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THE SMASH COPRODUCT FOR HOPF QUASIGROUPS

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Abstract. In this paper, the smash coproduct of Hopf quasigroups are discussed. Necessary and sufficient conditions for the smash coproduct of Hopf quasigroups to be a Hopf quasigroup are derived. The dual situation for Hopf coquasigroups are also discussed.

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1. Introduction

The concepts of Hopf quasigroups and Hopf coquasigroups were introduced recently in [4] in order to capture the quasigroup features of the (algebraic) 7-sphere. As the new generalizations of Hopf algebras, they are not required to be associative and coassociative. The lack of associativity and coassociativity is compensated by certain conditions involving the antipode. Some investigation related to Hopf quasigroups and Hopf coquasigroups can be found in [1,2,3,4,5,6].

Smash products of Hopf quasigroups and smash coproducts of Hopf coquasigroups were introduced in [1,4] as the 'quasi' versions of their Hopf algebra predecessors [7]. It is natural to ask what conditions are needed for construction of smash coproducts of Hopf quasigroups and smash products of Hopf coquasigroups. The aim of this paper is to discuss possible definitions of coactions of Hopf quasigroups and actions of Hopf coquasigroups and to study the construction of smash coproducts of Hopf quasigroups and products of Hopf coquasigroups. Since Hopf quasigroups are not required to be associative, we introduce the 'quasi' version of comodule coalgebra for Hopf quasigroups, for which a different (stronger) condition is needed for construction of smash coproduct of Hopf quasigroups. The 'coquasi' version of module algebra for Hopf coquasigroups is introduced in a similar manner.

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All algebras and coalgebras are over a field $\Bbbk$. Unadorned tensor product symbol represents the tensor product of $\Bbbk$-vector spaces. We use Sweedler notation for coproduct, that is, for all $h \in H$, $\Delta(h) = h_{(1)} \otimes h_{(2)}$ (summation implicit)[8].

2. Preliminaries on Hopf (co)quasigroups

In this section, we recall the definitions of a Hopf quasigroup and a Hopf coquasigroup from [4].

Definition 2.1. Let H be a vector space that is a unital (not necessarily associative) algebra with multiplication $\mu : H \otimes H \to H$ and unit $1 : \Bbbk \to H$. H is called a *Hopf quasigroup* if H is a counital coassociative coalgebra with comultiplication $\Delta : H \to H \otimes H$ and counit $\varepsilon : H \to \Bbbk$ that are algebra homomorphisms and there exists a linear map $S : H \to H$ such that

$$\mu \circ (id \otimes \mu) \circ (S \otimes id \otimes id) \circ (\Delta \otimes id) = \varepsilon \otimes id = \mu \circ (id \otimes \mu) \circ (id \otimes S \otimes id) \circ (\Delta \otimes id),$$

$$\mu \circ (\mu \otimes id) \circ (id \otimes S \otimes id) \circ (id \otimes \Delta) = id \otimes \varepsilon = \mu \circ (\mu \otimes id) \circ (id \otimes id \otimes S) \circ (id \otimes \Delta).$$

One can write these more explicitly as

$$S(h_{(1)})(h_{(2)}g) = \varepsilon_H(h)g = h_{(1)}(S(h_{(2)})g), \tag{2.1}$$

$$(gS(h_{(1)}))h_{(2)} = \varepsilon_H(h)g = (gh_{(1)})S(h_{(2)}) \tag{2.2}$$

for all $h, g \in H$.

Definition 2.2. Let H be a vector space that is a unital associative algebra with multiplication $\mu : H \otimes H \to H$ and unit $1 : \Bbbk \to H$ and a counital (not necessarily coassociative) coalgebra with comultiplication $\Delta : H \to H \otimes H$ and counit $\varepsilon : H \to \Bbbk$ that are algebra homomorphisms. H is called a *Hopf coquasigroup* if there exists a linear map $S : H \to H$ such that

$$(\mu \otimes id) \circ (S \otimes id \otimes id) \circ (id \otimes \Delta) \circ \Delta = 1 \otimes id = (\mu \otimes id) \circ (id \otimes S \otimes id) \circ (id \otimes \Delta) \circ \Delta,$$

$$(id \otimes \mu) \circ (id \otimes id \otimes S) \circ (\Delta \otimes id) \circ \Delta = id \otimes 1 = (id \otimes \mu) \circ (id \otimes S \otimes id) \circ (\Delta \otimes id) \circ \Delta.$$

One can write these more explicitly as

$$S(h_{(1)})h_{(2)(1)} \otimes h_{(2)(2)} = 1 \otimes h = h_{(1)}S(h_{(2)(1)}) \otimes h_{(2)(2)}, \tag{2.3}$$

$$h_{(1)(1)} \otimes S(h_{(1)(2)})h_{(2)} = h \otimes 1 = h_{(1)(1)} \otimes h_{(1)(2)}S(h_{(2)}) \tag{2.4}$$

for all $h \in H$.

