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MULTIPLIERS OF THE TERMS IN THE LOWER CENTRAL SERIES OF THE LIE ALGEBRA OF STRICTLY UPPER TRIANGULAR MATRICES

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Abstract. A Lie algebra multiplier parallels the idea of group theory's Schur multiplier. This paper classifies the Lie algebra multipliers for all Lie algebras in the lower central series of strictly upper triangular matrices. Multipliers are central, so the classification is focused on computing their dimensions. The calculations are lengthy because balancing various matrix positions plays an important role in determining these dimensions. The result divides into six cases and the dimensions are given as polynomials in the size of the matrices and the position in the lower central series.

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1. Introduction

We begin with a few definitions from [2] that we need to discuss multipliers, just as we did in [4]. Suppose L is a finite dimensional Lie algebra over a field with characteristic not equal to two.

Definition 1.1. A pair of Lie algebras (C, M) is called *a defining pair for* L if

(1) $L \cong C/M$
(2) $M \subset Z(C) \cap C^1$

where $C^1 = [C, C]$.

For a Lie algebra L, suppose $\dim L = n$. In [1] we see that $\dim M$ and $\dim C$ have upper bounds of $\frac{1}{2}n(n-1)$ and $\frac{1}{2}n(n+1)$ respectively. Therefore if L is finite dimensional this implies C and M are finite dimensional also. Furthermore for M maximal we attain $\dim M = \frac{1}{2}n(n-1) \Leftrightarrow L$ is abelian.

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Definition 1.2. If (C, M) is a defining pair for L, then a C of maximal dimension is called *a cover for* L. Likewise an M of maximal dimension is called *a multiplier.*

The Lie algebra multiplier is analogous to group theory's Schur multiplier. Please see [3] for more information about the group theory. Notice that a multiplier is central and hence abelian, so classifying it reduces to finding the dimension. Additionally since the bracket on any two elements in M is always trivial this allows us to easily establish an isomorphism between any two multipliers, therefore we will use $M(L)$ to denote the unique multiplier of L.

Let L be the Lie algebra of $n \times n$ strictly upper triangular matrices. It is shown in [2] that $\dim M(L) = 2(n-2) + \frac{(n-3)(n-2)}{2}$. Consider the lower central series $L \supset L^1 \supset L^2 \supset \cdots \supset L^{n-2} \supset L^{n-1} = 0$, where $L^1 = [L, L]$, $L^{j+1} = [L, L^j]$, $j = 1, 2, \ldots, n-2$. Notice that $L^{n-2} \neq 0$. Defining the superscripts this way causes L^k to be the Lie algebra of $n \times n$ strictly upper triangular matrices with k diagonals of zeros above the main diagonal. We will determine $\dim M(L^k)$ for $k = 0, 1, 2, \ldots, n-2$ with the convention that $k = 0$ corresponds to the multiplier of the full Lie algebra, a result already obtained in [2]. For a fixed $k \geq 2$ the result divides into six cases and the dimensions of the multipliers $M(L^k)$ are given as polynomials in the size n of the matrices and the position k in the lower central series of L:

(1) $k + 2 \leq n < 2k + 3$
L is abelian, hence $\dim M(L^k) = \frac{1}{2}(\dim L^k)(\dim L^k - 1) =$
$\frac{1}{8}(n-k-1)(n-k)(n-k+1)(n-k-2)$

(2) $n = 2k + 3, \ldots, 3k + 1$
$\dim M(L^k) = -4 - \frac{3}{2}nk^3 + 2n - \frac{13}{4}k - 2nk - 4nk^2 + \frac{15}{4}k^3 + \frac{27}{8}k^2 + \frac{1}{2}n^2k^2 + n^2k + \frac{9}{8}k^4$

(3) $n = 3k + 2$
$\dim M(L^k) = -4 + 2n - \frac{27}{4}k + \frac{5}{4}nk + \frac{5}{4}nk^2 - \frac{21}{4}k^3 - \frac{55}{8}k^2 + \frac{1}{4}n^2k^2 + \frac{1}{4}n^2k - \frac{9}{8}k^4$
$= \frac{11}{4}k + \frac{27}{8}k^2 + \frac{15}{4}k^3 + \frac{9}{8}k^4$

(4) $n = 3k + 3, \ldots, 4k + 1$
$\dim M(L^k) = -1 - \frac{3}{2}nk^3 - \frac{1}{2}n + \frac{17}{4}k - 5nk + \frac{1}{2}n^2 - 4nk^2 + \frac{15}{4}k^3 + \frac{63}{8}k^2 + \frac{1}{2}n^2k^2 + n^2k + \frac{9}{8}k^4$

(5) $n = 4k + 2$
$\dim M(L^k) = -2 + \frac{55}{6}nk^3 + \frac{19}{12}n - \frac{49}{12}k + \frac{20}{3}nk - \frac{23}{24}n^2 + 16nk^2 + \frac{5}{12}n^3 - \frac{275}{12}k^3 - \frac{371}{24}k^2 - \frac{7}{2}n^2k^2 + \frac{2}{3}n^3k - 4n^2k - \frac{229}{24}k^4 - \frac{1}{24}n^4$
$= \frac{17}{4}k + \frac{47}{8}k^2 + \frac{35}{4}k^3 + \frac{25}{8}k^4$

(6) $n \geq 4k+3$
$\dim M(L^k) = -1 - \frac{3}{2}nk^3 - \frac{1}{2}n + \frac{17}{4}k - 5nk + \frac{1}{2}n^2 - 4nk^2 + \frac{15}{4}k^3 + \frac{63}{8}k^2 + \frac{1}{2}n^2k^2 + n^2k + \frac{9}{8}k^4$

If $k = 1$, the second and fourth case have to be omitted, and if $k = 0$, only the first ($n < 3$) and last case ($n \geq 3$) apply.

