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GENERALIZED SECOND CHANGE OF RINGS THEOREMS FOR HOMOLOGICAL DIMENSIONS

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Abstract. The goal of this paper is to generalize the second change of rings theorem for the injective dimension and the projective dimension.

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1. Introduction

Throughout this paper, R denotes an associative ring with identity element. All modules, if not otherwise specified, are assumed to be left R-modules. If x is a central element of R, when no confusion is likely, R^* denotes the factor ring $\dfrac{R}{xR}$ and, for any R-module A, $Z(A)$ denotes the set of all zero-divisors of A.

Recall the following second change of rings theorems relative to the injective dimension and the projective dimension:

Theorem A [1, Theorem 205]**.** *Let x be a central non zero-divisor in R. If M is an R-module such that x is a non zero-divisor on M, then*

$$1 + \mathrm{id}_{R^*}\left(\frac{M}{xM}\right) \leq \mathrm{id}_R(M)$$

except when M is an injective R-module (in which case $M = xM$).

Theorem B [1, Exercise 1, page 155]**.** *Let x be a central element of R such that $x \notin Z(R)$. Let M be an R-module such that $M = xM$. Then*

$$1 + \ \mathrm{pd}_{R^*}({}_xM) \leq \ \mathrm{pd}_R(M)$$

except when M is projective over R in which case ${}_xM = 0$.

The aim of this paper is to generalize Theorem A and Theorem B dropping the hypothesis "$x \notin Z(M)$" and involving instead the submodule ${}_xM := \{z \in M : xz = 0\}$ of M annihilated by x. Our generalized second change of rings theorem for the injective dimension, Theorem 2.5, extends Theorem A by stating the following:

Let x be a central element of R such that $x \notin Z(R)$. Let M be an R-module which is not an injective R-module. Then

$$1 + \mathrm{id}_{R^*}\left(\frac{M}{xM}\right) \leq \mathrm{id}_R(M) \text{ if and only if } \mathrm{id}_{R^*}({}_xM) - 1 \leq \mathrm{id}_R(M).$$

As for the projective dimension, via Theorem 2.7, we prove the following general version of Theorem B: *Let x be a central element of R such that $x \notin Z(R)$. Let M be an R-module which is not projective over R. Then*

$$1 + \mathrm{pd}_{R^*}({}_xM) \leq \ \mathrm{pd}_R(M) \textit{ if and only if } \mathrm{pd}_{R^*}\left(\frac{M}{xM}\right) - 1 \leq \ \mathrm{pd}_R(M).$$

2. Main results

This section aims at giving a general version of the second change of rings theorem for the homological dimensions, that is, Theorem A and Theorem B.

First, we establish the following results which will be useful in the sequel. Recall that for a central element x of R, R^* denotes the factor ring $\dfrac{R}{xR}$.

Lemma 2.1. *Let M be an R-module and x a central element of R such that $x \notin Z(R)$. Then $Hom_R(R^*, M) \cong {}_xM$, $Ext^1_R(R^*, M) \cong \dfrac{M}{xM}$ and $Tor^R_1(R^*, M) \cong {}_xM$.*

Proof. As $x \notin Z(R)$, the following sequence is exact $0 \longrightarrow R \xrightarrow{x} R \longrightarrow R^* \longrightarrow 0$. Applying the functor $\mathrm{Hom}_R(-, M)$, we get the next exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(R^*, M) \longrightarrow \mathrm{Hom}_R(R, M) \xrightarrow{x} \mathrm{Hom}_R(R, M) \longrightarrow \mathrm{Ext}^1_R(R^*, M) \longrightarrow 0.$$

Since $\mathrm{Hom}_R(R, M) \cong M$, this latter sequence turns out to be the following exact one

$$0 \longrightarrow \mathrm{Hom}_R(R^*, M) \longrightarrow M \xrightarrow{x} M \longrightarrow \mathrm{Ext}^1_R(R^*, M) \longrightarrow 0.$$

Then the first two isomorphisms easily follows. Applying the functor $\otimes_R M$ instead of $\mathrm{Hom}_R(-, M)$ to the initial exact sequence yields the last isomorphism. $\square$

