.15. *Let R be an integral domain with the representation $R = \bigcap_{M \in \mathrm{Max}(R)} R_M$ being locally finite. If R is a GCD-Bezout domain, then R_M is a GCD-Bezout domain for each $M \in \mathrm{Max}(R)$.*

Proof. Let M be a maximal ideal of R and consider the ring R_M. Let J be a proper finitely generated integral ideal of R_M; then $J = IR_M$ for some finitely generated ideal I of R contained in M. As in the proof of Proposition 2.14, let $S = \{M_1, \cdots, M_r\}$ be the set of maximal ideals of R containing I and let $M = M_1$. Take $a \in M \setminus \bigcup_{i=2}^{r} M_i$ and consider the ideal $I' = (I, a)$ of R. Since R is a GCD-Bezout domain and I' is a proper finitely generated ideal of R, I' is not primitive, i.e., $I' \subseteq bR$ for some nonunit element $b \in R$. By construction of I', $b \in M$, and hence $J \subseteq bR_M \subsetneq R_M$. Thus any proper finitely generated ideal of R_M is not primitive. Therefore, R_M is a GCD-Bezout domain by Proposition 2.6. □

Remark 2.16. *The converse of Proposition 2.15 does not hold in general. Consider a Dedekind domain R which is not a PID. Then R_M is a GCD-Bezout domain for each $M \in \mathrm{Max}(R)$, but R is not (Example 2.8).*

As a consequence of Remark 2.16 and Corollary 2.12 combined with the fact that a locally DW-domain is DW, we have that a locally PSP-domain is not necessarily a PSP-domain.

3. A semistar approach

In [2] D.F.Anderson introduced the *p-operation*. For a nonzero integral ideal I of R, I^p is the intersection of all the principal integral ideals of R containing I. It is well-known that for any star operation $\star$, $I^\star \subseteq I_v \subseteq I^p$. Since $R^p = R$, the p-operation defines a semistar operation on R if and only if it defines a star operation on R, if and only if $p = v$. An easy observation is that an ideal I is primitive if and only if I is finitely generated and $I^p = R$.

Modeling after the construction of the w-operation, which is equal to $\tilde{v}$, we define another operation associated to the p-operation.

Definition 3.1. For each nonzero R-module $E \in \overline{\mathbf{F}}(R)$, we define the $\tilde{p}$-closure of E to be the set

$$\begin{aligned} E^{\tilde{p}} :&= \bigcup \{(E : J) \mid J \in \mathbf{f}(R),\ J^p = R\} \\ &= \bigcup \{(E : J) \mid J \text{ is primitive}\}. \end{aligned}$$

Like the p-operation, the $\tilde{p}$-operation, $E \mapsto E^{\tilde{p}}$, is not in general a semistar operation. It may happen even the case that $E^{\tilde{p}}$ is not an R-module for some $E \in \overline{\mathbf{F}}(R)$. We will investigate the domains in which the $\tilde{p}$-operation is a semistar or a star operation.

For all $x \in K \setminus \{0\}$ and $E, F \in \overline{\mathbf{F}}(R)$, it is easy to check that:

- $(xE)^{\tilde{p}} = xE^{\tilde{p}}$;
- $E \subseteq F$ implies $E^{\tilde{p}} \subseteq F^{\tilde{p}}$;
- $E \subseteq E^{\tilde{p}}$.

Thus, in order to have that $\tilde{p}$ is semistar it remains to determine when:

($\tilde{p}\,1$) the set $E^{\tilde{p}}$ is an R-module;
($\tilde{p}\,2$) $(E^{\tilde{p}})^{\tilde{p}} = E^{\tilde{p}}$, for each $E \in \overline{\mathbf{F}}(R)$.

Let us look into the reasons why $E^{\tilde{p}}$ may not be an R-module. For elements $x, y \in E^{\tilde{p}}$, we have that $xJ \subseteq E$ and $yH \subseteq E$ for some primitive ideals J and H. It is obvious that for any element $d \in R$, $(dx)J \subseteq dE \subseteq E$ and hence that $dx \in E^{\tilde{p}}$. The problem occurs with $x \pm y$. In fact, it is not guaranteed that there exists a primitive ideal L such that $(x \pm y)L \subseteq E$. The best candidate for L is the ideal JH. If the product JH is a primitive ideal, then clearly $(x \pm y)JH \subseteq E$ and so $x \pm y \in E^{\tilde{p}}$. But, while it is well-known that the product of superprimitive ideals is superprimitive, this is not the case for the primitive ideals.

