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ON NIL-SEMICOMMUTATIVE RINGS

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Abstract. Semicommutative and Armendariz rings are a generalization of reduced rings, and therefore, nilpotent elements play an important role in this class of rings. There are many examples of rings with nilpotent elements which are semicommutative or Armendariz. In fact, in [1], Anderson and Camillo prove that if R is a ring and $n \geq 2$, then $R[x]/(x^n)$ is Armendariz if and only if R is reduced. In order to give a noncommutative generalization of the results of Anderson and Camillo, we introduce the notion of nil-semicommutative rings which is a generalization of semicommutative rings. If R is a nil-semicommutative ring, then we prove that $ni\ell(R[x]) = ni\ell(R)[x]$. It is also shown that nil-semicommutative rings are 2-primal, and when R is a nil-semicommutative ring, then the polynomial ring $R[x]$ over R and the rings $R[x]/(x^n)$ are weak Armendariz, for each positive integer n, generalizing related results in [12].

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1. Introduction

Throughout this paper, all rings are associative with identity. Given a ring R, we denote by $ni\ell(R)$ the subset of all nilpotent elements of R. Recall that a ring R is called *reduced* if $a^2 = 0$ implies that $a = 0$, for all $a \in R$; R is *symmetric* if $abc = 0$ implies $acb = 0$, for all $a, b, c \in R$; R is *reversible* if $ab = 0$ implies $ba = 0$, for all $a, b \in R$; R is *semi-commutative* if $ab = 0$ implies $aRb = 0$, for all $a, b \in R$. In H.E. Bell's paper [4], semicommutative property is called the insertion-of-factors-property, or IFP. Rings satisfying IFP was later studied vis-à-vis QF-3 rings in [6] by J.M. Habeb (who referred to rings satisfying IFP as zero insertive or zi), see also [9]. By Rege and Chhawchharia [14], a ring R is called *Armendariz* if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j. In [12], Liu and Zhao introduce weak Armendariz rings as a generalization of Armendariz rings. A ring is *weak Armendariz* if whenever the product of two polynomials is zero then the product of their coefficients is nilpotent. This further motivates the study of the nilpotent elements in this class of rings. In [2], Ramon Antoine initiates the notion of nil-Armendariz rings. A ring R is said to be *nil-Armendariz* if whenever two polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) \in ni\ell(R)[x]$ then $ab \in ni\ell(R)$ for all $a \in coef(f(x))$ and $b \in coef(g(x))$, where $coef(g(x))$ denotes the set of all coefficients of $g(x)$. What we observe is that in all the examples found in the literature of Armendariz and weak Armendariz rings, the set of nilpotent elements forms an ideal. One may think that this is true for weak Armendariz rings or at least Armendariz rings, but this is not the case, and Antoine [2] provided a counterexample in the case of Armendariz rings. We have the following implications:

$$\begin{array}{cccccccccc}
\textit{commutative} \Rightarrow & \textit{symmetric} & \Rightarrow \textit{reversible} \Rightarrow & \textit{semicommu} & \Rightarrow & \textit{nil-semicommu} & \Rightarrow & \textit{2-Primal} \\
 & \Uparrow & & & & & & \Downarrow \\
 & \textit{reduced} & \Rightarrow \textit{Armendariz} & \Rightarrow & & \textit{weak-Armendariz} & \Leftarrow & \textit{nil-Armendariz}
\end{array}$$

In general, each of these implications is irreversible. Semicommutative rings are a generalization of reduced rings, and therefore, nilpotent elements play an important role in this class of rings. There are many examples of rings with nilpotent elements which are semicommutative or Armendariz. In fact, in [1], Anderson and Camillo proved that if $n \geq 2$, then $R[x]/(x^n)$ is an Armendariz ring if and only if R is reduced. This further motivates the study of the nilpotent elements in this class of rings. What we observe is that in semicommutative rings, the set of nilpotent elements forms an ideal. If the set of nilpotent elements forms an ideal, then it is easy to see that the ring is nil-Armendariz.

In a commutative ring, the set of nilpotent elements coincides with the intersection of all prime ideals. This property is also possessed by certain noncommutative rings, which are known as 2-primal rings. A ring R is called 2-*primal* if its prime radical contains every nilpotent element of R. Research on 2-primal rings was inaugurated by G. Shin in [15] (though the name "2-primal" was not coined until later). Shin proved in [15, Proposition 1.11] that a ring is 2-primal if and only if each of its minimal prime ideals is completely prime, i.e., the corresponding prime factor ring is a domain. In order to give a noncommutative generalization of the results of Anderson and Camillo, we introduce the notion of nil-semicommutative rings which is a generalization of semicommutative rings. We use this to define a new class of rings strengthening the condition for semicommutative rings. This property between semicommutative and 2-primal is what we call nil-semicommutative rings. Most of the results found in [12] for semicommutative rings can be extended to nil-semicommutative rings. We prove that if R is a nil-semicommutative ring, then the set of nilpotent elements of R is an ideal of R. This allows us to study the conditions under which the polynomial ring over a nil-semicommutative ring is also nil-semicommutative. These conditions are strongly connected to the question of Amitsur of whether or not a polynomial ring over a nil ring is nil. This problem was solved in the negative by Agata Smoktunowicz in [16]. Another property between commutative and 2-primal is what Cohn in [5] calls *reversible* rings: those rings R with the property that $ab = 0$, $ba = 0$ for all $a; b \in R$. Cohn shows that the Köthe Conjecture is true for the class of reversible rings. Indeed, all reversible rings are 2-primal, and the Köthe Conjecture is clearly true for 2-primal rings more generally, for the class of rings whose nilpotent elements form an ideal.

Hirano's claim [8] assumed that if R is semi-commutative then $R[x]$ is semi-commutative, and this was later shown to be false in [10, Example 2]. Due to an example of Kim and Lee [11, Example 2.1], we know that if R is reversible then $R[x]$ may not even be semi-commutative. In this paper we define nil-semicommutative rings which is a stronger condition than semicommutative rings and investigate the properties of several extensions of nil-semicommutative rings. If R is a nil-semicommutative ring, then we prove that $ni\ell(R[x]) = ni\ell(R)[x]$. It is shown that nil-semicommutative rings are 2-primal and hence satisfy the Köthe conjecture. If R is nil-semicommutative, then we prove that the polynomial ring $R[x]$ over R and the rings $R[x]/(x^n)$ are weak Armendariz, for each positive integer n. Since semicommutative rings are nil-semicommutative, this generalizes [12]. Notice that, using 2.2, 2.3 and 2.4, we can provide various examples of weak α-rigid nil-semicommutative rings that are not semicommutative.