Lemma 2.2. *Let x be a central element of R. Let $0 \longrightarrow N \xrightarrow{i} P \xrightarrow{\alpha} M \longrightarrow 0$ be an exact sequence of R-modules such that P is a projective R-module. Then the natural sequence of R^*-modules*

$$0 \longrightarrow {}_xM \longrightarrow \frac{N}{xN} \xrightarrow{\bar{i}} \frac{P}{xP} \xrightarrow{\bar{\alpha}} \frac{M}{xM} \longrightarrow 0$$

is exact with $\dfrac{P}{xP}$ is a projective R^*-module.

Proof. Tensoring with R^* the sequence $0 \longrightarrow N \xrightarrow{i} P \xrightarrow{\alpha} M \longrightarrow 0$ yields the exact sequence

$$0 \longrightarrow \operatorname{Tor}_1^R(R^*, M) \longrightarrow \frac{N}{xM} \xrightarrow{\bar{i}} \frac{P}{xP} \xrightarrow{\overline{\alpha}} \frac{M}{xM} \longrightarrow 0.$$

As, by Lemma 2.1, $\operatorname{Tor}_1^R(R^*, M) \cong {}_xM$, we get the desired exact sequence. Moreover, as P is R-projective, $\dfrac{P}{xP} \cong R^* \otimes_R P$ is an R^*-projective module. The proof is complete. □

Lemma 2.3. *Let x be a central element of R such that $x \notin Z(R)$. Let $0 \longrightarrow M \xrightarrow{i} I \xrightarrow{\alpha} N \longrightarrow 0$ be an exact sequence of R-modules such that I is an injective R-module. Then the natural sequence of R^*-modules*

$$0 \longrightarrow {}_xM \xrightarrow{{}_xi} {}_xI \xrightarrow{{}_x\alpha} {}_xN \longrightarrow \frac{M}{xM} \longrightarrow 0$$

is exact with ${}_xI$ is an injective R^*-module.

Proof. Applying the functor $\operatorname{Hom}_R(R^*, -)$ to the sequence $0 \longrightarrow M \xrightarrow{i} I \xrightarrow{\alpha} N \longrightarrow 0$, we get the exact sequence

$$0 \longrightarrow {}_xM \xrightarrow{{}_xi} {}_xI \xrightarrow{{}_x\alpha} {}_xN \longrightarrow \operatorname{Ext}_R^1(R^*, M).$$

As, by Lemma 2.1, $\operatorname{Ext}_R^1(R^*, M) \cong \dfrac{M}{xM}$, we obtain the desired exact sequence. Moreover, as I is R-injective, by [2, Theorem 3.44], ${}_xI := \operatorname{Hom}_R(R^*, I)$ is an injective R^*-module completing the proof. □

Next, through Theorem 2.4 and Theorem 2.5, we generalize the second change of rings theorem for the injective dimension, that is Theorem A. First, notice that if M is an injective R-module and x is a central element of R such that $x \notin Z(R)$, then $M = xM$ and ${}_xM$ is injective over R^*.

Theorem 2.4. *Let x be a central element of R such that $x \notin Z(R)$. Let M be an R-module which is not injective over R. Then*
1) $1 + id_{R^*}\left(\dfrac{M}{xM}\right) \leq max\{id_{R^*}({}_xM) - 1,\ id_R(M)\}$.
2) If $id_R(M) < +\infty$, then $id_{R^*}\left(\dfrac{M}{xM}\right)$ and $id_{R^*}({}_xM)$ are simultaneously finite.
3) Assume that $id_{R^*}({}_xM) > id_R(M)$. Then $id_{R^*}({}_xM) = 2 + id_{R^*}\left(\dfrac{M}{xM}\right)$.