In [1], the authors introduce the notion of *Gauss Lemma domain*, in short, GL-domain. This is an integral domain in which the product of primitive ideals is also a primitive ideal.

It is straightforward that PSP-domains are GL-domains.

Thus, in GL-domains, the first condition ($\tilde{p}\,1$) for $\tilde{p}$ to be semistar is verified. Moreover, it will be shown in the next proposition that the second point ($\tilde{p}\,2$) is also settled.

Proposition 3.2. *Let R be a GL-domain. Then the $\tilde{p}$-operation is a semistar operation which is stable and of finite type.*

Proof. In the paragraph above, we have seen that if R is a GL-domain then $(\tilde{p}\,1)$ holds. To prove that the $\tilde{p}$-operation is a semistar operation it only remains to show that $(\tilde{p}\,2)$ holds, that is $(E^{\tilde{p}})^{\tilde{p}} = E^{\tilde{p}}$ for all $E \in \overline{\mathbf{F}}(R)$. Since $E^{\tilde{p}} \in \overline{\mathbf{F}}(R)$ as shown above, $E^{\tilde{p}} \subseteq (E^{\tilde{p}})^{\tilde{p}}$. Now let $x \in (E^{\tilde{p}})^{\tilde{p}}$. Then, $xJ \subseteq E^{\tilde{p}}$ for some primitive ideal $J = (a_1, \cdots, a_n)$ of R. Then for each $i = 1, \cdots, n$, $xa_i \in E^{\tilde{p}}$ and hence $xa_iH_i \subseteq E$ for some primitive ideal H_i of R. Put $H := H_1H_2\cdots H_n$. Since R is a GL-domain, H is primitive and $xa_iH \subseteq E$ for all $i = 1, \cdots, n$. This implies that $xJH \subseteq E$. Again by the assumption that R is a GL-domain, JH is primitive and hence $x \in E^{\tilde{p}}$. Thus we have $(E^{\tilde{p}})^{\tilde{p}} = E^{\tilde{p}}$.

That the $\tilde{p}$-operation is of finite type follows at once from the equality $(E : J) = \bigcup\{(F : J) \mid F \subseteq E,\ F \in \mathbf{f}(R)\}$ for each $E \in \overline{\mathbf{F}}(R)$ and primitive ideal J of R.

As regards the stability, we have to prove that $(E \cap F)^{\tilde{p}} = E^{\tilde{p}} \cap F^{\tilde{p}}$ for each $E, F \in \overline{\mathbf{F}}(R)$. The inclusion $(\subseteq)$ is obvious (it is an easy consequence of the fact that if $F \subseteq E$, then $F^{\tilde{p}} \subseteq E^{\tilde{p}}$). For the opposite inclusion $(\supseteq)$, let $x \in E^{\tilde{p}} \cap F^{\tilde{p}}$. Then $xJ \subseteq E$, $xH \subseteq F$ for some primitive ideals J, H of R. Put $L = JH$, then L is primitive and $xL \subseteq E \cap F$. Thus we have $x \in (E \cap F)^{\tilde{p}}$. □

The next proposition states that the condition R is a GL-domain is not only sufficient but also necessary for the $\tilde{p}$-operation to be a semistar operation.

Proposition 3.3. *An integral domain R is a GL-domain if and only if the $\tilde{p}$-operation is a semistar operation on R.*

Proof. The "only if" part was shown in Proposition 3.2.

For the "if" part, suppose that R is not a GL-domain. Then there exist primitive ideals J and H in R such that their product JH is not primitive. Thus $JH \subseteq aR$ for some nonunit element $a \in R$. Then $(JH)^{\tilde{p}} \subsetneq R^{\tilde{p}}$, i.e., $1 \notin (JH)^{\tilde{p}}$. Otherwise, there would exist a primitive ideal L contained in JH. But then $L \subseteq aR$, which contradicts that L is primitive.