2. Nil-semicommutative rings

In this section we introduce the class of nil-semicommutative rings which contains the class of semicommutative rings. On the other hand, every semicommutative ring is both 2-primal and nil-Armendariz. We show that there is a large class of nil-semicommutative rings which are not semicommutative. For a ring R, $T_n(R)$ and E_{ij} denote the upper triangular matrix ring and the elementary matrix, respectively.

Definition 2.1. We say that a ring R is *nil-semicommutative* if for every $a, b \in ni\ell(R)$, $ab = 0$ implies $aRb = 0$.

Clearly semicommutative rings are nil-semicommutative and that every subring of a nil-semicommutative ring is nil-semicommutative. We first provide a large class of nil-semicommutative rings which are not semicommutative.

Example 2.2. *For every reduced ring R, the upper triangular matrix ring $T_3(R)$ is a nil-semicommutative ring which is not semicommutative.*

Proof. Let $\begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & b_{12} & b_{13} \\ 0 & 0 & b_{23} \\ 0 & 0 & 0 \end{pmatrix} \in ni\ell(T_3(R))$ and let $\begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & b_{12} & b_{13} \\ 0 & 0 & b_{23} \\ 0 & 0 & 0 \end{pmatrix}$=0. Then $\begin{pmatrix} 0 & 0 & a_{12}b_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$=0. Since R is reduced, $a_{12}Rb_{23} = 0$, and that

$$\begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & 0 & c_{33} \end{pmatrix}\begin{pmatrix} 0 & b_{12} & b_{13} \\ 0 & 0 & b_{23} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_{12}c_{22}b_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$= 0$. Hence $T_3(R)$ is nil-semicommutative. Now to see that $T_3(R)$ is not semicommutative, we have $E_{11}E_{22} = 0$, but $E_{11}E_{12}E_{22} \neq 0$. □

Anderson and Camillo prove that Armendariz rings are abelian (i.e. all idempotents are central). By [3, Corollary 2.8] semicommutative rings are *abelian*, but by Example 2.2, it is clear that nil-semicommutative rings need not be abelian.

For a ring R, let

$$V(R) = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{pmatrix} \mid a_{ij} \in R, 0 \leq i, j \leq 4 \right\}.$$

Then $V(R)$ forms a subring of $T_4(R)$.

Example 2.3. *For every reduced ring R, $V(R)$ is a nil-semicommutative ring which is not semicommutative.*

Proof. Suppose that $\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b_{12} & b_{13} & b_{14} \\ 0 & 0 & b_{23} & b_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in ni\ell(V(R))$, and that

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & b_{12} & b_{13} & b_{14} \\ 0 & 0 & b_{23} & b_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_{12}b_{23} & a_{12}b_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0.$$

Since R is reduced,

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ 0 & c_{22} & c_{23} & c_{24} \\ 0 & 0 & c_{33} & 0 \\ 0 & 0 & 0 & c_{44} \end{pmatrix} \begin{pmatrix} 0 & b_{12} & b_{13} & b_{14} \\ 0 & 0 & b_{23} & b_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & a_{12}c_{22}b_{23} & a_{12}c_{22}b_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0.$$

We have $E_{11}E_{22} = 0$, but $E_{11}E_{12}E_{22} \neq 0$, so $V(R)$ is not semicommutative. □

For a ring R, let

$$S(R) = \left\{ \begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix} \mid a_{ij} \in R, 0 \leq i, j \leq 4 \right\}.$$

Then $S(R)$ forms a subring of $T_4(R)$.

Example 2.4. *For every reduced ring R, $S(R)$ is a nil-semicommutative ring which is not semicommutative.*

Proof. The proof is similar to that of Example 2.3. □

We also observe that in all the examples found in the literature of Armendariz and weak Armendariz rings, the set of nilpotent elements forms an ideal. One may think that this is true for weak Armendariz rings or at least Armendariz rings, but this is not the case, and Antoine [2] provided a counterexample in the case of Armendariz rings. However we show that $ni\ell(R)$ is an ideal in a nil-semicommutative ring R.

Theorem 2.5. *If R is a nil-semicommutative ring, then $ni\ell(R)$ is an ideal of R.*

Proof. Suppose that $a^{2m} = 0$. Then $a^m r a^m$=0, for each $r \in R$, since R is nil-semicommutative. It is clear that $a^{m-1}, ara^m \in ni\ell(R)$. By nil-semicommutativity, $a^{m-1}rara^m = 0$, $a^{m-1}(ra)^2, a^{m-1} \in ni\ell(R)$, which yields that $a^{m-1}(ra)^2ra^{m-1} = 0$, and that $a^{m-1}(ra)^3, a^{m-2} \in ni\ell(R)$. By nil-semicommutativity, $a^{m-1}(ra)^3ra^{m-2} = 0$, and that $a^{m-2}, (ar)^4a^{m-2} \in ni\ell(R)$. Continuing in this process we deduce that $(ar)^{2m} = 0$. Therefore $ar, ra \in ni\ell(R)$. Now suppose that $a^m = 0, b^n = 0$, let $k = m + n + 1$. Then $(a + b)^k = \sum_{i_1+j_1+\cdots+i_s+j_s=k}(a^{i_1}b^{j_1}a^{i_2}b^{j_2}\cdots a^{i_s}b^{j_s})$,

$0 \leq i_1, j_1, \ldots, i_s, j_s \leq k$. If $i_1 + i_1 + \cdots + i_s \geq m$, then $a^{i_1}a^{i_2}\cdots a^{i_s} = 0$,$a^{i_p} \in ni\ell(R)$, for each p with $0 \leq p \leq s$. We have $a^{i_1}b^{j_1}a^{i_2}b^{j_2}\cdots a^{i_s}b^{j_s} = 0$, since R is nil-semicommutative. If $i_1 + i_1 + \cdots + i_s \leq m$, then $j_1 + j_1 + \cdots + j_s \geq n$, thus $b^{j_1+j_2+\cdots j_s} = 0$ and similarly we have $a^{i_1}b^{j_1}a^{i_2}b^{j_2}\cdots a^{i_s}b^{j_s} = 0$. Hence $(a+b)^k = 0$. □

Corollary 2.6. [12, Lemma 3.1] *If R is a semicommutative ring, then $ni\ell(R)$ is an ideal of R.*

Based on Artin and Wedderburn, the Wedderburn radical of a ring R means the sum of all nilpotent ideals in R (in spite of this sum being not a radical, it was given the name), written by $N_0(R)$. We also denote the lower nilradical of R by $Nil_*(R)$.