2. Constructing $M(L^k)$

For some $k \in \{1, \cdots, n-2\}$, let (C, M) be a defining pair for L^k. Therefore $C/M \cong L^k$ and $M \subset Z(C) \cap C^1$. Let E_{ab} denote the usual matrix units that form a basis for L^k. Since k counts the diagonals of zeros, we have $a + (k+1) \leq b$. Each basis element E_{ab} also corresponds to a coset in C. For each of these E_{ab}'s, choose an element from C in the corresponding coset and denote it F_{ab}. This F is called a transversal element. We can now define a map $u : L^k \to C$ that takes each E_{ab} to its F_{ab} and extend u linearly. We can now describe the bracket on C as

$$[F_{st}, F_{ab}] = \begin{cases} F_{sb} + y(s,t,a,b) & \text{if } t = a \\ y(s,t,a,b) & \text{if } t \neq a \end{cases}$$

where $y(s,t,a,b) \in M$. We will often write $y(s,t,a,b)$ more concisely as y_{stab}. To avoid double counting elements, we assume that either $s < a$ or $s = a$ and $t < b$. This is consistent with the convention established in [2].

As in [4] first make a change in the choice of F_{rt} following the model of [2]. Set

$$G_{rt} = \begin{cases} F_{rt} & \text{if } t - r < 2(k+1) \\ F_{rt} + y_{r,t-(k+1),t-(k+1),t} = [F_{r,t-(k+1)}, F_{t-(k+1),t}] & \text{otherwise} \end{cases}$$

Notice F_{rt} and G_{rt} only differ by a central element y, so the $F's$ or $G's$ describe the same multiplication in C. Similar to [4] we take advantage of this fact when computing; that is, multiplying by $F's$ or $G's$ gives the same result and we use which ever is most convenient. After removing any dependencies among the $y's$ we can conclude as in [2] that "These $y's$ are completely arbitrary. We can assume them to be a set of linearly independent vectors and no contradiction arises. In this case they would be a basis for a multiplier M." Thus we proceed to find all dependencies among the $y's$, using the Jacobi identity as our tool. After doing this, we have a basis for $M(L^k)$ and it remains to count the number of elements in this basis. To find these dependencies we consider two scenarios: elements produced by $[G_{rs}, G_{st}]$ and elements produced by $[G_{st}, G_{ab}]$, where $t \neq a$. Let $J(x,y,z) = 0$ denote the Jacobi identity.

3. $[G_{rs}, G_{st}]$ elements

Consider $[G_{rs}, G_{st}] = F_{rt} + y_{rsst} = G_{rt} + \widehat{y_{rsst}}$, where $\widehat{y_{rsst}} = y_{rsst} - y_{r,t-(k+1),t-(k+1),t}$. When $[G_{rs}, G_{st}] = G_{rt}$ there is no contribution to the multiplier. Hence we will count the occurrences of $\widehat{y} \neq 0$, or equivalently $y_{rsst} \neq y_{r,t-(k+1),t-(k+1),t}$. Notice that $s-r, t-s \geq k+1$ give $r+2k+2 \leq t$. For a general k (and r fixed) we get every $\widehat{y} = 0$ when $t = r+2k+2$ and $t \geq r+4k+3$. For the $k = 0$ case this implies $[G_{rs}, G_{st}] = G_{rt}$ is always true, which is consistent with [2]. If $k > 0$ then as t traverses the $2k$ values between $r+2k+2$ and $r+4k+3$, the number of non-trivial $y's$ we attain are $1, 2, 3, \ldots, k-3, k-2, k-1, k, k, k-1, k-2, k-3, \ldots, 3, 2, 1$ respectively. The theorem below shows all of this, and takes into account what happens as r changes to count all $y_{rsst} \neq y_{r,t-(k+1),t-(k+1),t}$ elements.

Theorem 3.1. *As r, s, and t range over all values for which the bracket is defined, the number of nonzero $\widehat{y}'s$ produced from $[G_{rs}, G_{st}]$ is* $\sum_{i=1}^{k} i \cdot (n - (2k+2+i)) + \sum_{j=1}^{k} (k-j+1) \cdot (n - (3k+2+j))$.

Proof. In general if $s - r \geq 2k+2$ then $\exists c$ to define F_{rc} and F_{cs} so that $J(F_{rc}, F_{cs}, F_{st}) = 0 \Rightarrow y_{rsst} = y_{rcct}$. Similarly if $t - s \geq 2k+2$ then $\exists c$ such that $J(F_{rs}, F_{sc}, F_{ct}) = 0 \Rightarrow y_{rsst} = y_{rcct}$.