Since the $\tilde{p}$-operation is a semistar operation on R, it is a star operation on $R^{\tilde{p}}$. By the same argument used in the proof of Proposition 3.2, the $\tilde{p}$-operation is of finite type. So every proper $\tilde{p}$-ideal of $R^{\tilde{p}}$ is contained in a $\tilde{p}$-maximal ideal and each $\tilde{p}$-maximal ideal is a prime ideal. Let N be a $\tilde{p}$-maximal ideal of $R^{\tilde{p}}$ containing $(JH)^{\tilde{p}}$. Then $JH \subseteq (JH)^{\tilde{p}} \subseteq N$, and hence $J \subseteq N$ or $H \subseteq N$. It follows that $J^{\tilde{p}} \subseteq N^{\tilde{p}} = N$ or $H^{\tilde{p}} \subseteq N^{\tilde{p}} = N$. But, either case is impossible, because J and H are primitive and hence $J^{\tilde{p}} = R^{\tilde{p}} = H^{\tilde{p}}$, while $N \subsetneq R^{\tilde{p}}$. □

From Propositions 3.2 and 3.3 we deduce the following corollary:

Corollary 3.4. *In an integral domain R, if the $\tilde{p}$-operation is a semistar operation, then $\tilde{p}$ is stable and of finite type.*

We can also characterize the domains R in which the $\tilde{p}$-operation (restricted to $\mathbf{F}(R)$) is a star operation.

Proposition 3.5. *Let R be an integral domain. The following conditions are equivalent:*

(i) *R is a PSP-domain;*

(ii) *R is a GL-domain and $R^{\tilde{p}} = R$;*

(iii) *the $\tilde{p}$-operation is a star operation on R; in this case, $\tilde{p} = w$.*

Proof. (i) $\Rightarrow$ (ii) PSP-domains are always GL-domains. Moreover,

$$\begin{aligned} R^{\tilde{p}} &= \bigcup\{(R : J) \mid J \text{ is primitive}\} \\ &= \bigcup\{(R : J) \mid J \text{ is superprimitive}\} \\ &= R_w = R. \end{aligned}$$

(ii) $\Rightarrow$ (iii) It directly follows from Proposition 3.3.

(iii) $\Rightarrow$(i) Assume that the $\tilde{p}$-operation is a star operation. Then

$$R^{\tilde{p}} = \bigcup\{(R : J) \mid J \text{ is primitive}\} = R,$$

and hence $(R\colon J) = R$ for each primitive ideal J of R. This implies that each primitive ideal J is superprimitive (since $J_v = (R\colon (R\colon J))$). Therefore, R is a PSP-domain. □

[1, Example 2.5] is an example of a GL-domain which does not have the PSP-property. So it may happen that $\tilde{p}$ is semistar but not star.

Corollary 3.6. *An integral domain R is a GCD-Bezout domain if and only if $\tilde{p} = d$ as star operations on R.*

Proof. It follows directly from Corollary 2.12 and Proposition 3.5. □

Remark 3.7. *If p is a star operation, i.e., $p = v$, then from the definition of $\tilde{p}$, it follows that $\tilde{p} = \tilde{v} = w$. But the converse does not hold, i.e., the condition $p = v$ is not necessary to have that $\tilde{p} = w$. An example is given in* [2, p 171]*: Let K be a field and let $V = K + M$ be a 1-dimensional nondiscrete valuation domain, where M is the maximal ideal of V. For a proper subfield F of K, let $R = F + M$. Then R is a PSP-domain (*[1, Lemma 3.8]*) and hence $\tilde{p} = w = \tilde{v}$, but $p \neq v$ by* [2, Proposition 2.3]. *Thus we can have that $\tilde{p}$ is a star operation even if p is not a star operation.*

Next, we will provide new characterizations of UFDs in terms of the $\tilde{p}$-operation.

Theorem 3.8. *Let R be an integral domain. The following conditions are equivalent:*

(i) *R is a UFD;*

(ii) *R is a completely integrally closed domain and $\tilde{p} = v$.*

(iii) *R is a Krull domain and $\tilde{p} = v$.*

Proof. (i) $\Rightarrow$ (ii) If R is a UFD, then it is a Krull domain. Hence R is completely integrally closed and $v = t = w$. Moreover, since R is a GCD-domain, it is a PSP-domain (Corollary 2.12). By Proposition 3.5, $\tilde{p} = w$, whence $\tilde{p} = v$.

(ii) $\Rightarrow$ (iii) By Corollary 3.4, the hypothesis $\tilde{p} = v$ implies that v is stable and of finite type. Hence $v = w$. By [4, Proposition 3.7], R is a Krull domain.