As for standard Hopf algebras, the map S is called an *antipode*. It is proven in [4] that the antipode is antimultiplicative and anticomultiplicative and it immediately follows from (any of) equations (2.1)-(2.4) that, for all $h \in H$,

$$S(h_{(1)})h_{(2)} = h_{(1)}S(h_{(2)}) = \varepsilon_H(h)1,$$

i.e., S enjoys the standard antipode property.

3. The smash coproduct for Hopf quasigroups

In [1], it was given the definition of action of Hopf quasigroup. Now we will define the coaction of Hopf quasigroup.

Definition 3.1. Let H be a Hopf quasigroup. A vector space M is called a *left H-comodule* if there is a linear map $\rho : M \to H \otimes M, \rho(m) = m^{(1)} \otimes m^{(2)}$ such that, for all $m \in M$

$$m^{(1)} \otimes m^{(2)(1)} \otimes m^{(2)(2)} = m^{(1)}{}_{(1)} \otimes m^{(1)}{}_{(2)} \otimes m^{(2)}, \tag{3.1}$$

$$\varepsilon_H(m^{(1)})m^{(2)} = m. \tag{3.2}$$

An algebra (not necessarily associative) A is a *left H-comodule algebra* if A is a left H-comodule and for all $a, a' \in A$

$$\rho(aa') = a^{(1)}a'^{(1)} \otimes a^{(2)}a'^{(2)}, \tag{3.3}$$

$$\rho(1_A) = 1_H \otimes 1_A. \tag{3.4}$$

A coalgebra C is a *left quasi H-comodule coalgebra* if C is a left H-comodule and for all $c \in C, h \in H$

$$c^{(1)}h \otimes c^{(2)}{}_{(1)} \otimes c^{(2)}{}_{(2)} = c_{(1)}{}^{(1)}(c_{(2)}{}^{(1)}h) \otimes c_{(1)}{}^{(2)} \otimes c_{(2)}{}^{(2)}, \tag{3.5}$$

$$c^{(1)}\varepsilon_C(c^{(2)}) = \varepsilon_C(c)1_H. \tag{3.6}$$

A Hopf quasigroup A is called a *left H-comodule Hopf quasigroup* if it is a left H-comodule algebra and a left quasi H-comodule coalgebra simultaneously, denoted it by (A,ρ), in which ρ is the comodule action.

Let H be a Hopf quasigroup, (A, ρ) a left H-comodule Hopf quasigroup. Define $A \rtimes H = A \otimes H$ as a vector space, with tensor product multiplication, unit and counit, and comultiplication

$$\Delta_{A \rtimes H}(a \otimes h) = a_{(1)} \otimes a_{(2)}{}^{(1)} h_{(1)} \otimes a_{(2)}{}^{(2)} \otimes h_{(2)} \tag{3.7}$$

for all $a \in A, h \in H$.

Definition 3.2. With the notation above, $A \bowtie H$ is called a *smash coproduct* of A and H if $A \bowtie H$ is a Hopf quasigroup with antipode $S : A \otimes H \to A \otimes H$

$$S(a \otimes h) = S_A(a^{(2)}) \otimes S_H(a^{(1)}h) \tag{3.8}$$

for all $a \in A, h \in H$.

In the following, we will study the necessary and sufficient conditions for $A \bowtie H$ to be a smash coproduct Hopf quasigroup.

Theorem 3.3. *Let H be a Hopf quasigroup, (A, ρ) is a left H-comodule Hopf quasigroup, then the following statements are equivalent for all $a \in A, h, g \in H$.*

(1) *$A \bowtie H$ is a smash coproduct Hopf quasigroup for A and H;*
(2) *The following conditions hold*
  (c1) $ha^{(1)} \otimes a^{(2)} = a^{(1)}h \otimes a^{(2)}$;
  (c2) $a^{(1)}(hg) \otimes a^{(2)} = (a^{(1)}h)g \otimes a^{(2)}$;
  (c3) $h(a^{(1)}g) \otimes a^{(2)} = (ha^{(1)})g \otimes a^{(2)}$.
(3) *The following conditions hold*
  (cp1) $a^{(1)}(hg) \otimes a^{(2)} = h(a^{(1)}g) \otimes a^{(2)}$;
  (c2) $a^{(1)}(hg) \otimes a^{(2)} = (a^{(1)}h)g \otimes a^{(2)}$.