<u>**Case 1:**</u> $t = r + 2k + 2 + i$ where $1 \leq i \leq k$

We do not need to consider $i = 0$ because $t = r+2k+2 \Rightarrow s = r+k+1 = t-(k+1) \Rightarrow [G_{rs}, G_{st}] = G_{rt}$. For any value of i from 1 to k, notice $t-r = 2k+2+i \leq 3k+2$. In order to define G_{ab} there must be at least k integers between a and b. Given a distance from r to t of at most $3k+2$ does not provide sufficient room to define either of the Jacobi identities above. While a number c may still be defined to produce other Jacobi identities, none of these will involve y_{rsst} and hence none of the $\widehat{y}'s$ may be eliminated. Thus for each value of $i \in \{1, \ldots, k\}$ and a fixed r there will be i distinct values for s, giving i distinct $y's$, namely y_{rsst} where $s \in \{r+k+1, \ldots, r+k+i\}$. Notice we have excluded the final case when $s = r+k+i+1 = t-(k+1)$, since $[G_{rs}, G_{st}] = F_{rt} + y_{rsst} = G_{rt}$.

Now for a fixed number i, it is necessary to count the $y's$ as r varies. Notice that $1 \leq r < r+2k+2+i = t \leq n$ and so $1 \leq r \leq n-(2k+2+i)$. Therefore each of the $n-(2k+2+i)$ values of r produce i distinct values for y_{rsst} as s varies. In total this produces $\sum_{i=1}^{k} i \cdot (n - (2k+2+i))$ distinct values for y_{rsst} as r, s and t vary.

Case 2: $t = r + 3k + 2 + j$ where $j \geq 1$

Note: $j = 0$ would correspond to the maximum i from the previous case and that s may now assume j values more than it did in this maximal $i = k$ case since we widened the gap from r to t. However we wish to show that this will decrement the count of nonzero $\widehat{y}'s$ rather than continue to increment it.

It is now possible to define G_{rc}, G_{cd}, and G_{dt}. For convenience we will allow c to vary and force d to be $d = c + (k+1)$ for the remainder of this case. Please observe that c is taking the place of s from case 1 and d indicates a new value of s for case 2. Enforcing these relationships, we are interested to count the number of $c's$ that will allow us to define G_{rc}, G_{cd}, and G_{dt}. Notice we need $c \geq r+k+1$ and $d \leq t-(k+1)$ which together imply $c \leq t - 2(k+1)$ and since $t = r + 3k + 2 + j$ we get $c \leq r + k + j$ and hence $c \in \{r+k+1, \ldots, r+k+j\}$, so there are j choices for c and d.

Notice that (1) $J(F_{rc}, F_{cd}, F_{dt}) = 0 \Rightarrow y_{rcct} = y_{rddt}$, so d (a new s value in y_{rsst} not present in case 1) does not describe a new y. Furthermore, (2) $J(F_{rc}, F_{c,t-(k+1)}, F_{t-(k+1),t}) = 0$ gives $y_{rcct} = y_{r,t-(k+1),t-(k+1),t}$ and hence eliminates the y_{rsst} terms for the $j-1$ values of c where these two Jacobi identities are different (i.e. $c \neq t-(2k+2)$). Since c describes an s found in case 1, we lose $j-1$ of the k $y'_{rsst}s$ found in the maxmimal case 1 setting.

Therefore for a fixed r there are $k-j+1$ values of y when $1 \leq j \leq k$, and if $j > k$ (i.e. $t \geq r+4k+3$), all the $y'_{rsst}s$ are equal to each other and to $y_{r,t-(k+1),t-(k+1),t}$ hence producing no non-trivial values for $\widehat{y_{rsst}}$.

For a fixed number $j \in \{1, \ldots, k\}$, it is necessary to see how many $y's$ can be produced as r varies. Since $1 \leq r < r + 3k + 2 + j = t \leq n$ it follows that $r \in \{1, \ldots, n-(3k+2+j)\}$. So each of the $n-(3k+2+j)$ values of r produce $k-j+1$ distinct values of y_{rsst} as s fluctuates when $j \leq k$ and zero otherwise. In total case 2 produces $\displaystyle\sum_{j=1}^{k}(k-j+1)\cdot(n-(3k+2+j))$ distinct values for y_{rsst} as r and j vary. So together there are

$$\sum_{i=1}^{k} i\cdot(n-(2k+2+i))+\sum_{j=1}^{k}(k-j+1)\cdot(n-(3k+2+j)) \text{ different } y_{rsst} \text{ values.} \quad \square$$

Notice both sums involve a term of the form $n-x$. Since x denotes a distance between subscripts of the $G's$ and hence a distance between matrix positions in the original $n \times n$ matrices, whenever $n-x \leq 0$ occurs it should be replaced with a zero since the necessary matrix units were not available to produce the corresponding multiplier elements.

4. $[G_{st}, G_{ab}]$, $t \neq a$ elements

Consider the second case where $[G_{st}, G_{ab}] = y_{stab}$ since $t \neq a$. Notice that $s \neq b$ as a result of the assumption that $s < a$ or $s = a$ and $t < b$. Hence $[G_{st}, G_{ab}] = y_{stab}$ where $y_{stab} \in M(L^k)$, since no F is produced by the bracket. For convenience we will work with the $F's$ rather than the $G's$. We will establish all the relationships between the values of the subscripts s, t, a, and b where $y_{stab} = 0$, otherwise we assume $y_{stab} \neq 0$ to get $M(L^k)$ of maximal dimension.