(iii) $\Rightarrow$ (i) By [12, Theorem 5], it is enough to show that each nonzero prime ideal contains a prime element. Since $\tilde{p} = v$ and $\tilde{p}$ is stable and of finite type, it follows that $v = t = w$. By [4, Theorem 3.3], each t-maximal ideal P of R is t-invertible, whence P is t-finite, i.e., $P = I_t$ for some finitely generated ideal I of R. Since $P = P_t = P^{\tilde{p}}$, each finitely generated ideal contained in P is not primitive (otherwise $1 \in P^{\tilde{p}}$). Hence $I \subseteq (a)$ for some nonunit element $a \in R$. Now, $P = I_t \subseteq (a)_t = (a)$. By the t-maximality of P, $P = (a)$. Thus any t-maximal ideal of R is principal. Also, since R is a Krull domain, the t-maximal ideals of R are exactly the height-one primes. It follows that each nonzero prime ideal of R contains a prime element, and so R is a UFD. $\square$

Remark 3.9. *We notice that the condition "completely integrally closed" (respectively, "$\tilde{p} = v$") in the theorem above cannot be weakened by using the condition "integrally closed" (respectively "$\tilde{p} = t$"). Take a nondiscrete rank-one valuation domain V. Then V is completely integrally closed and $t = d$. Since V is a Bezout domain (hence a GCD-Bezout domain), $\tilde{p} = d$, so $\tilde{p} = t$. But V is not a UFD.*

Now consider a valuation domain V with principal maximal ideal M and $\dim V > 1$. Then V is integrally closed but not completely integrally closed. Since $\tilde{p} = d$ (as V being a Bezout domain) and $d = v$ (as M being principal), we have that $\tilde{p} = v$. But, also in this case, V is not a UFD.

The following proposition gives a relationship between the GCD-domains and the integrally closed domains in which $\tilde{p} = v$ or $\tilde{p} = t$.

Proposition 3.10. *Let R be an integral domain and consider the following conditions:*

(i) *R is integrally closed and $\tilde{p} = v$;*
(ii) *R is a GCD-domain;*
(iii) *R is integrally closed and $\tilde{p} = t$.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)*, but* (iii) $\not\Rightarrow$ (ii) $\not\Rightarrow$ (i)*.*

Proof. (i) $\Rightarrow$ (ii) The condition $\tilde{p} = v$ implies that v is stable and of finite type. Hence $v = t = w$. By [4, Theorem 3.3] R is a PvMD and each t-maximal ideal of R is t-invertible. Following the same argument used in the proof of (iii) $\Rightarrow$ (i) of Theorem 3.8, we have that all the t-maximal ideals of R are principal. By [8, Corollary 1.10], the class group of R is generated by the classes of the t-maximal ideals, so the class group of R is trivial. Therefore, R is a GCD-domain ([3]).

(ii) $\Rightarrow$ (iii) It is obvious that a GCD domain is integrally closed. Moreover, a GCD-domain is PSP and PvMD, whence $\tilde{p} = w$ (Proposition 3.5) and $t = w$ ([11, Theorem 3.5]). It follows that $\tilde{p} = t$.

We will give two counterexamples to show that (iii) $\not\Rightarrow$ (ii) $\not\Rightarrow$ (i).

(ii) $\not\Rightarrow$ (i) A valuation domain is always a GCD-domain since it is Bezout, hence $\tilde{p} = d$. If we take a valuation domain V with nonprincipal maximal ideal, then $v \neq d$; whence $\tilde{p} \neq v$.

(iii) $\not\Rightarrow$ (ii) We have mentioned that there exist Prüfer domains which are GCD-Bezout but not Bezout (see Remark 2.13). Take one of these domains R. Then R is integrally closed and $\tilde{p} = t = d$, but R is not a GCD-domain, because this would force R to be Bezout. $\square$

Remark 3.11. *It is an open question to decide whether a completely integrally closed domain in which $p = v$ is necessarily a GCD-domain* ([2, p 169]).

Next, our intent is to study GCD-Bezout domains by using semistar operations, as we have done with the PSP-domains. In view of Propositions 2.1 and 2.6, we need to define another operation closely related to primitive ideals.

Definition 3.12. For each nonzero R-module $E \in \overline{\mathbf{F}}(R)$, we define

$$E^q := \bigcup \{(E : J) \mid J = H_1 \cdots H_n, \text{ where each } H_i \text{ is primitive}\}.$$

Note that for each $E \in \overline{\mathbf{F}}(R)$, $E^{\tilde{p}} \subseteq E^q$ (i.e., $\tilde{p} \leq q$).

Lemma 3.13. *Let R be an integral domain. Then the q-operation is a semistar operation which is stable and of finite type.*

Proof. We follow the same arguments used in Proposition 3.2.