Proof. (2) $\Leftrightarrow$ (3) Assume that (2) holds, then for all $a \in A, h, g \in H$,

$$a^{(1)}(hg) \otimes a^{(2)} \overset{(c2)}{=} (a^{(1)}h)g \otimes a^{(2)} \overset{(c1)}{=} (ha^{(1)})g \otimes a^{(2)} \overset{(c3)}{=} h(a^{(1)}g) \otimes a^{(2)},$$

hence (cp1) holds.

Conversely, suppose that (3) holds, then (c1) follows by letting $g = 1_H$ in (cp1). For all $a \in A, h, g \in H$, we have

$$h(a^{(1)}g) \otimes a^{(2)} \overset{(cp1)}{=} a^{(1)}(hg) \otimes a^{(2)} \overset{(c2)}{=} (a^{(1)}h)g \otimes a^{(2)} \overset{(c1)}{=} (ha^{(1)})g \otimes a^{(2)}.$$

so (c3) follows.

(3)$\Rightarrow$ (1) It is not hard to verify that $1_{A \bowtie H} = 1_A \otimes 1_H$ is the unit of $A \bowtie H$, $\varepsilon_{A \bowtie H} = \varepsilon_A \otimes \varepsilon_H$ is the counit of $A \bowtie H$ and $\varepsilon_{A \bowtie H}$ is a algebra homomorphism.

To show that $A \bowtie H$ is a smash coproduct Hopf quasigroup for A and H, we only need to prove that $\Delta_{A \bowtie H}$ is coassociative, $\Delta_{A \bowtie H}$ is a algebra homomorphism and antipode S satisfies equations (2.1) and (2.2).

For all $a \in A, h \in H$, we have

$$\begin{aligned}
& (\Delta_{A\bowtie H}\otimes id)\Delta_{A\bowtie H}(a\otimes h) \\
= \;& a_{(1)}\otimes a_{(2)}{}^{(1)}(a_{(3)}{}^{(1)}{}_{(1)}h_{(1)})\otimes a_{(2)}{}^{(2)}\otimes a_{(3)}{}^{(1)}{}_{(2)}h_{(2)}\otimes a_{(3)}{}^{(2)}\otimes h_{(3)} \\
\stackrel{(3.1)}{=} \;& a_{(1)}\otimes a_{(2)}{}^{(1)}(a_{(3)}{}^{(1)}h_{(1)})\otimes a_{(2)}{}^{(2)}\otimes a_{(3)}{}^{(2)(1)}h_{(2)}\otimes a_{(3)}{}^{(2)(2)}\otimes h_{(3)} \\
= \;& a_{(1)}\otimes a_{(2)(1)}{}^{(1)}(a_{(2)(2)}{}^{(1)}h_{(1)})\otimes a_{(2)(1)}{}^{(2)}\otimes a_{(2)(2)}{}^{(2)(1)}h_{(2)}\otimes a_{(2)(2)}{}^{(2)(2)}\otimes h_{(3)} \\
\stackrel{(3.5)}{=} \;& a_{(1)}\otimes a_{(2)}{}^{(1)}h_{(1)}\otimes a_{(2)}{}^{(2)}{}_{(1)}\otimes a_{(2)}{}^{(2)}{}_{(2)}{}^{(1)}h_{(2)}\otimes a_{(2)}{}^{(2)}{}_{(2)}{}^{(2)}\otimes h_{(3)} \\
= \;& (id\otimes\Delta_{A\bowtie H})\Delta_{A\bowtie H}(a\otimes h),
\end{aligned}$$

which means $\Delta_{A\bowtie H}$ is coassociative. Next we will prove $\Delta_{A\bowtie H}$ is a algebra homomorphism. In fact,

$$\Delta_{A\bowtie H}(1_A\otimes 1_H) \stackrel{(3.4)}{=} 1_A\otimes 1_H\otimes 1_A\otimes 1_H$$