4.1. Eliminating y_{stab} trivial.

Theorem 4.1. *If $b \geq a + 2k + 3$ or $t \geq s + 2k + 3$, then $y_{stab} = 0$.*

Proof. Suppose $b \geq a + 2k + 3$. If $t \neq a + k + 1$ then let $c = a + k + 1$, otherwise choose $c = a+k+2$. This gives $c \neq s, t, a, b$, the last three by construction and the first by $s \leq a < c \Rightarrow c \neq s$. Therefore $J(F_{st}, F_{ac}, F_{cb}) = 0 \Rightarrow y_{stab} = 0$.

Similarly suppose $t \geq s + 2k + 3$, if $a = s + k + 1$ let $c = s + k + 2$ otherwise let $c = s + k + 1$. By construction $c \neq s, t, a$. If $s = a$ then $c < t < b$ gives $c \neq b$. When $s < a$ then $a + k + 1 \leq b$ gives $b \geq s + k + 2$. If the inequality is strict then $c \neq b$. If $b = s + k + 2$ then $a = s + 1$ so $c = s + k + 1 \neq b$ Therefore in all cases $c \neq b$. Having $c \neq s, t, a, b$ gives $J(F_{sc}, F_{ct}, F_{ab}) = 0 \Rightarrow y_{stab} = 0$. □

Theorem 4.2. *If $b = a + 2k + 2$, then $y_{stab} = 0 \Leftrightarrow t \neq a + k + 1$ or $s < a$ $(s \neq a)$. Similarly if $t = s + 2k + 2$ then $y_{stab} = 0 \Leftrightarrow a \neq s + k + 1$ or $b \neq t$.*

Proof. $(\Leftarrow)$ Suppose $b = a + 2k + 2$. If $t \neq a + k + 1$ then let $c = a + k + 1$, and furthermore $s \leq a < c \Rightarrow c \neq s$. Therefore $c \neq s, t, a, b$ and $J(F_{st}, F_{ac}, F_{cb}) = 0 \Rightarrow y_{stab} = 0$. On the other hand if $t = a+k+1$ and $s < a$ then let $c = t-1$. In this case $J(F_{st}, F_{tb}, F_{at}) = 0 \Rightarrow y_{stab} = -y_{sbat}$ and $J(F_{sc}, F_{cb}, F_{at}) = 0 \Rightarrow y_{sbat} = 0$ so $y_{stab} = 0$.
$(\Rightarrow)$ Suppose $b = a + 2k + 2$, $t = a + k + 1$, and $s = a$. There is no value of c such that F_{ac} and F_{cb} are both defined while $c \neq t$. As such, there are no Jacobi identities available to zero out y_{stab}, thus $y_{stab} \neq 0$.

$(\Leftarrow)$ Suppose $t = s+2k+2$. If $a \neq s+k+1$ then let $c = s+k+1$, so $c \neq s, t, a$. Notice when $s = a$ that $s < c < t < b \Rightarrow c \neq b$. If $s < a$ then $c < a + k + 1 \leq b \Rightarrow c \neq b$. Therefore $J(F_{sc}, F_{ct}, F_{ab}) = 0 \Rightarrow y_{stab} = 0$. On the other hand if $a = s + k + 1$ and $b > t$ then let $c = a + 1$. In this case $J(F_{sa}, F_{at}, F_{ab}) = 0 \Rightarrow y_{stab} = y_{sbat}$ and $J(F_{sc}, F_{cb}, F_{at}) = 0 \Rightarrow y_{sbat} = 0$ and so $y_{stab} = 0$. If $a = s + k + 1$ and $b < t$ then $s < s + k + 1 = a < a + k + 1 \leq b < t \Rightarrow s + 2k + 2 < t$ and hence by Theorem 4.1, $y_{stab} = 0$.

($\Rightarrow$) Suppose $t = s + 2k + 2$, $a = s + k + 1$, and $b = t$. There is no value of c such that F_{sc} and F_{ct} are both defined while $c \neq a$. As such, there are no Jacobi identities available to zero out y_{stab}, thus $y_{stab} \neq 0$. □

Now we have reduced the problem to a finite number of ways the non-trivial y_{stab} values may be produced. For convenience we will investigate the non-trivial possibilities by separating the variable relationships into three cases. Either (1) $s = a$, (2) $a > t$, or (3) $s < a < t$, sorted from easiest to hardest.

4.2. Types of elements.

<u>**Case 1:**</u> $s = a$

For a fixed value of s, suppose $s = a$. Theorems 4.1 and 4.2 discuss $b \geq s+2k+2$, so consider $b < s + 2k + 2$. Let $t_{min} = s + k + 1$, the minimum possible value of t. Since $s = a \Rightarrow t < b$ this gives $t_{min} < b < s+2k+2 = t_{min}+k+1$, so $b = t_{min}+j$, for $j \in \{1, \ldots, k\}$.

Theorem 4.3. *When $b = t_{min} + j$, $j \in \{1, \ldots, k\}$ we get j new non-trivial values for y_{stab}.*

Proof. For any value of j, $t < b < s+2k+2$. Therefore $\nexists c$ such that F_{ac} and F_{cb} are both defined, similarly $\nexists c$ such that F_{sc} and F_{ct} are both defined, which forces $y_{stab} \neq 0$.