Proof. 1) (and (2)) If $\operatorname{id}_R(M) = +\infty$, then we are done. Assume that $1 \leq \operatorname{id}_R(M) = n < +\infty$. Let $0 \longrightarrow M \longrightarrow I \longrightarrow A \longrightarrow 0$ $(*)$ be an exact sequence of R-modules such that I is injective over R. Note that, being a quotient of the injective R-module I, A is a divisible R-module, so that, as $x \notin Z(R)$, $A = xA$. So, by [1, Theorem 204], $\operatorname{id}_{R^*}({}_xA) \leq \operatorname{id}_R(A) = n - 1$. On the other hand, applying the functor $\mathrm{Hom}_R(R^*, -)$ to the sequence $(*)$ yields, by Lemma 2.3, the exact sequence of R^*-modules

$$0 \longrightarrow {}_xM \longrightarrow {}_xI \longrightarrow {}_xA \longrightarrow \frac{M}{xM} \longrightarrow 0.$$

Let $K = \mathrm{Im}({}_xI \to {}_xA)$. Then we have the next two exact sequences of R^*-modules

$$\begin{cases} 0 \longrightarrow {}_xM \longrightarrow {}_xI \longrightarrow K \longrightarrow 0 \ (**) \\ \\ 0 \longrightarrow K \longrightarrow {}_xA \longrightarrow \dfrac{M}{xM} \longrightarrow 0 \ (***). \end{cases}$$

It is then easy to see that $\mathrm{id}_{R^*}\Big(\dfrac{M}{xM}\Big) < +\infty$ if and only if $\mathrm{id}_{R^*}(K) < +\infty$ if and only if $\mathrm{id}_{R^*}({}_xM) < +\infty$ establishing (2). If ${}_xM$ is injective over R^*, then K is injective over R^*, so that, ${}_xA \cong K \oplus \dfrac{M}{xM}$ and thus $\mathrm{id}_{R^*}\Big(\dfrac{M}{xM}\Big) = \mathrm{id}_{R^*}({}_xA) \leq n-1$, as claimed. Next, suppose that $\mathrm{id}_{R^*}({}_xM) \geq 1$. Consider the following portion of the long exact sequence associated to the sequence $(***)$

$$\mathrm{Ext}^j_{R^*}(B, {}_xA) \longrightarrow \mathrm{Ext}^j_{R^*}\Big(B, \frac{M}{xM}\Big) \longrightarrow \mathrm{Ext}^{j+1}_{R^*}(B, K)$$

for each positive integer j and each R^*-module B. Then $\mathrm{id}_{R^*}\Big(\dfrac{M}{xM}\Big) \leq \max\Big\{\mathrm{id}_{R^*}(K) - 1, \mathrm{id}_{R^*}({}_xA)\Big\}$. Moreover, by the sequence $(**)$, as $\mathrm{id}_{R^*}({}_xM) \geq 1$, we get $\mathrm{id}_{R^*}({}_xM) = 1 + \mathrm{id}_{R^*}(K)$. It follows that

$$\mathrm{id}_{R^*}\Big(\frac{M}{xM}\Big) \leq \max\{\mathrm{id}_{R^*}({}_xM) - 2, n-1\}, \text{ as desired.}$$

3) Assume that $\mathrm{id}_{R^*}({}_xM) > \mathrm{id}_R(M)$. Then, applying (1), $1 + \mathrm{id}_{R^*}\Big(\dfrac{M}{xM}\Big) \leq \mathrm{id}_{R^*}({}_xM) - 1$, that is, $2 + \mathrm{id}_{R^*}\Big(\dfrac{M}{xM}\Big) \leq \mathrm{id}_{R^*}({}_xM)$. On the other hand, consider the above two exact sequences of R^*-modules

$$\begin{cases} 0 \longrightarrow {}_xM \longrightarrow {}_xI \longrightarrow K \longrightarrow 0 \ (**) \\ \\ 0 \longrightarrow K \longrightarrow {}_xA \longrightarrow \dfrac{M}{xM} \longrightarrow 0 \ (***). \end{cases}$$

In view of the following portion of the long exact sequence associated to the sequence $(***)$

$$\mathrm{Ext}^j_{R^*}\Big(B, \frac{M}{xM}\Big) \longrightarrow \mathrm{Ext}^{j+1}_{R^*}(B, K) \longrightarrow \mathrm{Ext}^{j+1}_{R^*}(B, {}_xA)$$

for each positive integer j and each R^*-module B, we get

$$\mathrm{id}_{R^*}(K) \leq \max\Big\{1 + \mathrm{id}_{R^*}\Big(\frac{M}{xM}\Big), \ \mathrm{id}_{R^*}({}_xA)\Big\}.$$