We only need to show that if I and J are nonzero finitely generated ideals in R which are finite products of primitive ideals, then the same holds for IJ. But this is straightforward. □

Proposition 3.14. *Let R be an integral domain. The following conditions are equivalent:*

(i) *R is a PSP-domain;*
(ii) *q is a star operation;*
(iii) *$\tilde{p}$ is a star operation;*
(iv) *$q = w = \tilde{p}$.*

Proof. (i) $\Rightarrow$ (ii) Assume that R is a PSP-domain. Then R is a GL-domain and by Proposition 3.5 we have that $\tilde{p} = w$. Moreover, since in a GL-domain the product of primitive ideals is primitive we also have that $\tilde{p} = q$ (this follows directly from the definitions of $\tilde{p}$ and q). Therefore $q = w$ is a star operation.

(ii) $\Rightarrow$ (i) Assume that the q-operation is a star operation. Then $R^q = R$. Thus $(R\colon H_1 \cdots H_n) = R$ for each finite set $\{H_1, \cdots, H_n\}$ of primitive ideals of R. In particular, $(R\colon H) = R$ for each primitive ideal H of R. Thus R is a PSP-domain.

(ii) $\Leftrightarrow$ (iii) From Proposition 3.5, $\tilde{p}$ is a star operation if and only if R is PSP. From the equivalence (i) $\Leftrightarrow$ (ii), we have that $\tilde{p}$ is a star operation if and only if q is a star operation.

(iv) $\Rightarrow$ (iii) It is obvious.

(iii) $\Rightarrow$ (iv) From Proposition 3.5 $\tilde{p} = w$. From (ii), q is a star operation and from Lemma 3.13 q is stable and of finite type. Since $w = \tilde{p} \leq q$, and w is maximal among the star operations on a domain R which are stable and of finite type, it follows that $q = w$. □

Corollary 3.15. *An integral domain R is a GCD-Bezout domain if and only if $q = d$ as star operations on R.*

Proof. It follows directly from Corollary 2.12 and Proposition 3.14. □

In the following diagram, we summarize the principal implications among the classes of domains that we have considered:

$$\begin{array}{rclcl}
\text{GCD-Bezout domain} & \Leftrightarrow & \text{DW-domain + PSP-domain} & \Leftrightarrow & \tilde{p} = d \\
 & & \Downarrow \not\Uparrow & & \\
\text{PSP-domain} & \Leftrightarrow & \tilde{p} \text{ is a star operation} & \Leftrightarrow & \tilde{p} = w \\
 & & \Updownarrow & & \\
 & & q \text{ is a star operation} & \Leftrightarrow & q = w \\
 & & \Downarrow \not\Uparrow & & \\
\text{GL-domain} & \Leftrightarrow & \tilde{p} \text{ is a semistar operation} & &
\end{array}$$

4. Pullbacks

In this section we are interested in the GCD-Bezout-property of the integral domain R arising from the following pullback diagram of canonical homomorphisms:

$$\begin{array}{ccc}
R := \varphi^{-1}(D) & \longrightarrow & D \\
\Big\downarrow & & \Big\downarrow \\
T & \xrightarrow{\varphi} & k = T/M
\end{array} \qquad (\Box)$$

where T is an integral domain, M a nonzero maximal ideal of T, k the residue field T/M, $\varphi : T \to k$ the canonical projection, and D a proper subring of k.

We start with the case in which T is a quasi-local domain.

Proposition 4.1. *Consider a pullback diagram of type* $(\Box)$, *where* T *is quasi-local and* D *is a field. Then* R *is a GCD-Bezout domain if and only if* T *is a GCD-Bezout domain and* $M = M^2$.

Proof. $(\Rightarrow)$ Suppose that $M \neq M^2$. Take $a \in M \setminus M^2$, $x \in T \setminus R$, and consider the ideal $I = (a, ax)$ of R. Then $I \subseteq M$, and moreover, I is a primitive ideal of R. In fact, if I is not primitive, then $I \subseteq bR$ for some nonunit element $b \in R$. Since T is quasi-local and D is a field, R is quasi-local with maximal ideal M. Therefore, $b \in M$. Since $a = bc$ for some $c \in R$ and $a \notin M^2$, we have $c \notin M$, i.e., c is a unit element of R. Then $ax \in I \subseteq bR = aR$, which contradicts that $x \notin R$. Thus if we assume that $M \neq M^2$, then R has a proper primitive ideal, which contradicts that R is a GCD-Bezout domain (Proposition 2.6).