For $b = t_{min} + j, t < b$ we get $t \in \{t_{min}, t_{min} + 1, \ldots, t_{min} + j - 1\}$ and so t may take on j values for a fixed b, and hence giving j new distinct possible non-trivial y_{stab} elements as t fluctuates. □

<u>**Case 2:**</u> $a > t$

Theorem 4.4. *If $a > t$, then $y_{stab} \neq 0$ for all t and b such that both $t < s+2k+2$ and $b < a + 2k + 2$. Otherwise $y_{stab} = 0$ when $a > t$.*

Proof. If $t \geq s + 2k + 3$ or $b \geq a + 2k + 3$ then Theorem 4.1 $\Rightarrow$ $y_{stab} = 0$. If $t = s + 2k + 2$ or $b = a + 2k + 2$ then Theorem 4.2 $\Rightarrow$ $y_{stab} = 0$ since $a > t \Rightarrow t \neq a + k + 1$ and $a \neq s + k + 1$.

If $t < s + 2k + 2$ and $b < a + 2k + 2$ then there is no value of c, such that F_{sc} and F_{ct} are both defined for $s < c < t$. Similarly there is no value of c, such that F_{ac} and F_{cb} are both defined for $a < c < b$. Therefore the idea in Theorem 4.1, of using $J(F_{sc}, F_{ct}, F_{ab}) = 0$ or $J(F_{st}, F_{ac}, F_{cb}) = 0$ will not work here. Also, placing a c such that $s < t < c < a < b$ will not provide any helpful Jacobi identities, no matter how large the gap between t and a. Thus y_{stab} will always be non-zero in this case. □

<u>**Case 3:**</u> $s < a < t$

Theorems 4.1 and 4.2 discuss $b \geq a + 2k + 2$ and $t \geq s + 2k + 2$.

Theorem 4.5. *If $s < a < t$, then $y_{stab} \neq 0$ if $b < a + 2k + 2$ and $t < s + 2k + 2$.*

Proof. There is not enough space between s, t, a, b to define a suitable c to use any Jacobi identities previously mentions. Therefore we cannot zero out y_{stab} when $b \in \{a + k + 1, \ldots, a + 2k + 1\}$ and $t \in \{s + k + 1, \ldots, s + 2k + 1\}$. □

Collecting all this information, Table 1 lists all non-zero y_{stab} possibilities.

TABLE 1. $y_{stab} \neq 0$ possibilities

Theorem 4.2	1. $b = a + 2k + 2$, $t = a + k + 1$, and $s = a$ 2. $t = s + 2k + 2$, $a = s + k + 1$ and $b = t$
Theorem 4.3	$s = a$, $b = t_{min} + j$ where $j \in \{1, \ldots, k\}$
Theorem 4.4	$a > t$, $t < s + 2k + 2$, and $b < a + 2k + 2$
Theorem 4.5	$s < a < t$, $t < s + 2k + 2$, and $b < a + 2k + 2$

5. Counting the multiplier elements for $[G_{st}, G_{ab}], t \neq a$

As a reminder, n denotes the number of rows and columns in the matrices. We are interested in counting all the cases when $y_{stab} \neq 0$. There are two types of elements: (1) $y(s, s+x_1, s+x_2, s+x_3)$ where x_1, x_2, x_3 are all fixed, which produce $n - w$ multiplier elements for $w = max\{x_1, x_2, x_3\}$ as s traverses $1, 2, \ldots, n - w$ and (2) $y(s, s + x_1, a, a + x_2)$ where $a > s + x_1$ and x_1, x_2 are both fixed, which produce $\frac{1}{2}(n - (x_2 + x_1 + 1))(n - (x_2 + x_1))$ multiplier elements as a traverses $s + x_1 + 1, \ldots, n - x_2$ for $s \in \{1, \ldots, n - (x_2 + x_1 + 1)\}$.

We now use this to count the number of non-trivial values for y_{stab} when $t \neq a$. As in section 3, notice that the element counts take the form $n - x$ where x denotes a distance between matrix positions. For the calculations below we assume n to be sufficiently large. Once we have the result, if ever $n - x$ is not positive then replace it with zero since the necessary matrix units would not have been available to produce the corresponding multiplier elements with the appropriate subscripts.

5.1. Counting the cases in Theorem 4.2.

Since the two y_{stab} values listed in Table 1 simplify to $y(s, s+k+1, s, s+2k+2)$ and $y(s, s + 2k + 2, s + k + 1, s + 2k + 2)$, in total they contribute $2(n - (2k + 2))$ values to the multiplier.

5.2. Counting the cases in Theorem 4.3.

Recall that $t_{min} = s + (k+1)$ and $s = a \;\Rightarrow\; t < b$. Hence $y_{stab} = y(s, t_{min} + i, s, t_{min} + j)$ where $i < j$, which gives $i \in \{0, 1, 2, \ldots, j-1\}$. Therefore $t_{min} + j = max\{s, t_{min} + i, s, t_{min} + j\}$ and since $t_{min} + j = s + k + 1 + j$ this means that y_{stab} will assume $n - (k+1+j)$ values as s varies, for a fixed i and j. For j fixed, i may assume the j distinct values mentioned above, yielding $n - (k+1+j)$ values of y_{stab} for each i, and $j \times (n - (k+1+j))$ values for y_{stab} as s and i both vary. Since j can range from 1 up to k, in total this produces $\sum_{j=1}^{k} j \times (n - (k+1+j))$ non-trivial values for y_{stab} in this situation.

5.3. Counting the cases in Theorem 4.4.

Let $b_{min} = a + (k+1) =$ the minimum possible value for b. So $y_{stab} = y(s, t_{min} + i, a, b_{min} + j) = y(s, s+k+1+i, a, a+k+1+j)$ where $i, j \in \{0, 1, 2, \ldots, k\}$.