It follows, using the sequence $(**)$, that

$$\begin{array}{rcl} \mathrm{id}_{R^*}({}_xM) & \leq & 1+\mathrm{id}_{R^*}(K) \\ & \leq & 1+\ \max\Big\{1+\mathrm{id}_{R^*}\Big(\dfrac{M}{xM}\Big),\ \mathrm{id}_{R^*}({}_xA)\Big\} \\ & \leq & 1+\ \max\Big\{1+\mathrm{id}_{R^*}\Big(\dfrac{M}{xM}\Big),\ \mathrm{id}_R(M)-1\Big\}. \end{array}$$

As, by hypotheses, $\mathrm{id}_{R^*}({}_xM) > \mathrm{id}_R(M)$, we get $\mathrm{id}_{R^*}({}_xM) \leq 2+\mathrm{id}_{R^*}\Big(\dfrac{M}{xM}\Big)$ yielding the desired equality and completing the proof. □

Our next result represents the generalized version of Theorem A of the introduction.

Theorem 2.5. *Let x be a central element of R such that $x \notin Z(R)$. Let M be an R-module which is not an injective R-module. Then*

$$1+id_{R^*}\Big(\frac{M}{xM}\Big) \leq id_R(M) \text{ if and only if } id_{R^*}({}_xM)-1 \leq id_R(M).$$

Proof. If $\mathrm{id}_{R^*}({}_xM)-1 \leq \mathrm{id}_R(M)$, then, by Theorem 2.4(1), $1+\mathrm{id}_{R^*}\Big(\dfrac{M}{xM}\Big) \leq \mathrm{id}_R(M)$. Conversely, assume that $1+\mathrm{id}_{R^*}\Big(\dfrac{M}{xM}\Big) \leq \mathrm{id}_R(M)$. Let us resume the notation of the proof of Theorem 2.4 and, thus, considering the above two exact sequences of R^*-modules

$$\begin{cases} 0 \longrightarrow {}_xM \longrightarrow {}_xI \longrightarrow K \longrightarrow 0 \ (**) \\ \\ 0 \longrightarrow K \longrightarrow {}_xA \longrightarrow \dfrac{M}{xM} \longrightarrow 0 \ (***) \end{cases}$$

we show in the proof of Theorem 2.4(3) that

$$\mathrm{id}_{R^*}(K) \leq \ \max\Big\{1+\mathrm{id}_{R^*}\Big(\frac{M}{xM}\Big),\ \mathrm{id}_{R^*}({}_xA)\Big\}.$$

It follows that

$$\mathrm{id}_{R^*}(K) \leq \ \max\Big\{\mathrm{id}_R(M), \mathrm{id}_R(M)-1\Big\} = \mathrm{id}_R(M).$$

Hence, using the sequence $(**)$, we get $\mathrm{id}_{R^*}({}_xM) \leq 1+\mathrm{id}_{R^*}(K) \leq 1+\mathrm{id}_R(M)$, as desired. □

Next, through Theorem 2.6 and Theorem 2.7, we generalize the second change of rings theorem for the projective dimension, that is Theorem B. Note that if M is a projective R-module and x is a central element of R such that $x \notin Z(R)$, then $\dfrac{M}{xM}$ is projective over R^* and ${}_xM = 0$.

Theorem 2.6. *Let x be a central element of R such that $x \notin Z(R)$. Let M be an R-module. Then*
1) $pd_{R^*}\Big(\dfrac{M}{xM}\Big) \leq max\{2 + pd_{R^*}({}_xM),\ pd_R(M)\}$.
2) If $pd_R(M) < +\infty$, then $pd_{R^*}\Big(\dfrac{M}{xM}\Big)$ and $pd_{R^*}({}_xM)$ are simultaneously finite.
3) If $1+pd_{R^*}({}_xM) > pd_R(M)$ and M is not projective over R, then $pd_{R^*}\Big(\dfrac{M}{xM}\Big) = 2+pd_{R^*}({}_xM)$.