Lemma 2.7. *Nil-semicommutative rings are 2-primal.*

Proof. It is sufficient to prove that $ni\ell(R) \subseteq Nil_*(R)$. Let $a \in ni\ell(R)$. Since by Theorem 2.5, $ni\ell(R)$ is an ideal R, $RaR \subseteq ni\ell(R)$. Since R is nil-semicommutative, RaR is a nilpotent ideal, so $RaR \in N_0(R) \subseteq Nil_*(R)$. Hence each nilpotent element is contained in an arbitrary prime ideal. □

By Theorem 2.5, we see that the Köthe Conjecture is true for the class of nil-semicommutative rings. Indeed, all nil-semicommutative rings are 2-primal, and the Köthe Conjecture is clearly true for 2-primal rings.

We now give an example of a 2-primal ring, which is not nil-semicommutative.

Example 2.8. *Let R be a reduced ring. By Example 2.11, $T_5(R)$ is not nil-semicommutative, as it is not abelian, but it is clear that $T_5(R)$ is 2-primal.*

Corollary 2.9. *Nil-semicommutative rings are nil-Armendariz.*

Proof. Let R be a nil-semicommutative ring. By Theorem 2.5, $ni\ell(R)$ is an ideal of R. So by [2, Proposition 2.1], R is nil-Armendariz. □

Corollary 2.10. *Nil-semicommutative rings are weak Armendariz.*

Proof. Nil-semicommutative rings are nil-Armendariz and nil-Armendariz rings are weak Armendariz, by [2]. □

Hence nil-semicommutative rings stand as a generalization of semicommutative rings and a particular case of weak Armendariz rings.

One may suspect that the nil-semicommutative property is inherited by $T_n(R)$. But the following example erases the possibility. Observe that

$$ni\ell(T_n(R)) = \begin{pmatrix} ni\ell(R) & R & R \\ 0 & \ddots & R \\ 0 & 0 & ni\ell(R) \end{pmatrix}.$$

Example 2.11. *For any ring R, the triangular matrix ring $T_5(R)$ is not nil-semicommutative. Take $0 \neq a \in ni\ell(R)$, then we have $aE_{11}, E_{23} \in ni\ell(T_5(R))$ and $aE_{11}E_{23} = 0$, but $aE_{11}E_{12}E_{23} = aE_{13} \neq 0$.*

Now we give an example of a ring R such that $ni\ell(R)$ is an ideal and that R is not nil-semicommutative.

Example 2.12. *Let R be a reduced ring. By [12, Proposition 2.2], $T_4(R)$ is weak-Armendariz, and $ni\ell(T_4(R))$ is an ideal of $T_4(R)$. However we see that $T_4(R)$ is not nil-semicommutative. Consider $E_{12}, E_{34} \in ni\ell(T_4(R))$, and $E_{12}E_{34} = 0$, but $E_{12}E_{23}E_{34} \neq 0$.*

Proposition 2.13. *Finite product of nil-semicommutative rings is nil-semicomm-utative.*

Proof. First we observe that $ni\ell(\prod_{i=0}^{n} R_i)$=$\prod_{i=0}^{n} ni\ell(R_i)$. To see this let $(a_1, a_2, \dots, a_n) \in ni\ell(\prod_{i=0}^{n} R_i)$, then $(a_1, a_2, \dots, a_n)^k = 0$, $a_i^k = 0$ and hence $(a_1, a_2 \dots, a_n) \in \prod_{i=0}^{n} ni\ell(R_i)$. If $(b_1, b_2, \dots, b_n) \in \prod_{i=0}^{n} ni\ell(R_i)$, then $b_i^{k_i} = 0$. Let $k = max\{k_1, k_2, \dots, k_n\}$, then $(b_1, b_2, \dots, b_n)^k = 0$, so $(b_1, b_2, \dots, b_n) \in ni\ell(\prod_{i=0}^{n} R_i)$. If $(a_1, a_2, \dots, a_n)$,$(b_1, b_2, \dots, b_n) \in ni\ell(\prod_{i=0}^{n} R_i)$, $(a_1, a_2, \dots, a_n) \cdot (b_1, b_2, \dots, b_n) = 0$, then for each $i, j = 1, 2, \dots, n$, $a_ib_j = 0$. Since R_i is nil-semicommutative, $a_iR_ib_i = 0$, for each i. So we get $(a_1, a_2, \dots, a_n) \prod_{i=0}^{n} R_i(b_1, b_2, \dots, b_n)$=0. □

The ring of Laurent polynomials in x with coefficients in a ring R, consists of all formal sums $\sum_{i=k}^{n} m_ix^i$ with obvious addition and multiplication, where $m_i \in R$ and k, n are (possibly negative) integers. We denote this ring by $R[x; x^{-1}]$.

Proposition 2.14. *Let R be a ring and Ω be a multiplicatively closed subset of R consisting of central regular elements. Then R is nil-semicommutative if and only if $\Omega^{-1}R$ is nil-semicommutative.*

Proof. It suffices to prove the necessary condition because subrings of nil-semicomm utative are also nil-semicommutative. Let $\alpha\beta = 0$ with $\alpha = u^{-1}a, \beta = v^{-1}b \in ni\ell(\Omega^{-1}R)$, then $u, v \in \Omega a, b \in ni\ell(R)$, since Ω is contained in the center of R.

We have $0 = \alpha\beta = u^{-1}av^{-1}b = u^{-1}v^{-1}(ab) = (uv)^{-1}(ab)$ and so $ab = 0$. It follows that $arb = 0$ for all $r \in R$ because R is nil-semicommutative. Now for each $\gamma = \omega^{-1}r \in \Omega^{-1}R$ with $\omega \in \Omega$ and $r \in R$, $\alpha\gamma\beta = (u\omega v)^{-1}arb = (u\omega v)^{-1}0 = 0$. Hence $\Omega^{-1}R$ is nil-semicommutative. □

Corollary 2.15. *For a ring R, $R[x]$ is nil-semicommutative if and only if $R[x; x^{-1}]$ is nil-semicommutative.*

Proof. Suppose that $R[x]$ is nil-semicommutative. Let $\Omega = \{1, x, x^2, \cdots\}$, then clearly Ω is a multiplicatively closed subset of $R[x]$. Since $R[x; x^{-1}] = \Omega^{-1}R[x]$. It follows that $R[x; x^{-1}]$ is nil-semicommutative by Proposition, 2.14, The sufficient condition is proved straightforwardly since subrings of nil-semicommutative rings are also nil-semicommutative. □

A classical right quotient ring for R is a ring Q which contains R as a subring in such a way that every regular element (i.e., non-zero-divisor) of R is invertible in Q and $Q = \{ab^{-1} \mid a, b \in R, \text{b regular}\}$.

By the Goldie's Theorem, if R is a semiprime right Goldie ring, then R has classical right quotient rings. Hence there exists a class of rings satisfying the following hypothesis.