If we let $x_1 = k+1+i$ and $x_2 = k+1+j$, then $y_{stab} = y(s, s+x_1, a, a+x_2)$ and we know from our earlier discussion that this may assume $\frac{(n-(x_2+x_1+1))(n-(x_2+x_1))}{2}$ different values for a fixed x_1 and x_2 (fixed i and j here). Since i and j both range in value from 0 to k, in total this produces

$$\sum_{j=0}^{k}\sum_{i=0}^{k} \frac{(n - (2k+i+j+3)) \times (n - (2k+i+j+2))}{2}$$

non-trivial values for y_{stab} in this situation.

5.4. Counting the cases in Theorem 4.5.

This is the most difficult case to count. Even though y_{stab} takes the form $y(s, s+x_1, s+x_2, s+x_3)$, we can have either $t < b$ or $b \le t$ making it difficult to choose $max\{x_1, x_2, x_3\}$ which is needed to count this occurrence. The former is more numerous but the larger the k, the more common the latter. Theorem 4.5 produces 4 distinct patterns, one from the $b \le t$ scenario and three from the $t < b$ scenario. We will separate these into the 4 cases that follow and use $t_{min} = s + k + 1$ as before, in addition to $t_{max} = t_{min} + k$ for the minimum and maximum t values. Please notice that the largest possible value of b is $t_{min} + 3k$ which occurs when $t < b$, $a = t_{max} - 1$, and $b = a + 2k + 1$.

5.4.1. *Case 1: $b \le t$.*

Since we have $b \in \{a+k+1, \ldots, a+2k+1\}$ and $t \in \{s+k+1, \ldots, s+2k+1\}$, then $s < a < t$ forces $b > t_{min}$. Therefore it is sufficient to consider $t = t_{min} + j$ where $1 \le j \le k$ since $t_{max} = t_{min} + k$. This allows for $b = t_{min} + i$ with $1 \le i \le j$

which causes $a \in \{s+1, \ldots, s+i\}$. So as i fluctuates we get $\sum_{i=1}^{j} i = \frac{j(j+1)}{2}$ possible occurences of $b \leq t$ for each j since a may take on i different values for each fixed $b = t_{min} + i$. Fortunately $\frac{j(j+1)}{2}$ is counting how a and b move around for a fixed t. Since t is the largest number we get $n - (k+1+j)$ y_{stab} possibilities for a fixed $t = t_{min} + j = s + k + 1 + j$ as s varies. Since $1 \leq j \leq k$ we get a total of $\sum_{j=1}^{k} \frac{j(j+1)}{2} \cdot (n - (k+1+j))$ possibilities.

5.4.2. *Case 2: $t < b$ and $b \in \{t_{min}+1, \ldots, t_{min}+k\}$.*

Suppose $b = t_{min} + i$, then $t < b$ gives $t \in \{t_{min}, \ldots, t_{min} + i - 1\}$. Having $s < a$ implies $a \in \{s+1, \ldots, s+i\}$ and the upper bound on b causes $a < t_{min}$. Notice that a and t may both assume i different values and neither letter's position affects the other, giving i^2 possible arrangements of a and t. With $b = t_{min} + i = s + k + 1 + i$ we count $n - (k+1+i)$ occurences of this for a fixed i as s fluctuates. So as i also fluctuates we get a total of $\sum_{i=1}^{k} i^2 \cdot (n - (k+1+i))$ possibilities.

5.4.3. *Case 3: $t < b$ and $b \in \{t_{min}+k+1, \ldots, t_{min}+2k\}$.*

Now that $b > t_{min} + k = t_{max}$, we get $t < b$ for free but we introduce the possibility that $a \geq t_{min}$ and must enforce $s < a < t$. If $b = t_{min} + k + i$ then $a \in \{s+i, \ldots, s+k+i\}$ in order to guarantee $s < a$ and $a+k+1 \leq b \leq a+2k+1$. Additionally $t_{min} \leq t \leq t_{max}$ so a and t may each assume $k+1$ different values. This would give $(k+1)^2$ possible interactions, however we must eliminate some to ensure $s < a < t$. If $a = s+k+1 = t_{min}$ then we cannot allow $t = t_{min}$, similarly if $a = s+k+2$ then we cannot allow $t = t_{min}, t_{min}+1$, and so on. If $a = s+k+i$, disregard $t = t_{min}, \ldots, t_{min}+i-1$. This gives a total of $1+2+\ldots+i$ eliminations as a traverses $s+k+1, \ldots, s+k+i$. Therefore we will allow $(k+1)^2 - \frac{i(i+1)}{2}$ possible interaction of a and t. Since $b = t_{min}+k+i = s+2k+i+1$ exceeds t, we get $n - (2k+i+1)$ occurences of this for a fixed i as s fluctuates. Thus as i varies the total possibilities we get are $\sum_{i=1}^{k} \left(\left((k+1)^2 - \frac{i(i+1)}{2} \right) \cdot (n - (2k+i+1)) \right)$.