Proof. 1) (and (2)) If $\mathrm{pd}_R(M) = +\infty$, then we are done. Also, if M is projective over R, then $\dfrac{M}{xM}$ is projective over R^* and ${}_xM = 0$. Then, assume that $1 \leq \mathrm{pd}_R(M) = n < +\infty$. Let $0 \longrightarrow A \longrightarrow P \longrightarrow M \longrightarrow 0$ $(*)$ be an exact sequence of R-modules such that P is projective over R. Note that as A is a submodule of the projective module P and $x \notin Z(R)$, $x \notin Z(A)$, that is, ${}_xA = 0$. So, by [3, Theorem 4.3.5], $\mathrm{pd}_{R^*}\Big(\dfrac{A}{xA}\Big) \leq \mathrm{pd}_R(A) \leq n-1$. On the other hand, tensoring the sequence $(*)$ with R^* yields, by Lemma 2.2, the exact sequence of R^*-modules

$$0 \longrightarrow {}_xM \longrightarrow \frac{A}{xA} \longrightarrow \frac{P}{xP} \longrightarrow \frac{M}{xM} \longrightarrow 0.$$

Now, let $H = \mathrm{Im}\Big(\dfrac{A}{xA} \to \dfrac{P}{xP}\Big)$. We have the next two exact sequences of R^*-modules

$$\begin{cases} 0 \longrightarrow {}_xM \longrightarrow \dfrac{A}{xA} \longrightarrow H \longrightarrow 0 \ (**) \\ \\ 0 \longrightarrow H \longrightarrow \dfrac{P}{xP} \longrightarrow \dfrac{M}{xM} \longrightarrow 0 \ (***). \end{cases}$$

It is then clear that $\mathrm{pd}_{R^*}\Big(\dfrac{M}{xM}\Big) < +\infty$ if and only if $\mathrm{pd}_{R^*}(H) < +\infty$ if and only if $\mathrm{pd}_{R^*}({}_xM) < +\infty$ establishing (2). Consider the following portion of the long exact sequence associated to the sequence $(**)$

$$\mathrm{Ext}^j_{R^*}({}_xM, B) \longrightarrow Ext^{j+1}_{R^*}(H, B) \longrightarrow \mathrm{Ext}^{j+1}_{R^*}(\frac{A}{xA}, B)$$

for each integer $j \geq 0$ and each R^*-module B. Then $\mathrm{pd}_{R^*}(H) \leq \max\Big\{1+\mathrm{pd}_{R^*}({}_xM), \mathrm{pd}_{R^*}\Big(\dfrac{A}{xA}\Big)\Big\}$. Moreover, by the sequence $(***)$, we have $\mathrm{pd}_{R^*}\Big(\dfrac{M}{xM}\Big) \leq 1+\mathrm{pd}_{R^*}(H)$. It follows that

$$\begin{aligned} \mathrm{pd}_{R^*}\Big(\frac{M}{xM}\Big) &\leq 1 + \max\{1 + \mathrm{pd}_{R^*}({}_xM), n-1\} \\ &= \max\{2 + \mathrm{pd}_{R^*}({}_xM),\ \mathrm{pd}_R(M)\}, \text{ as desired.} \end{aligned}$$

3) Assume that $1+\mathrm{pd}_{R^*}({}_xM) > \mathrm{pd}_R(M) \geq 1$. Then, by (1), $\mathrm{pd}_{R^*}\Big(\dfrac{M}{xM}\Big) \leq 2+\mathrm{pd}_{R^*}({}_xM)$. Conversely, proceeding as in (1), consider the above-mentioned two

exact sequences

$$\begin{cases} 0 \longrightarrow {}_xM \longrightarrow \dfrac{A}{xA} \longrightarrow H \longrightarrow 0 \ (**) \\ \\ 0 \longrightarrow H \longrightarrow \dfrac{P}{xP} \longrightarrow \dfrac{M}{xM} \longrightarrow 0 \ (***). \end{cases}$$