and for all $a, b \in A, h, g \in H$, we have

$$\begin{aligned}
& \Delta_{A\bowtie H}((a\otimes h)(b\otimes g)) \\
= \;& a_{(1)}b_{(1)}\otimes(a_{(2)}b_{(2)})^{(1)}(h_{(1)}g_{(1)})\otimes(a_{(2)}b_{(2)})^{(2)}\otimes h_{(2)}g_{(2)} \\
\stackrel{(3.3)}{=} \;& a_{(1)}b_{(1)}\otimes(a_{(2)}{}^{(1)}b_{(2)}{}^{(1)})(h_{(1)}g_{(1)})\otimes a_{(2)}{}^{(2)}b_{(2)}{}^{(2)}\otimes h_{(2)}g_{(2)} \\
\stackrel{(c2)}{=} \;& a_{(1)}b_{(1)}\otimes a_{(2)}{}^{(1)}(b_{(2)}{}^{(1)}(h_{(1)}g_{(1)}))\otimes a_{(2)}{}^{(2)}b_{(2)}{}^{(2)}\otimes h_{(2)}g_{(2)} \\
\stackrel{(cp1)}{=} \;& a_{(1)}b_{(1)}\otimes a_{(2)}{}^{(1)}(h_{(1)}(b_{(2)}{}^{(1)}g_{(1)}))\otimes a_{(2)}{}^{(2)}b_{(2)}{}^{(2)}\otimes h_{(2)}g_{(2)} \\
\stackrel{(c2)}{=} \;& a_{(1)}b_{(1)}\otimes(a_{(2)}{}^{(1)}h_{(1)})(b_{(2)}{}^{(1)}g_{(1)})\otimes a_{(2)}{}^{(2)}b_{(2)}{}^{(2)}\otimes h_{(2)}g_{(2)} \\
= \;& (a^{(1)}\otimes a_{(2)}{}^{(1)}h_{(1)}\otimes a_{(2)}{}^{(2)}\otimes h_{(2)})(b^{(1)}\otimes b_{(2)}{}^{(1)}g_{(1)}\otimes b_{(2)}{}^{(2)}\otimes g_{(2)}) \\
= \;& \Delta_{A\bowtie H}(a\otimes h)\Delta_{A\bowtie H}(b\otimes g).
\end{aligned}$$

Finally, we will check identities (2.1) and (2.2) hold. For all $a, b \in A, h \in H$, we have

$$\begin{aligned}
& S((a\otimes h)_{(1)})((a\otimes h)_{(2)}(b\otimes g)) \\
= \;& S(a_{(1)}\otimes a_{(2)}{}^{(1)}h_{(1)})((a_{(2)}{}^{(2)}\otimes h_{(2)})(b\otimes g)) \\
= \;& (S_A(a_{(1)}{}^{(2)})\otimes S_H(a_{(1)}{}^{(1)}(a_{(2)}{}^{(1)}h_{(1)})))(a_{(2)}{}^{(2)}b\otimes h_{(2)}g) \\
= \;& S_A(a_{(1)}{}^{(2)})(a_{(2)}{}^{(2)}b)\otimes S_H(a_{(1)}{}^{(1)}(a_{(2)}{}^{(1)}h_{(1)}))(h_{(2)}g) \\
\stackrel{(3.5)}{=} \;& S_A(a^{(2)}{}_{(1)})(a^{(2)}{}_{(2)}b)\otimes S_H(a^{(1)}h_{(1)})(h_{(2)}g) \\
\stackrel{(2.1)}{=} \;& \varepsilon_A(a^{(2)})b\otimes S_H(a^{(1)}h_{(1)})(h_{(2)}g) \\
\stackrel{(3.6)(2.1)}{=} \;& \varepsilon_A(a)\varepsilon_H(h)b\otimes g,
\end{aligned}$$

and

$$
\begin{aligned}
& (a\otimes h)_{(1)}(S((a\otimes h)_{(2)})(b\otimes g)) \\
= \quad & a_{(1)}(S_A(a_{(2)}{}^{(2)(2)})b)\otimes(a_{(2)}{}^{(1)}h_{(1)})(S_H(a_{(2)}{}^{(2)(1)}h_{(2)})g) \\
\overset{(3.1)}{=} \quad & a_{(1)}(S_A(a_{(2)}{}^{(2)})b)\otimes(a_{(2)}{}^{(1)}{}_{(1)}h_{(1)})(S_H(a_{(2)}{}^{(1)}{}_{(2)}h_{(2)})g) \\
\overset{(2.1)}{=} \quad & a_{(1)}(S_A(a_{(2)}{}^{(2)})b)\otimes\varepsilon_H(a_{(2)}{}^{(1)}h)g \\
\overset{(3.2)(2.1)}{=} \quad & \varepsilon_A(a)\varepsilon_H(h)b\otimes g.
\end{aligned}
$$