5.4.4. *Case 4: $t < b$ and $b \in \{t_{min}+2k+1, \ldots, t_{min}+3k\}$.*

Since b is larger than the previous case, $t < b$ is still automatic but guaranteeing $s < a < t$ needs extra care. Now we have a new concern to keep $a < t_{max} = s+2k+1$. Suppose $b = t_{min}+2k+i$, so $a \in \{s+k+i, \ldots, s+2k\}$. Suppose $a = s+2k = t_{max} - 1$, then t must be t_{max}. If $a = s+2k-1$ then t must be t_{max} or $t_{max} - 1$. Each additional decrement of a will increment the allowed values of t. Notice there are $k+1-i$ numbers a may take, yielding $1+2+\ldots+(k+1-i) = \frac{(k+1-i)(k+2-i)}{2}$ interactions of a and t for this fixed b. Since $b = t_{min} + 2k + i = s + 3k + i + 1$ we get $n - (3k + i + 1)$ occurences of this for a fixed i as s fluctuates. Thus as i fluctuates we get a total of $\sum_{i=1}^{k} \left(\frac{(k+1-i)(k+2-i)}{2} \cdot (n - (3k + i + 1)) \right)$ values of y_{stab}.

6. $\dim M(L^k)$ formula

Putting all possible cases together gives the open form result

$$\dim M(L^k) = \sum_{i=1}^{k} i \cdot (n - (2k+2+i)) + \sum_{j=1}^{k} (k-j+1) \cdot (n - (3k+2+j)) +$$
$$2(n - (2k+2)) + \sum_{j=1}^{k} j \times (n - (k+1+j)) +$$
$$\sum_{j=0}^{k} \sum_{i=0}^{k} \frac{(n - (2k+i+j+3)) \times (n - (2k+i+j+2))}{2} +$$
$$\sum_{j=1}^{k} \frac{j(j+1)}{2} \cdot (n - (k+1+j)) + \sum_{i=1}^{k} i^2 \cdot (n - (k+1+i)) +$$
$$\sum_{i=1}^{k} \left(\left((k+1)^2 - \frac{i(i+1)}{2} \right) \cdot (n - (2k+i+1)) \right) +$$
$$\sum_{i=1}^{k} \left(\frac{(k+1-i)(k+2-i)}{2} \cdot (n - (3k+i+1)) \right)$$

Notice that the $4^{th}, 6^{th}$, and 7^{th} terms combine into a single sum nicely to give

$$\dim M(L^k) = \sum_{i=1}^{k} i \cdot (n - (2k+2+i)) + \sum_{j=1}^{k} (k-j+1) \cdot (n - (3k+2+j)) +$$
$$2(n - (2k+2)) + \sum_{j=0}^{k} \sum_{i=0}^{k} \frac{(n - (2k+i+j+3)) \times (n - (2k+i+j+2))}{2} +$$
$$\sum_{j=1}^{k} \frac{3j(j+1)}{2} \cdot (n - (k+1+j)) +$$
$$\sum_{i=1}^{k} \left(\left((k+1)^2 - \frac{i(i+1)}{2} \right) \cdot (n - (2k+i+1)) \right) +$$
$$\sum_{i=1}^{k} \left(\frac{(k+1-i)(k+2-i)}{2} \cdot (n - (3k+i+1)) \right)$$

Assuming that n is sufficiently large, at least $4k+3$, the previous formula will work without modification. In the event $n < 4k+3$ some terms should be zeroed off to ensure all $n-x$ terms in the open form are positive. In order to develop a formula to work for any n and k pair it will be helpful to reindex these sums to better see when terms should be zeroed off. Additionally the double sum can be rewritten as single sums by reindexing over the counter $i+j+1$ and then reindexing again with the other single sums. The reindexing yields $\dim M(L^k) =$

$$\sum_{j=k+2}^{2k+1} \left(\frac{3}{2} \cdot (j-k-1)(j-k)(n-j) \right) + 2(n-(2k+2))+$$

$$\sum_{j=2k+2}^{3k+1} \left((k+1)^2 - \frac{(j-2k-1)(j-2k)}{2} \right) \cdot (n-j) + \sum_{j=2k+3}^{3k+2} (j-2k-2) \cdot (n-j) +$$

$$\sum_{j=2k+3}^{3k+2} \frac{j-2k-2}{2} \cdot (n-j)\,(n-j+1) + \sum_{j=3k+2}^{4k+1} \frac{1}{2} \cdot (4k+2-j)(4k+3-j)(n-j)+$$

$$\sum_{j=3k+3}^{4k+2} (4k+3-j) \cdot (n-j) + \sum_{j=3k+3}^{4k+3} \frac{4k+4-j}{2} \cdot (n-j)\,(n-j+1)$$

for sufficiently large n. Now it is easier to see when $n-j$ will not be positive in one of the sums above. Whenever this happens a sum should be terminated since further increments of j also cause this dilema. If $k \geq 2$ then we can separate the above open form into six cases based on when sums should terminate early, so that within each case the open form no longer needs to be modified. This will allow us to expand the open form in each case into a closed polynomial form. Below is the result, computational algebra omitted.