If $\dfrac{M}{xM}$ is projective over R^*, then H is projective over R^*, and thus $\mathrm{pd}_{R^*}({}_xM) = \mathrm{pd}_{R^*}\Big(\dfrac{A}{xA}\Big) \leq \mathrm{pd}_R(M) - 1$ which is contradictory to our initial assumption. Then $\mathrm{pd}_{R^*}\Big(\dfrac{M}{xM}\Big) \geq 1$, so that $\mathrm{pd}_{R^*}(H) = \mathrm{pd}_{R^*}\Big(\dfrac{M}{xM}\Big) - 1$. Consider the following portion of the long exact sequence associated to the sequence $(**)$

$$\mathrm{Ext}^j_{R^*}\Big(\frac{A}{xA}, B\Big) \longrightarrow \mathrm{Ext}^j_{R^*}({}_xM, B) \longrightarrow \mathrm{Ext}^{j+1}_{R^*}(H, B)$$

for each integer $j \geq 0$ and each R^*-module B. Hence $\mathrm{pd}_{R^*}({}_xM) \leq \max\Big\{\mathrm{pd}_{R^*}\Big(\dfrac{A}{xA}\Big), \mathrm{pd}_{R^*}(H) - 1\Big\} \leq \max\Big\{\mathrm{pd}_R(M) - 1, \mathrm{pd}_{R^*}\Big(\dfrac{M}{xM}\Big) - 2\Big\}$. This ensures, as $1+\mathrm{pd}_{R^*}({}_xM) > \mathrm{pd}_R(M)$, that $\mathrm{pd}_{R^*}({}_xM) \leq \mathrm{pd}_{R^*}\Big(\dfrac{M}{xM}\Big) - 2$. It follows that $\mathrm{pd}_{R^*}\Big(\dfrac{M}{xM}\Big) = 2+\mathrm{pd}_{R^*}({}_xM)$ establishing (3) and completing the proof. □

The following stands for the dual result of Theorem 2.5 and it represents the generalized version of Theorem B of the introduction.

Theorem 2.7. *Let x be a central element of R such that $x \notin Z(R)$. Let M be an R-module which is not projective over R. Then*

$$1 + pd_{R^*}({}_xM) \leq \ pd_R(M) \text{ if and only if } pd_{R^*}\Big(\frac{M}{xM}\Big) - 1 \leq \ pd_R(M).$$

Proof. Let $\mathrm{pd}_{R^*}\Big(\dfrac{M}{xM}\Big) - 1 \leq \mathrm{pd}_R(M)$. If $1+\mathrm{pd}_{R^*}({}_xM) > \mathrm{pd}_R(M)$, then, by Theorem 2.6(3),

$$\begin{aligned} \mathrm{pd}_{R^*}\Big(\frac{M}{xM}\Big) &= 2 + \mathrm{pd}_{R^*}({}_xM) \\ &> 1 + \mathrm{pd}_R(M) \text{ which is absurd.} \end{aligned}$$

It follows that $1+\mathrm{pd}_{R^*}({}_xM) \leq \mathrm{pd}_R(M)$. Conversely, assume that $1+\mathrm{pd}_{R^*}({}_xM) \leq \mathrm{pd}_R(M)$. Then, using Theorem 2.6(1),

$$\begin{aligned} \mathrm{pd}_{R^*}\Big(\frac{M}{xM}\Big) &\leq 1 + \max\{1 + \mathrm{pd}_{R^*}({}_xM),\ \mathrm{pd}_R(M)\} \\ &\leq 1 + \mathrm{pd}_R(M), \text{ as desired.} \end{aligned}$$

□

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References

[1] I. Kaplansky, Commutative Rings, The University of Chicago Press, Chicago, 1974.

[2] J. Rotman, An Introduction to Homological Algebra, in Pure and Appl. Math., Academic Press, 1979.

[3] C.A. Weibel, An Introduction to Homological Algebra, Cambridge University Press, Cambridge, Cambridge Studies in Advanced Mathematics, 38, 1994.

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