By similar computations, we get

$$((b\otimes g)S((a\otimes h)_{(1)}))(a\otimes h)_{(2)} = \varepsilon_A(a)\varepsilon_H(h)b\otimes g,$$

$$((b\otimes g)(a\otimes h)_{(1)})S((a\otimes h)_{(2)}) = \varepsilon_A(a)\varepsilon_H(h)b\otimes g.$$

(1)$\Rightarrow$ (3) For all $a, b \in A, g, h \in H$,

$$
\begin{aligned}
& \Delta_{A\bowtie H}((a\otimes h)(b\otimes g)) \\
= \quad & a_{(1)}b_{(1)}\otimes(a_{(2)}{}^{(1)}b_{(2)}{}^{(1)})(h_{(1)}g_{(1)})\otimes a_{(2)}{}^{(2)}b_{(2)}{}^{(2)}\otimes h_{(2)}g_{(2)},
\end{aligned}
$$

and

$$
\begin{aligned}
& \Delta_{A\bowtie H}(a\otimes h)\Delta_{A\bowtie H}(b\otimes g) \\
= \quad & a_{(1)}b_{(1)}\otimes(a_{(2)}{}^{(1)}h_{(1)})(b_{(2)}{}^{(1)}g_{(1)})\otimes a_{(2)}{}^{(2)}b_{(2)}{}^{(2)}\otimes h_{(2)}g_{(2)}.
\end{aligned}
$$

Since $\Delta_{A\bowtie H}$ is a algebra homomorphism, we obtain

$$
\begin{aligned}
& a_{(1)}b_{(1)}\otimes(a_{(2)}{}^{(1)}b_{(2)}{}^{(1)})(h_{(1)}g_{(1)})\otimes a_{(2)}{}^{(2)}b_{(2)}{}^{(2)}\otimes h_{(2)}g_{(2)} \\
&= a_{(1)}b_{(1)}\otimes(a_{(2)}{}^{(1)}h_{(1)})(b_{(2)}{}^{(1)}g_{(1)})\otimes a_{(2)}{}^{(2)}b_{(2)}{}^{(2)}\otimes h_{(2)}g_{(2)}).
\end{aligned}
\tag{3.9}
$$

Taking $b = 1_A$ in (3.9), we have

$$a_{(1)}\otimes a_{(2)}{}^{(1)}(h_{(1)}g_{(1)})\otimes a_{(2)}{}^{(2)}\otimes h_{(2)}g_{(2)} = a_{(1)}\otimes(a_{(2)}{}^{(1)}h_{(1)})g_{(1)}\otimes a_{(2)}{}^{(2)}\otimes h_{(2)}g_{(2)}. \tag{3.10}$$

Taking $a = 1_A$ in (3.9), we get

$$b_{(1)}\otimes b_{(2)}{}^{(1)}(h_{(1)}g_{(1)})\otimes b_{(2)}{}^{(2)}\otimes h_{(2)}g_{(2)} = b_{(1)}\otimes h_{(1)}(b_{(2)}{}^{(1)}g_{(1)})\otimes b_{(2)}{}^{(2)}\otimes h_{(2)}g_{(2)}. \tag{3.11}$$

Applying $\varepsilon_A\otimes id_H\otimes id_A\otimes\varepsilon_H$ to both sides of (3.10) and (3.11), respectively, we find that

$$a^{(1)}(hg)\otimes a^{(2)} = (a^{(1)}h)g\otimes a^{(2)},$$

$$h(b^{(1)}g)\otimes b^{(2)} = b^{(1)}(hg)\otimes b^{(2)}.$$

So (3) follows. This completes the proof. □

4. The smash product for Hopf coquasigroups

The results of Section 3 can be dualized to the Hopf coquasigroup case.