Lemma 2.16. *Suppose R is a semiprime right Goldie ring. Then the following statements are equivalent:*

(1) *R is reduced.*

(2) *R is semicommutative.*

(3) *R is nil-semicomutative.*

(4) *Q is reduce.*

(5) *Q is semicommutative.*

(6) *Q is finite direct product of division rings.*

Proof. (1) ⇒ (2) ⇒ (4) ⇒ (5) ⇒ (6) ⇒ (1) See [10, Corollary 13].

(2) ⇔ (3) One can prove this, using Proposition 2.18. □

The trivial extension of a ring R is the ring $T(R, R) = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in R\}$, with the usual matrix operations. It is clear that $ni\ell(T(R, R)) = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a \in ni\ell(R), b \in R\}$. For a nil-semicommutative ring R, we give an example that $T(R, R)$ may not be nil-semicommutative.

Example 2.17. *Let R be a reduced ring. By Example 2.2 , $S = T_3(R)$ is nil-semicommutative. But $T(S,S)$ is not nil-semicommutative. To see this, take* $a = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, c = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, d = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}$, $e = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ *and* $f = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *Then $(a,b),(c,d) \in ni\ell(T(S,S)), (e,f) \in T(S,S)$ and $(a,b)(c,d) = 0$ but $(a,b)(e,f)(c,d) \neq 0$.*

The following results show that, for a semiprime ring, the properties of reduced, symmetric, reversible, semicommutative, 2-primal and nil-semicommutative are coincide.

Proposition 2.18. *For a semiprime ring R, the following statements are equivalent:*

(1) *R is reduced.*
(2) *R is symmetric.*
(3) *R is reversible.*
(4) *R is semicommutative*
(5) *R is nil-semicommutative.*
(6) *R is 2-primal.*

Proof. (1)-(4) are equivalent by [11, Lemma 2.7]. (1) $\Rightarrow$ (5) It is clear.

(5) $\Rightarrow$ (1) Let $a^2 = 0$ then $a \in ni\ell(R)$, by nil-semicommutative we have $aRa = 0$, since R is semiprime then $a = 0$.

(6) $\Leftrightarrow$ (1) By definition a ring R is 2-primal if and only if $Nil_*(R) = ni\ell(R)$. This yields that, a ring R is reduced if and only if it is semiprime and 2-primal. $\square$

Corollary 2.19. *For a von Neumann regular ring R, the following statements are equivalent:*

(1) *R is reduced.*
(2) *R is symmetric.*
(3) *R is reversible.*
(4) *R is semicommutative.*
(5) *R is nil-semicommutative.*
(6) *R is 2-primal.*

We say a right (or left) ideal I of a ring R is *nil-semicommutative* if $ab \in I$ implies $aRb \subseteq I$ for $a, b \in \sqrt{I}$, where $\sqrt{I} = \{s \in R \mid s^n \in I, \text{ for some positive integer n}\}$.

Lemma 2.20. *Let I be an ideal of a ring R. Then R/I is a nil-semicommutative ring if only if I is a nil-semicommutative ideal.*

Proof. It is clear. $\square$

We denote $r_R(U) = \{r \in R \,|\, Ur = 0\})$ and $\ell_R(U) = \{r \in R \,|\, rU = 0\})$.

Proposition 2.21. *For a ring R the following conditions are equivalent:*

(1) *R is nil-semicommutative.*

(2) *$r_{ni\ell(R)}(U)$ is an ideal of R for each $U \subseteq ni\ell(R)$.*

(3) *$\ell_{ni\ell(R)}(V)$ is an ideal of R for each $V \subseteq ni\ell(R)$.*

Proof. (1) $\Rightarrow$ (2) Let $r \in r_{ni\ell(R)}(U)$. Since R is nil-semicommutative, $URr = 0$. So $Rr \subseteq r_{ni\ell(R)}(U)$ and that $rR \subseteq r_{ni\ell(R)}(U)$ for each $r \in r_{ni\ell(R)}(U)$.
(2) $\Rightarrow$ (1) It is clear.
(1) $\Leftrightarrow$ (3) It is similar to (1) $\Leftrightarrow$ (2). $\square$

Proposition 2.22. *Let R be a nil-semicommutative ring. Then*

(1) *$R/r_{ni\ell(R)}(U)$ is a nil-semicommutative ring for each $U \subseteq ni\ell(R)$.*

(2) *$R/l_{ni\ell(R)}(V)$ is a nil-semicommutative ring for each $V \subseteq ni\ell(R)$.*

Proof. 1) If $\overline{a}, \overline{b} \in ni\ell(R/r_{ni\ell(R)}(U))$, then there exist positive integers m, n such that $a^n, b^m \in r_{ni\ell(R)}(U)$. So $a, b \in ni\ell(R)$. If $\overline{a}.\overline{b} = \overline{0}$, then $Uab = 0$. Since R is nil-semicommutative, $Uarb = 0$, for each $r \in R$. Since $ni\ell(R)$ is an ideal, $arb \in r_{ni\ell(R)}(U)$ for each $r \in R$. So $\overline{ar}\overline{b} = \overline{arb} = \overline{0}$, for each $r \in R$. Thus $R/r_{ni\ell(R)}(U)$ is a nil-semicommutative ring.
(2) is similar to (1). $\square$

3. Polynomial extension of nil-semicommutative rings

Due to an example of Kim and Lee [11, Example 2.1], we know that if R is reversible then $R[x]$ may not even be semi-commutative. Hirano's claim [8] assumed that if R is semi-commutative then $R[x]$ is semi-commutative, and this was later shown to be false in [10, Example 2]. By [2, Theorem 5.3], the question of whether $ni\ell(R[x]) = ni\ell(R)[x]$ for nil-Armendariz rings is equivalent to the question of whether polynomial rings over nil rings are nil. Amitsur, proved that this is true for K-algebras over uncountable fields. But recently, Agata Smoktunowicz, in [16], has proven that the result is not true for algebras over countable fields.

Lemma 3.1. *Let R be a nil-semicommutative ring. If $f_1 f_2 \cdots f_n \in R[x]$, $C_{f_1 f_2 \cdots f_n} \in ni\ell(R)$, then $C_{f_1} C_{f_2} \cdots C_{f_n} \in ni\ell(R)$, where C_f denotes the set of coefficients of f.*

Proof. It is similar to the proof of [12, Proposition, 3.3]. $\square$

Corollary 3.2. *If R is a nil-semicommutative ring, then $ni\ell(R[x]) \subseteq ni\ell(R)[x]$.*

Theorem 3.3. *If R is a nil-semicommutative ring, then $ni\ell(R[x]) = ni\ell(R)[x]$.*

Proof. By Corollary 3.2, $ni\ell(R[x]) \subseteq ni\ell(R)[x]$. Now suppose that $a_i^{m_i} = 0$, for $i = 0, 1, \dots, n$. Let $k = m_0 + m_1 + \cdots + m_n + 1$, then

$(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)^k = \sum_{s=0}^{nk} \left(\sum_{i_1+i_2+\cdots+i_k=s} a_{i_1}a_{i_2}\cdots a_{i_k}\right) x^s$.