Now let J be a proper finitely generated ideal of T. Then $J \subseteq M$, and hence $J = IT$ for some proper finitely generated ideal I of R. Since R is a GCD-Bezout domain, $I \subseteq aR$ for some $a \in M$. Then $J \subseteq aT$ and hence J is not primitive. Therefore, T is a GCD-Bezout domain (Proposition 2.6).

$(\Leftarrow)$ Take a nonzero proper finitely generated ideal I of R. Since T is a GCD-Bezout domain, $IT \subseteq aT$ for some $a \in M$. By the assumption that $M = M^2$,

$a = \sum_{i=1}^{n} b_i c_i$, where $b_i, c_i \in M$. From the fact that T is a GCD-Bezout domain, it follows that $(b_1, \cdots, b_n) \subseteq bT$ for some $b \in M$. Thus $I \subseteq aT \subseteq bM \subseteq bR$, and hence I is not primitive. Therefore, R is a GCD-Bezout domain (Proposition 2.6). □

A particularly interesting case is when T is a valuation domain and D is a field. Note that the integral domain R arising from a pullback diagram of such type is a pseudo-valuation domain with associated valuation domain T.

Since a valuation domain is a Bezout domain, it is always a GCD-Bezout domain. Thus we get the following corollary to Proposition 4.1.

Corollary 4.2. *Let R be a pseudo-valuation domain which is not a valuation domain and let M be the maximal ideal of R. Then R is a GCD-Bezout domain if and only if $M = M^2$.*

We will now give an example of an integrally closed GCD-Bezout domain in which $t \neq d$ (cf. Example 2.7). Hence, this domain is not a Prüfer domain.

Example 4.3. *Let V be a 1-dimensional nondiscrete valuation domain with residue field $k = V/M$ containing $\mathbb{Q}(X)$. (For the existence of such a valuation domain, see* [9, Proposition 18.4 and Corollary 18.5]*.) Let F be the algebraic closure of $\mathbb{Q}$ in k, then F is a proper subfield of k.*

Consider the pullback diagram of type (□) with $T = V$, $D = F$. Then $R = \varphi^{-1}(D)$ is an integrally closed pseudo-valuation domain which is not a valuation domain, and hence $t \neq d$ on R.

Since V is a 1-dimensional nondiscrete valuation domain, $M = M^2$. Therefore, by Corollary 4.2, R is a GCD-Bezout domain.

In the next result, we will see that if T is quasi-local and D is not a field, then any other condition on T and on M is not needed for R to be a GCD-Bezout domain.

Proposition 4.4. *Consider a pullback diagram of type (□), where T is quasi-local and D is not a field. Then R is a GCD-Bezout domain if and only if D is a GCD-Bezout domain.*

Proof. (⇒) Let J be a nonzero proper finitely generated ideal of D. Then $\varphi^{-1}(J)$ is a proper finitely generated ideal of R containing M by [6, Corollary 1.7 (b)]. Since R is a GCD-Bezout domain, $\varphi^{-1}(J) \subseteq aR \subsetneq R$ for some $a \in R$. Thus we have $J = \varphi(\varphi^{-1}(J)) \subseteq \varphi(a)D \subsetneq D$. This proves that D does not have proper primitive ideals, and hence D is a GCD-Bezout domain (Proposition 2.6).

(⇐) Note first that since T is quasi-local, every ideal of R is comparable with M, and that since D is not a field, M is not a maximal ideal of R.

Let I be a nonzero proper finitely generated ideal of R. Consider the case $I \subseteq M$. Choose a nonunit element $a \in R \setminus M$. Then $I \subseteq M \subseteq aR \subsetneq R$, whence I is not primitive. Now assume that $M \subsetneq I$. Then $\varphi(I)$ is a nonzero proper finitely generated ideal of D. Since D is a GCD-Bezout domain, $\varphi(I) \subseteq dD$ for some nonunit element $d \in D$. Let a be an element of R such that $\varphi(a) = d$. Then $I = \varphi^{-1}(\varphi(I)) \subseteq \varphi^{-1}(dD) = aR + M = aR \subsetneq R$. Thus I is not primitive. Therefore, R is a GCD-Bezout domain. $\square$