Definition 4.1. Let H be a Hopf coquasigroup. A vector space M is called a *right H-module* if there is a linear map $\alpha : M\otimes H \to M$, $\alpha(m\otimes h) = m \cdot h$ such that, for all $m \in M, h, g \in H$

$$(m \cdot h) \cdot g = m \cdot (hg), \tag{4.1}$$

$$m \cdot 1_H = m. \tag{4.2}$$

A algebra A is called a *right quasi H-module algebra* if A is a right H-module and for all $a, a^{'} \in A$

$$h_{(1)}\otimes(aa^{'}) \cdot h_{(2)} = h_{(1)(1)}\otimes(a \cdot h_{(1)(2)})(a^{'} \cdot h_{(2)}), \tag{4.3}$$

$$1_A \cdot h = \varepsilon_H(h)1_A. \tag{4.4}$$

A coalgebra (not necessarily coassociative) C is a *right H-module coalgebra* if C is a right H-module and for all $c \in C, h \in H$

$$\Delta(c \cdot h) = c_{(1)} \cdot h_{(1)}\otimes c_{(2)} \cdot h_{(2)}, \tag{4.5}$$

$$\varepsilon_C(c \cdot h) = \varepsilon_C(c)\varepsilon_H(h). \tag{4.6}$$

A Hopf coquasigroup C is called a *right H-module Hopf coquasigroup* if C is both a right quasi H-module algebra and a right H-module coalgebra .

Let H be a Hopf coquasigroup, C is a right H-module Hopf coquasigroup. Let $H \ltimes C$ be equal to $H \otimes C$ as a vector space, with tensor product comultiplication, unit and counit, and multiplication

$$(h\otimes c)(g\otimes d) = hg_{(1)}\otimes(c \cdot g_{(2)})d$$

for all $h, g \in H, c, d \in C$.

Definition 4.2. With notation as above, $H \ltimes C$ is called a *smash product* for H and C if $H \ltimes C$ is a Hopf coquasigroup with antipode $S : H \otimes C \to H \otimes C$

$$S(h \otimes c) = S_H(h)_{(1)} \otimes S_C(c) \cdot S_H(h)_{(2)}$$

for all $h \in H, c \in C$.

In what follows, we will present necessary and sufficient conditions for $H \ltimes C$ to be a smash product Hopf coquasigroup. The proof will be omitted since it is dual to the proof of Theorem 3.3.

Theorem 4.3. *Let H be a Hopf coquasigroup, C is a right H-module Hopf coquasigroup, then the followings are equivalent for all $h \in H, c \in C$.*

(1) $H \ltimes C$ *is a smash product Hopf coquasigroup for H and C;*

(2) the following conditions hold

(dc1) $h_{(1)} \otimes c \cdot h_{(2)} = h_{(2)} \otimes c \cdot h_{(1)}$;

(dc2) $h_{(1)(1)} \otimes h_{(1)(2)} \otimes c \cdot h_{(2)} = h_{(1)} \otimes h_{(2)(1)} \otimes c \cdot h_{(2)(2)}$;

(dc3) $h_{(1)(1)} \otimes c \cdot h_{(1)(2)} \otimes h_{(2)} = h_{(1)} \otimes c \cdot h_{(2)(1)} \otimes h_{(2)(2)}$.

(3) the following conditions hold

(dcp1) $h_{(1)(1)} \otimes h_{(1)(2)} \otimes c \cdot h_{(2)} = h_{(1)(1)} \otimes h_{(2)} \otimes c \cdot h_{(1)(2)}$;

(dc2) $h_{(1)(1)} \otimes h_{(1)(2)} \otimes c \cdot h_{(2)} = h_{(1)} \otimes h_{(2)(1)} \otimes c \cdot h_{(2)(2)}$.

In [4], the structure of $\Bbbk[S^{2^n-1}]$ is introduced as an example of Hopf coquasigroup. The Hopf coquasigroup $\Bbbk[S^{2^n-1}]$ has a left action of $\Bbbk\mathbb{Z}_2^n$ defined by

$$\sigma_a \cdot x_b = (-1)^{a \cdot b} x_a$$

such that the resulting cross product $\Bbbk[S^{2^n-1}] \rtimes \Bbbk\mathbb{Z}_2^n$ is a Hopf coquasigroup (see [4, Proposition 5.10 and Example 5.11] for the detail). Then we have an example that fits in the Theorem 4.3.

Example 4.4. *Let $C = \Bbbk[S^{2^n-1}]$ be a Hopf coquasigroup equipped with an action of $\Bbbk\mathbb{Z}_2^n$ by*

$$x_a \triangleleft \sigma_b = S(\sigma_b) \cdot x_a = \sigma_{-b} \cdot x_a = (-1)^{-b \cdot a} x_a,$$

then $\Bbbk[S^{2^n-1}]$ is a right $\Bbbk\mathbb{Z}_2^n$-module Hopf coquasigroup and $\Bbbk\mathbb{Z}_2^n \ltimes \Bbbk[S^{2^n-1}]$ is a smash product Hopf coquasigroup.

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