Consider $a_{i_1}, a_{i_2}, \cdots a_{i_k} \in \{a_0, a_1, \dots a_n\}$. If the number of a_0's in $a_{i_1}a_{i_2}\cdots a_{i_k}$ is more than m_0, then we write $a_{i_1}a_{i_2}\cdots a_{i_k}$ as $b_0a_0^{j_1}b_1a_0^{j_2}\cdots b_{t-1}a_0^{j_t}b_t$, where $1 \leq j_1, j_2, \dots, j_t$, $m_0 \leq j_1+j_2+\ldots+j_t$ and for each i that $0 \leq i \leq t$, b_i is product of some elements choosing from $\{a_0, a_1, \dots a_n\}$ or equal to 1. Since $a_0^{j_1+j_2+\cdots+j_t}$=0 and R is nil-semicommutative, $a_0^{j_1}b_1a_0^{j_2}a_0^{j_3}\cdots a_0^{j_t} = 0$. By Lemma 2.5, $a_0^{j_1}b_1$, $a_0^{j_2}a_0^{j_3}\cdots a_0^{j_t} \in ni\ell(R)$. Then by nil-semicommutativity we have $a_0^{j_1}b_1a_0^{j_2}b_2a_0^{j_3}a_0^{j_4}\cdots a_0^{j_{t-1}}a_0^{j_t} = 0$. Continuing this process we have $b_0a_0^{j_1}b_1a_0^{j_2}\cdots b_{t-1}a_0^{j_t}b_t$=0, thus $a_{i_1}a_{i_2}\ldots a_{i_k} = 0$. If the number of a_i's in $a_{i_1}a_{i_2}\ldots a_{i_k}$ is more than m_i, a similar discussion yields that $a_{i_1}a_{i_2}\ldots a_{i_k} = 0$. Hence $\sum_{i_1+i_2+\cdots+a_k=s} a_{i_1}a_{i_2}\ldots a_{i_k} = 0$. This implies that $(a_0 + a_1x + \cdots + a_nx^n)^k = 0$. $\square$

Notice that, in [12], Liu and Zhao proved that, if R is a semicommutative ring, then $ni\ell(R)[x] \subseteq ni\ell(R[x])$.

Corollary 3.4. *If R is a semicommutative ring, then $ni\ell(R[x]) = ni\ell(R)[x]$.*

Theorem 3.5. *If R is a nil-semicommutative and Armendariz ring, then the polynomial ring $R[x]$ is nil-semicommutative.*

Proof. Let $f(x) = \sum_{i=0}^{m} a_ix^i, g(x) = \sum_{j=0}^{n} b_jx^j \in ni\ell(R[x])$. Since R is nil-semicommutative, by Theorem 3.3, we have $ni\ell(R)[x] = ni\ell(R[x])$. So $a_i, b_j \in ni\ell(R)$ for $0 \leq i \leq m, 0 \leq j \leq n$ $(*)$. Suppose that $f(x)g(x) = 0$. Since R is Armendariz, a_ib_j=0 and by $(*)$ and nil-semicommutativity we have $a_iRb_j = 0$ for $0 \leq i \leq m, 0 \leq j \leq n$. For each $h(x) = \sum_{k=0}^{p} c_kx^k \in R[X]$ we have $f(x)h(x)g(x) = \sum_{s=0}^{m+n+p}(\sum_{i+j+k=s} a_ic_kb_j)x^s$=0. Thus $f(x)R[x]g(x) = 0$ and hence $R[x]$ is nil-semicommutative. $\square$

C. Huh, Y. Lee and A. Smoktunowicz [10, Example 2] gave an example of a semicommutative ring R such that $R[x]$ is not semicommutative. We see that in this case, $R[x]$ is also not nil-semicommutative.

Example 3.6. *Let $\mathbb{Z}_2$ be the field of integers modulo 2 and consider the free algebra of polynomials with zero constant terms in noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over $\mathbb{Z}_2$ denoted by $A = \mathbb{Z}_2[a_0, a_1, a_2, b_0, b_1, b_2, c]$. Note that A is a ring without identity and consider an ideal of $\mathbb{Z}_2 + A$, say I, generated by $a_0b_0, a_1b_2 + a_2b_1, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, a_2b_2, a_0rb_0, a_2rb_2, (a_0a_1a_2)r(b_0b_1b_2)$ with $r \in A$ and $r_1r_2r_3r_4$ with $r_1, r_2, r_3, r_4 \in A$. Then clearly $A^4 \subseteq I$. Let $R = \mathbb{Z}_2 + A \backslash I$. Notice that $a_0, a_1, a_2, b_0, b_1, b_2, \in ni\ell(R)$. By [10, Example 2] R is semicommutative and by Theorem 3.3, we have $(a_0 + a_1x + a_2x^2), (b_0 + b_1x + b_2x^2) \in ni\ell(R[x])$, $(a_0 + a_1x + a_2x^2)(b_0 + b_1x + b_2x^2) \in I[x] = 0_{R[x]}$, but $(a_0 + a_1x + a_2x^2)c(b_0 + b_1x + b_2x^2) \notin I[x] = 0_{R[x]}$ because $a_0cb_1 + a_1cb_0 \notin I$. Hence $R[x]$ is not nil-semicommutative.*

Theorem 3.7. *If R is a nil-semicommutative ring, then $R[x]$ is a weak Armendariz ring.*

Proof. Let $F = \sum_{i=0}^{p} f_i y^i, G = \sum_{j=0}^{q} g_j y^j \in R[x][y]$ such that $FG = 0$. Set $f_i = \sum_{s=0}^{m_i} a_s^i x^s, g_j = \sum_{t=0}^{n_j} b_t^j x^t$. Then, as in the proof of [1, Theorem 2], we see that $a_s^i b_t^j \in ni\ell(R)$ by Theorem 2.5. Thus $\sum_{s+t=k} a_s^i b_t^j$ is a nilpotent element of R for each i, j, s, t. Now, by Theorem 3.3, $f_i g_j = \left(\sum_{s=0}^{m_i} a_s^i x^s\right)\left(\sum_{t=0}^{n_j} b_t^j x^t\right) = \sum_{k=0}^{m_i+n_j}\left(\sum_{s+t=k} a_s^i b_t^j\right) x^k$ is a nilpotent element of $R[x]$. This means that $R[x]$ is weak Armendariz. □