Remark 4.5. *(1)* The fact that R is a GCD-Bezout domain does not imply that the integral closure $\bar{R}$ of R is a GCD-Bezout domain, even in the case when $\bar{R}$ is a finite R-module *: Let $T = k[X]_{(X)}$, where k is a number field such that its ring of integers A is a Dedekind domain that is not a PID. (For instance, if $k = \mathbb{Q}(\sqrt{-5})$, then $A = \mathbb{Z}[\sqrt{-5}]$ is not a PID.) Consider the pullback diagram of type ($\square$) with $D = \mathbb{Z}$. Then since $\mathbb{Z}$ is a GCD-Bezout domain, $R = \varphi^{-1}(\mathbb{Z})$ is also a GCD-Bezout domain by Proposition 4.4. But, since A, the integral closure of $\mathbb{Z}$ in k, is not a GCD-Bezout domain (Corollary 2.10), $\bar{R} = \varphi^{-1}(A)$ is not a GCD-Bezout domain by Proposition 4.4 again.*

(2) The GCD-Bezout-property is not preserved by localization *: Let $V = \mathbb{R}[\![X]\!] = \mathbb{R}+M$, where $M = X\mathbb{R}[\![X]\!]$. Consider the pullback diagram of type ($\square$) with $T = V$, $D = \mathbb{Z}$. Then $R = \varphi^{-1}(\mathbb{Z})$ is a GCD-Bezout domain by Proposition 4.4. Let $S = \mathbb{Z} \setminus \{0\}$. Then $R_S = \varphi^{-1}(\mathbb{Q})$ is a pseudo-valuation domain. Since $M \neq M^2$, R_S is not a GCD-Bezout domain by Corollary 4.2.*

The next example shows that in a pullback diagram of type ($\square$), the fact that R is a GCD-Bezout domain does not necessarily imply that T is a GCD-Bezout domain.

Example 4.6. *Let T be a Noetherian local domain which is not a PID. (For instance, $T := k[X,Y]_{(X,Y)}$.) Then T is not a GCD-Bezout domain by Corollary 2.10. Take a PID D which is a subring of the residue field k of T. Then $R = \varphi^{-1}(D)$ is a GCD-Bezout domain.*

Assume, now, that T is not necessarily quasi-local and D is not a field.

Proposition 4.7. *With the above hypotheses and notation, we have that if R is a GCD-Bezout domain, then D is a GCD-Bezout domain.*

Proof. It follows from the same argument used in the proof of the "only if" part of Proposition 4.4. $\square$

The next result deals with the relationship among the GCD-Bezout-properties of the domains R, D, and T, in the classical pullback situation with $T = k + M$. In this case, the domain R is of the form $D + M$.

Proposition 4.8. *Consider a pullback diagram of type* ($\square$) *with* $T = k + M$. *If* D *and* T *are GCD-Bezout domains, then* $R = D + M$ *is a GCD-Bezout domain.*

Proof. Let I be a nonzero proper finitely generated ideal of R. We will show that I is contained in a proper principal ideal of R.

Case 1. $M \subsetneq I$.

Then I/M is a nonzero proper finitely generated ideal of D. Since D is a GCD-Bezout domain, we have that $I/M \subseteq aD \subsetneq D$ for some nonzero nonunit element $a \in D$. Thus $I \subseteq \varphi^{-1}(aD) = aD + M = a(D + M) = aR \subsetneq R$.

Case 2. $M \not\subseteq I$.

By [6, Proposition 1.1], $IT \neq T$. So, IT is a nonzero proper finitely generated ideal of T. Since T is a GCD-Bezout domain, $IT \subseteq xT \subsetneq T$ for some nonunit element $x \in T$. Write $x = a + m$, with $a \in k$ and $m \in M$. If $a = 0$, then $I \subseteq M$. Choose any nonunit element $d \in D \setminus \{0\}$. Then $I \subseteq M \subseteq dD + M = dR \subsetneq R$. Thus we may assume that $a \neq 0$. Then $x = a(1 + \frac{m}{a})$, where a is a unit in T. Thus $\frac{m}{a} \in M$. Let $c = 1 + \frac{m}{a} \in R$. Then $cT = xT \subsetneq T$ and hence c is a nonunit element of R.

Case 2.1. k is the quotient field of D.

Put $R(M) := D + MT_M$, that is, $R(M)$ is the integral domain arising from the following pullback diagram:

$$\begin{array}{ccc} R(M) & \longrightarrow & D \\ \downarrow & & \downarrow \\ T_M = k + MT_M & \longrightarrow & k \end{array}$$

By [6, Lemma 1.3],

$$I = IR(M) \cap IT \subseteq R(M) \cap cT = cR(M) \cap cT = cR \subsetneq R,$$

because c is a unit in $R(M)$.