(1) Case 1: $k+2 \leq n < 2k+3$
L is abelian, hence $\dim M(L^k) = \frac{1}{2}(\dim L)(\dim L - 1) = \frac{1}{8}(n-k-1)(n-k)(n-k+1)(n-k-2)$

(2) Case 2: $n = 2k+3, \ldots, 3k+1$

$$\dim M(L^k) = \left(\sum_{j=k+2}^{2k+1} \frac{3}{2} \cdot (j-k-1)(j-k)(n-j) \right) + 2(n-(2k+2))+$$

$$\sum_{j=2k+2}^{n} \left((k+1)^2 - \frac{(j-2k-1)(j-2k)}{2} \right) \cdot (n-j)+$$

$$\sum_{j=2k+3}^{n} (j-2k-2) \cdot (n-j) + \sum_{j=2k+3}^{n} \frac{j-2k-2}{2} \cdot (n-j)\,(n-j+1)$$

$\dim M(L^k) = -4 - \frac{3}{2}nk^3 + 2n - \frac{13}{4}k - 2nk - 4nk^2 + \frac{15}{4}k^3 + \frac{27}{8}k^2 + \frac{1}{2}n^2k^2 + n^2k + \frac{9}{8}k^4$

(3) Case 3: $n = 3k + 2$

$$\dim M(L^k) = \left(\sum_{j=k+2}^{2k+1} \frac{3}{2} \cdot (j-k-1)(j-k)(n-j)\right) + 2(n-(2k+2)) +$$

$$\sum_{j=2k+2}^{3k+1} \left((k+1)^2 - \frac{(j-2k-1)(j-2k)}{2}\right) \cdot (n-j) +$$

$$\sum_{j=2k+3}^{3k+2} (j-2k-2) \cdot (n-j) + \sum_{j=2k+3}^{3k+2} \frac{j-2k-2}{2} \cdot (n-j)\,(n-j+1)$$

$\dim M(L^k) = -4 + 2n - \frac{27}{4}k + \frac{5}{4}nk + \frac{5}{4}nk^2 - \frac{21}{4}k^3 - \frac{55}{8}k^2 + \frac{1}{4}n^2k^2 + \frac{1}{4}n^2k - \frac{9}{8}k^4$
$= \frac{11}{4}k + \frac{27}{8}k^2 + \frac{15}{4}k^3 + \frac{9}{8}k^4$

(4) Case 4: $n = 3k + 3, \ldots, 4k + 1$

$$\dim M(L^k) = \left(\sum_{j=k+2}^{2k+1} \frac{3}{2} \cdot (j-k-1)(j-k)(n-j)\right) + 2(n-(2k+2)) +$$

$$\sum_{j=2k+2}^{3k+1} \left((k+1)^2 - \frac{(j-2k-1)(j-2k)}{2}\right) \cdot (n-j) +$$

$$\sum_{j=2k+3}^{3k+2} (j-2k-2) \cdot (n-j) + \sum_{j=2k+3}^{3k+2} \frac{j-2k-2}{2} \cdot (n-j)\,(n-j+1) +$$

$$\sum_{j=3k+2}^{n} \frac{1}{2} \cdot (4k+2-j)(4k+3-j)(n-j) + \sum_{j=3k+3}^{n} (4k+3-j) \cdot (n-j) +$$

$$\sum_{j=3k+3}^{n} \frac{4k+4-j}{2} \cdot (n-j)\,(n-j+1)$$

$\dim M(L^k) = -1 - \frac{3}{2}nk^3 - \frac{1}{2}n + \frac{17}{4}k - 5nk + \frac{1}{2}n^2 - 4nk^2 + \frac{15}{4}k^3 + \frac{63}{8}k^2 + \frac{1}{2}n^2k^2 + n^2k + \frac{9}{8}k^4$

(5) Case 5: $n = 4k + 2$

$$\dim M(L^k) = \left(\sum_{j=k+2}^{2k+1} \frac{3}{2} \cdot (j-k-1)(j-k)(n-j)\right) + 2(n-(2k+2)) +$$

$$\sum_{j=2k+2}^{3k+1} \left((k+1)^2 - \frac{(j-2k-1)(j-2k)}{2}\right) \cdot (n-j) +$$

$$\sum_{j=2k+3}^{3k+2}(j-2k-2)\cdot(n-j)+\sum_{j=2k+3}^{3k+2}\frac{j-2k-2}{2}\cdot(n-j)\,(n-j+1)+$$
$$\sum_{j=3k+2}^{4k+1}\frac{1}{2}\cdot(4k+2-j)(4k+3-j)(n-j)+\sum_{j=3k+3}^{n}(4k+3-j)\cdot(n-j)+$$
$$\sum_{j=3k+3}^{n}\frac{4k+4-j}{2}\cdot(n-j)\,(n-j+1)$$

$\dim M(L^k) = -2+\frac{55}{6}nk^3+\frac{19}{12}n-\frac{49}{12}k+\frac{20}{3}nk-\frac{23}{24}n^2+16nk^2+\frac{5}{12}n^3-\frac{275}{12}k^3-\frac{371}{24}k^2-\frac{7}{2}n^2k^2+\frac{2}{3}n^3k-4n^2k-\frac{229}{24}k^4-\frac{1}{24}n^4$
$=\frac{17}{4}k+\frac{47}{8}k^2+\frac{35}{4}k^3+\frac{25}{8}k^4$

(6) Case 6: $n \geq 4k+3$

No sums terminate early, hence the original calculation of $\dim M(L^k)$ may be used.
$\dim M(L^k) = -1-\frac{3}{2}nk^3-\frac{1}{2}n+\frac{17}{4}k-5nk+\frac{1}{2}n^2-4nk^2+\frac{15}{4}k^3+\frac{63}{8}k^2+\frac{1}{2}n^2k^2+n^2k+\frac{9}{8}k^4$

If $k=1$, notice that cases 2 and 4 get skipped as they count zero values of n. In the event that $k=0$ we can refer back to [2] or notice that the first and final cases describe this scenario: $n<3$ and $n \geq 3$.

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