Corollary 3.8. [12, Theorem 3.8] *If R is a semicommutative ring, then $R[x]$ is a weak Armendariz ring.*

Theorem 3.9. *If R is a nil-semicommutative ring, then $R[x]/(x^n)$ is a weak Armendariz ring, for each positive integer n.*

Proof. Denote $\overline{x}$ in $R[x]/(x^n)$ by u so $R[x]/(x^n) = R[u] = R + Ru + Ru^2 + \cdots + Ru^{n-1}$, where u commutes with elements of R and $u^n = 0$. Let $f, g \in R[u][y]$ be such that $fg = 0$. Suppose that $f = \sum_{i=0}^{p} f_i y^i$ and $g = \sum_{j=o}^{q} g_j y^j$. Let $f_i = \sum_{s=0}^{n-1} a_s^i u^s, g_j = \sum_{t=0}^{n-1} b_t^j u^t$. Then $f = \sum_{s=0}^{n-1}(\sum_{i=0}^{p} a_s^i y^i)u^s$,
$g = \sum_{t=0}^{n-1}(\sum_{j=o}^{q} b_t^j y^j)u^t$. From $fg = 0$, we have the following equations:

$$\sum_{s+t=k}\left(\sum_{i=0}^{p} a_s^i y^i\right)\left(\sum_{j=o}^{q} b_t^j y^j\right) \tag{1}$$

for $k = 0, 1, ..., n-1$. We will show by induction on $s + t$ that $a_s^i b_t^j \in ni\ell(R)$ for $0 \leq i \leq p, 0 \leq j \leq q$, and each s, t with $s + t = 0, 1, ..., n-1$. If $s + t = 0$, then $s = t = 0$. Thus $\left(\sum_{i=0}^{p} a_0^i y^i\right)\left(\sum_{j=o}^{q} b_0^j y^j\right) = 0$. Since R is nil-semicommutative, R

is weak Armendariz by Corollary 2.10. Thus $a_0^i b_0^j \in ni\ell(R)$ for $0 \leq i \leq p, 0 \leq j \leq q$. Now suppose that $k \leq n-1$ is such that $a_s^i b_t^j \in ni\ell(R)$ for $0 \leq i \leq p, 0 \leq j \leq q$ and s,t with $s+t < k$. We will show that $a_s^i b_t^j \in ni\ell(R)$ for $0 \leq i \leq p, 0 \leq j \leq q$ and each s,t with $s+t=k$. From (1) we have

$$0 = \sum_{s+t=k} \left(\sum_{i=0}^{p} a_s^i y^i\right)\left(\sum_{j=0}^{q} b_t^j y^j\right) = \sum_{s+t=k}\sum_{l=0}^{p+q}\left(\sum_{i+j=l} a_s^i b_t^j\right) y^l$$

$$= \sum_{l=0}^{p+q}\left(\sum_{s+t=k}\sum_{i+j=l} a_s^i b_t^j\right) y^l = \sum_{l=0}^{p+q}\left(\sum_{i+j=l}\sum_{s+t=k} a_s^i b_t^j\right) y^l.$$

Thus

$$\sum_{s+t=k} a_s^0 b_t^0 = 0, \sum_{s+t=k} a_s^0 b_t^1 + \sum_{s+t=k} a_s^1 b_t^0 = 0, ...,$$

$$\sum_{s+t=k} a_s^0 b_t^{p+q} + \sum_{s+t=k} a_s^1 b_t^{p+q-1} + \cdots + \sum_{s+t=k} a_s^{p+q} b_t^0 = 0.$$

If $s \geq 1$, then $k-s < k$. Thus by induction hypothesis, $a_0^0 b_{k-s}^0 \in ni\ell(R)$ and so $b_{k-s}^0 a_0^0 \in ni\ell(R)$. Hence $a_1^0 b_{k-1}^0 a_0^0 + a_2^0 b_{k-2}^0 a_0^0 + \cdots + a_k^0 b_0^0 a_0^0 \in ni\ell(R)$, since R is nil-semicommutative. Therefore, if we multiply $\sum_{s+t=k} a_s^0 b_t^0 = 0$ on the right side by a_0^0, then it follows that $a_0^0 b_k^0 a_0^0 \in ni\ell(R)$ and so $a_0^0 b_k^0 \in ni\ell(R)$. If we multiply $\sum_{s+t=k} a_s^0 b_t^0 = 0$ on the right side by a_1^0, then $a_1^0 b_{k-1}^0 a_1^0 = -a_0^0 b_k^0 a_1^0 - (a_2^0 b_{k-2}^0 a_1^0 + \cdots + a_k^0 b_0^0 a_1^0) = -(a_0^0 b_k^0) a_1^0 - (a_2^0 (b_{k-2}^0 a_1^0) + \cdots + a_k^0 (b_0^0 a_1^0)) \in ni\ell(R)$, since R is nil-semicommutative. Thus $a_1^0 b_{k-1}^0 \in ni\ell(R)$. Similarly, we can show that $a_2^0 b_{k-2}^0 \in ni\ell(R), ..., a_k^0 b_0^0 \in ni\ell(R)$. So we have shown that $a_s^i b_t^j \in ni\ell(R)$ for each s,t with $s+t=k$ and i,j with $i+j=0$. Suppose that $l \leq p+q$ is such that $a_s^i b_t^j \in ni\ell(R)$ for each s,t with $s+t=k$ and i,j with $i+j<l$. We will show that $a_s^i b_t^j \in ni\ell(R)$ for each s,t with $s+t=k$ and each i,j with $i+j=l$. If $t<k$ then by induction hypothesis, $a_0^0 b_t^j \in ni\ell(R)$ thus $b_t^j a_0^0 \in ni\ell(R)$. If $i \geq 1$, then $l-i < l$. Thus by induction hypothesis on $p+q$, $a_0^0 b_k^{l-i} \in ni\ell(R)$ for each $i \geq 1$, which implies $b_k^{l-i} a_0^0 \in ni\ell(R)$. Multiplying

$$\sum_{s+t=k} a_s^0 b_t^l + \sum_{s+t=k} a_s^1 b_t^{l-1} + \cdots + \sum_{s+t=k} a_s^l b_t^0 = 0$$

on the right side by a_0^0 we have $a_0^0 b_k^l a_0^0 \in ni\ell(R)$, so $a_0^0 b_k^l \in ni\ell(R)$. Similarly we can show that $a_s^i b_t^j \in ni\ell(R)$ for each s,t with $s+t=k$ and i,j with $i+j=l$. Therefore, by induction we have $a_s^i b_t^j \in ni\ell(R)$ for each $0 \leq i \leq p, 0 \leq j \leq q$, and s,t with $s+t = 0,1,2,...,n-1$. We have also

$$f_i g_j = \left(\sum_{s=0}^{n-1} a_s^i u^s\right)\left(\sum_{t=0}^{n-1} b_t^j u^t\right) = \sum_{k=0}^{2n-2}\left(\sum_{s+t=k} a_s^i b_t^j\right) u^k = \sum_{k=0}^{n-1}\left(\sum_{s+t=k} a_s^i b_t^j\right) u^k.$$