Case 2.2. k is not a quotient field of D.

Then

$$I \subseteq IT \cap R \subseteq cT \cap R = cR \subsetneq R.$$

In fact:

$$
\begin{aligned}
cT \cap R =& (\bigcap_{N \in \mathrm{Max}(T)} cT_N) \cap R \\
=& (\bigcap_{P \in \mathrm{Max}(R), P \not\supseteq M} cR_P) \cap cT_M \cap R \\
=& (\bigcap_{P \in \mathrm{Max}(R), P \not\supseteq M} cR_P) \cap cT_M \cap (\bigcap_{P \in \mathrm{Max}(R)} R_P) \\
=& (\bigcap_{P \in \mathrm{Max}(R), P \not\supseteq M} cR_P) \cap cT_M \cap (\bigcap_{P \in \mathrm{Max}(R), P \supseteq M} R_P) \\
=& (\bigcap_{P \in \mathrm{Max}(R), P \not\supseteq M} cR_P) \cap cT_M \cap (\bigcap_{P \in \mathrm{Max}(R), P \supseteq M} cR_P) \\
=& cR \cap cT_M \\
=& cR.
\end{aligned}
$$

The third to last equality follows from the observation that for each $P \in \mathrm{Max}(R)$ with $P \supseteq M$, $c = 1 + \frac{m}{a} \in R \setminus P$ and hence $cR_P = R_P$. □

The converse of Proposition 4.8 does not hold in general. In fact, we have already shown that if R is a GCD-Bezout domain, then D is always a GCD-Bezout domain (Proposition 4.7). But the GCD-Bezout-property for R does not imply the GCD-Bezout-property for T as shown in Example 4.6.

5. The Nagata and the polynomial rings

In this last section we consider the Nagata ring $R(X)$ and study the GCD-Bezout-property in $R(X)$. We recall that the Nagata ring (see, for instance, [9, § 33]) is defined as follows:

$$R(X) := \{\frac{f}{g} \mid f, g \in R[X],\ c(g) = R\},$$

where $c(g)$ is the content of the polynomial $g \in R[X]$, i.e., the ideal generated by the coefficients of g.

The Nagata ring is often considered in order to give new examples of domains with some desired properties, by carefully manipulating some aspects of the structure of R which is reflected on the structure of $R(X)$. For instance, it is well-known that R is a Prüfer domain if and only if $R(X)$ is a Prüfer domain ([9, Theorem 33.4]) and that R is a DW-domain if and only if $R(X)$ is a DW-domain ([16, Proposition 3.1]). In the following we will show that the GCD-Bezout-property transfers from R to $R(X)$.

Proposition 5.1. *Let R be an integral domain.*

(1) *If R is a GCD-Bezout domain, then so is $R(X)$.*

(2) *$R[X]$ is a GCD-Bezout domain if and only if R is a field.*

Proof. (1) Let J be a proper finitely generated ideal of $R(X)$. Then we can write $J = (f_1, \cdots, f_m)R(X)$, where $f_i \in R[X]$ for each $i = 1, \cdots, m$. Let

$$f := f_1 + f_2 X^{\deg(f_1)+1} + \cdots + f_m X^{(\deg(f_1)+\cdots+\deg(f_{m-1})+m-1)}.$$

Then $J \subseteq (c(f_1), \cdots, c(f_m))R(X) = c(f)R(X)$ and $f \in J$. Since J is a proper ideal of $R(X)$, f is not invertible in $R(X)$ and hence $c(f) \neq R$. Thus $c(f)$ is a proper finitely generated ideal of the GCD-Bezout domain R, and hence $c(f) \subseteq aR$ for some nonunit element $a \in R$. Then $J \subseteq c(f)R(X) \subseteq aR(X) \subsetneq R(X)$, and so J is not primitive. Thus it follows that $R(X)$ is GCD-Bezout.

(2) If $R[X]$ is a GCD-Bezout domain, then $R[X]$ is DW by Corollary 2.12. But it is known that $R[X]$ is DW if and only if R is a field ([14, Proposition 2.12]). Conversely, if R is a field, then $R[X]$ is a PID and hence a GCD-Bezout domain. □

Finally, we notice that $R(X)$ being a GCD-Bezout domain does not imply that R is a GCD-Bezout domain. Take a Prüfer domain R which is not GCD-Bezout, as in Example 2.8. Then $R(X)$ is a Bezout domain by [9, Theorems 32.7 & 33.4], whence it is GCD-Bezout, but R is not.

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