Since R is nil-semicommutative, by Theorem 2.5, $\sum_{s+t=k} a_s^i b_t^j \in ni\ell(R)$. Thus by Theorem 3.3, $f_i g_j \in ni\ell(R[u])$. This shows that $R[u]$ is weak Armendariz and the result follows. □

Corollary 3.10. [12, Theorem 3.9] *If R is a semicommutative ring, then $R[x]/(x^n)$ is a weak Armendariz ring, for each positive integer n.*

In [12, Theorem 3.6], Liu and Zhao, proved that, for a ring R, if R/I is weak Armendariz for some ideal I of R and I is semicommutative, then R is weak Armendariz. By a similar proof we can extend Liu and Zhao's result and obtain:

Proposition 3.11. *For a ring R suppose that R/I is weak Armendariz for some ideal I of R. If I is nil-semicommutative, then R is weak Armendariz.*

Proof. It is similar to the proof of [12, Theorem 3.6]. □

According to L. Ouyang [13], a ring R is called *weak α-skew Armendariz* if whenever polynomials $p = \sum_{i=0}^{m} a_i x^i$, and $q = \sum_{j=0}^{n} b_j x^j$ in $R[x;\alpha]$ satisfy $pq = 0$, then $a_i \alpha^i(b_j)$ is a nilpotent element of R for each $0 \le i \le m, 0 \le j \le n$.

A ring R is said to be *weak α-rigid* if $a\alpha(a) \in ni\ell(R) \Leftrightarrow a \in ni\ell(R)$.

Proposition 3.12. [13, Proposition 2.3] *Let R be a weak α-rigid ring and $ni\ell(R)$ be an ideal of R. Then we have the following:*

(1) If $ab \in ni\ell(R)$, then $a\alpha^m(b) \in ni\ell(R), \alpha^n(a)b \in ni\ell(R)$ for positive integers m and n.

(2) If $\alpha^k(a)b \in ni\ell(R)$ for some positive integer k, then $ab, ba \in ni\ell(R)$.

(3) If $a\alpha^t(b) \in ni\ell(R)$ for some positive integer t, then $ab, ba \in ni\ell(R)$.

In [13, Theorem 3.3], L. Ouyang proved that, if R is a weak α-rigid ring with $ni\ell(R)$ an ideal of R, then R is a weak α-skew Armendariz ring. L. Ouyang in [13, Theorem 3.9] proved that if R is a weak α-rigid and semicommutative ring, then $R[x]$ is a weak α-skew Armendariz ring. By a similar proof we can extend it to the following more general result. We notice that in all the examples 2.2, 2.3 and 2.4 if we take R any α-rigid ring, then by [13, Theorem 3.1] and the fact that each subring of a nil-semicommutative ring is nil-semicommutative, there are various examples of weak α-rigid nil-semicommutative rings that are not semicommutative. In [7], the authors introduced α-compatible rings and studied its properties. A ring R is *α-compatible* if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\alpha(b) = 0$. In this case, clearly the endomorphism α is injective. Also by [7, Lemma 2.2], a ring R is α-rigid if and only if R is α-compatible and reduced.

Theorem 3.13. *If R is a weak α-rigid and nil-semicommutative ring, then $R[x]$ is a weak α-skew Armendariz ring.*

Proof. Let $f = f_0 + f_1 y + \cdots + f_p y^p \in R[x][y;\alpha]$ and $g = g_0 + g_1 y + \cdots + g_q y^q \in R[x][y;\alpha]$ be such that $fg = 0$. Suppose that $f_i = \sum_{s=0}^{m_i} a_s^i x^s$. Let $m = Max\{m_i\}, i = 0, 1, \ldots, p$. Then each f_i can be written in the form of $f_i = \sum_{s=0}^{m} a_s^i x^s$. By [13, Proposition 2.4], $\alpha(1) = 1$, and we have $xy = yx$ and $xa = ax$ for each $a \in R$. Thus

$$f = \sum_{i=0}^{p}(\sum_{s=0}^{m} a_s^i x^s) y^i = \sum_{s=0}^{m}(\sum_{i=0}^{p} a_s^i y^i) x^s.$$

Similarly, each g_j can be written in the form of $g_j = \sum_{t=0}^{n} b_t^j x^t$, and thus

$$g = \sum_{j=0}^{q}(\sum_{t=0}^{n} b_t^j x^t) y^j = \sum_{t=0}^{n}(\sum_{j=0}^{q} b_t^j y^j) x^t.$$

From $fg = 0$, we have the following equation:

$$\sum_{s+t=k}(\sum_{i=0}^{p} a_s^i y^i)(\sum_{j=0}^{q} b_t^j y^j) = 0, \quad k = 0, 1, \ldots, m+n. \tag{2}$$

We will show by induction on $s+t$ that $a_s^i \alpha^i(b_t^j) \in nil(R)$ for each $0 \le i \le p$, and $0 \le j \le q$ and each s, t with $s+t = 0, 1, \ldots, m+n$. If $s+t = 0$, then $s = t = 0$. Thus $(\sum_{i=0}^{p} a_0^i y^i)(\sum_{j=0}^{q} b_0^j y^j) = 0$. Since R is nil-semicommutative, $nil(R)$ is an ideal of R by Theorem 2.5. Thus R is weak α-skew Armendariz by [13, Theorem 3.3]. So $a_0^i \alpha^i(b_0^j) \in nil(R)$ for each $0 \le i \le p$, and each $0 \le j \le q$. Now suppose that $k \le m+n$ is such that $a_s^i \alpha^i(b_t^j) \in nil(R)$ for each $0 \le i \le p$, and each $0 \le j \le q$, and each s, t with $s+t < k$. We will show that $a_s^i \alpha^i(b_t^j) \in nil(R)$ for each $0 \le i \le p$, and each $0 \le j \le q$, and each s, t with $s+t = k$. From (2), we have

$$\sum_{s+t=k}(\sum_{i=0}^{p} a_s^i y^i)(\sum_{j=0}^{q} b_t^j y^j) = \sum_{s+t=k}\sum_{l=0}^{p+q}(\sum_{i+j=l} a_s^i \alpha^i(b_t^j)) y^l$$

$$= \sum_{l=0}^{p+q}(\sum_{s+t=k}\sum_{i+j=l} a_s^i \alpha^i(b_t^j)) y^l = \sum_{l=0}^{p+q}(\sum_{i+j=l}\sum_{s+t=k} a_s^i \alpha^i(b_t^j)) y^l.$$

Thus

$$\sum_{s+t=k} a_s^0 b_t^0 = 0;$$

$$\sum_{s+t=k} a_s^0 b_t^1 + \sum_{s+t=k} a_s^1 \alpha(b_t^0) = 0;$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sum_{s+t=k} a_s^0 b_t^l + \sum_{s+t=k} a_s^1 \alpha(b_t^{l-1}) + \cdots + \sum_{s+t=k} a_s^l \alpha^l(b_t^0) = 0;$$

$$\sum_{s+t=k} a_s^p \alpha^p(b_t^q) = 0.$$

If $s < k$, then by the induction hypothesis, $a_s^0 b_0^0 \in ni\ell(R)$ and so $b_0^0 a_s^0 \in ni\ell(R)$ for $s < k$. Hence $b_0^0 a_0^0 b_k^0 + b_0^0 a_1^0 b_{k-1}^0 + \cdots + b_0^0 a_{k-1}^0 b_1^0 \in ni\ell(R)$, since R is nil-semicommutative. Therefore, if we multiply $\sum_{s+t=k} a_s^0 b_t^0 = 0$ on the left side by b_0^0, then it follows that $b_0^0 a_k^0 b_0^0 \in ni\ell(R)$, and so $b_0^0 a_k^0 \in ni\ell(R)$ and $a_k^0 b_0^0 \in ni\ell(R)$. If we multiply $\sum_{s+t=k} a_s^0 b_t^0 = 0$ on the left side by b_1^0, then $b_1^0 a_{k-1}^0 b_1^0 = (b_1^0 a_0^0 b_k^0 + b_1^0 a_1^0 b_{k-1}^0 + \cdots + b_1^0 a_{k-2}^0 b_2^0) - b_1^0 a_k^0 b_0^0 = -(b_1^0 a_0^0) b_k^0 - (b_1^0 a_1^0) b_{k-1}^0 - \cdots - (b_1^0 a_{k-2}^0) b_2^0 - b_1^0 (a_k^0 b_0^0) \in ni\ell(R)$, since R is nil-semicommutative. Thus $a_{k-1}^0 b_1^0 \in ni\ell(R)$. Similarly, we can show that $a_{k-2}^0 b_2^0 \in ni\ell(R), \cdots, a_0^0 b_k^0 \in ni\ell(R)$. So we show that $a_s^i \alpha^i(b_t^j) \in ni\ell(R)$ for each s, t with $s+t = k$ and each i, j with $i+j = 0$. Suppose that $l \leq p+q$ is such that $a_s^i \alpha^i(b_t^j) \in ni\ell(R)$ for each s, t with $s+t = k$ and each i, j with $i+j < l$. We will show that $a_s^i \alpha^i(b_t^j) \in ni\ell(R)$ for each s, t with $s+t = k$ and each i, j with $i+j = l$. If $s < k$, then by the induction hypothesis, $a_s^i \alpha^i(b_0^0) \in ni\ell(R)$. Thus $a_s^i b_0^0 \in ni\ell(R)$ by Proposition 3.12, and so $b_0^0 a_s^i \in ni\ell(R)$. If $i < l$, then by the induction hypothesis on l, $a_k^i \alpha^i(b_0^0) \in ni\ell(R)$ for each $i < l$, which implies $a_k^i b_0^0 \in ni\ell(R)$ and so $b_0^0 a_k^i \in ni\ell(R)$ for each $i < l$. Multiplying $\sum_{s+t=k} a_s^0 b_t^l + \sum_{s+t=k} a_s^1 \alpha(b_t^{l-1}) + \cdots + \sum_{s+t=k} a_s^l \alpha^l(b_t^0) = 0$ on the left side by b_0^0, we have $b_0^0 a_k^l \alpha^l(b_0^0) \in ni\ell(R)$, since $ni\ell(R)$ is an ideal of R by Theorem 2.5, Thus $b_0^0 a_k^l \alpha^l(b_0^0) \alpha^l(a_k^l) = b_0^0 a_k^l \alpha^l(b_0^0 a_k^l) \in ni\ell(R)$. Thus $b_0^0 a_k^l \in ni\ell(R)$ which implies $a_k^l b_0^0 \in ni\ell(R)$ and so $a_k^l \alpha^l(b_0^0) \in ni\ell(R)$ by Proposition 3.12, Similarly, we can show that $a_s^i \alpha^i(b_t^j) \in ni\ell(R)$ for each s, t with $s+t = k$ and each i, j with $i+j = l$. Therefore, by induction, we have $a_s^i \alpha^i(b_t^j) \in ni\ell(R)$ for each $0 \leq i \leq p$, and $0 \leq j \leq q$ and each s, t with $s+t = 0, 1, \ldots, m+n$. We have

$$f_i \alpha^i(g_j) = \sum_{s=0}^{m} a_s^i x^s \alpha^i(\sum_{t=0}^{n} b_t^j x^t) = \sum_{s=0}^{m} a_s^i x^s) \sum_{t=0}^{n} \alpha^i(b_t^j) x^t) = \sum_{k=0}^{m+n} (\sum_{s+t=k} a_s^i \alpha^i(b_t^j)) x^k.$$

Since R is nil-semicommutative, by Theorem 2.5, $\sum_{s+t=k} a_s^i \alpha^i(b_t^j) \in ni\ell(R)$. Thus by Theorem 3.3, $f_i \alpha^i(g_j) \in ni\ell(R[x])$. Therefore $R[x]$ is weak α-skew Armendariz. □

Corollary 3.14. [13, Theorem 3.9]*If R is a weak α-rigid and semicommutative ring, then $R[x]$ is a weak α-skew Armendariz ring.*

Corollary 3.15. *Let R be a weak α-rigid nil-semicommutative ring. Then $R[x]/\langle x^n \rangle$ is a weak α-skew Armendariz ring, where $\langle x^n \rangle$ is the ideal of $R[x]$ generated by x^n.*

Proof. It is similar to the proof of [13, Corollary 3.10]. □

Corollary 3.16. [13, Corollary 3.10] *Let R be a weak α-rigid semicommutative ring. Then $R[x]/\langle x^n\rangle$ is a weak α-skew Armendariz ring, where $\langle x^n\rangle$ is the ideal of $R[x]$ generated by x^n.*

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