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MIYASHITA ACTION IN STRONGLY GROUPOID GRADED RINGS

Johan Öinert and Patrik Lundström

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Abstract. We determine the commutant of homogeneous subrings in strongly groupoid graded rings in terms of an action on the ring induced by the grading. Thereby we generalize a classical result of Miyashita from the group graded case to the groupoid graded situation. In the end of the article we exemplify this result. To this end, we show, by an explicit construction, that given a finite groupoid G, equipped with a nonidentity morphism $t : d(t) \to c(t)$, there is a strongly G-graded ring R with the properties that each R_s, for $s \in G$, is nonzero and R_t is a nonfree left $R_{c(t)}$-module.

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1. Introduction

Let R be an associative ring. If R is unital, then the identity element of R is denoted 1_R. We say that a subset R' of R is a subring of R if it is itself a ring under the binary operations of R; note that even if R and R' are unital it may happen that $1_{R'} \neq 1_R$. However, we always assume that ring homomorphisms $R \to R''$ between unital rings R and R'' map 1_R to $1_{R''}$. The group of ring automorphisms of R is denoted $\mathrm{Aut}(R)$.

By the *commutant* of a subset X of R, denoted $C_R(X)$, we mean the set of elements of R that commute with each element of X. If Y is another subset of R, then XY denotes the set of all finite sums of products xy, for $x \in X$ and $y \in Y$. The task of calculating $C_R(X)$ is in general a difficult problem. However, if R is strongly group graded and X belongs to a certain class of subrings of R, then, by a classical result of Miyashita [12] (see Theorem 1.1), there is an elegant solution to

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this problem formulated in terms of a group action defined by the grading. Namely, recall that R is said to be *graded* by the group G, or G-graded, if there is a set of additive subgroups, R_s, for $s \in G$, of R such that $R = \bigoplus_{s\in G} R_s$ and $R_s R_t \subseteq R_{st}$, for $s,t \in G$. If H is a subgroup of G, then we let R_H denote the subring $\bigoplus_{s\in H} R_s$ of R; in particular, R_e is a subring of R, where e denotes the identity element of G. If R is graded by G and $R_s R_t = R_{st}$, for $s,t \in G$, then R is said to be *strongly graded*. If in addition R is unital, then there is a unique group action $G \ni s \mapsto \sigma_s \in \mathrm{Aut}(C_R(R_e))$ of G on $C_R(R_e)$ satisfying $r_s x = \sigma_s(x) r_s$, for $s \in G$, $r_s \in R_s$ and $x \in C_R(R_e)$. Indeed, $\sigma_s(x) = \sum_{i=1}^n a_i x b_i$, for $x \in R_e$, where $a_i \in R_s$ and $b_i \in R_{s^{-1}}$ are chosen so that $\sum_{i=1}^n a_i b_i = 1_R$. If $H \subseteq G$ and $Y \subseteq C_R(R_e)$, then we let Y^H denote the set of $y \in Y$ which are fixed by all σ_s, for $s \in H$.

Theorem 1.1 (Miyashita [12]). *Let R be a unital ring strongly graded by the group G. If H is a subgroup of G, then $C_R(R_H) = C_R(R_e)^H$.*

In fact, Miyashita proves a more general statement concerning G-actions on module endomorphisms (see Theorems 2.12 and 2.13 in [12]). For more details concerning this and related results, see e.g. [1, Section I.2], [2, Theorem (2.1)], [14, Section 3.4] and [16]. For more details about group graded rings in general, see e.g. [13] or [14].

The purpose of this article is to generalize Theorem 1.1 from groups to groupoids (see Theorem 1.2). To be more precise, suppose that G is a small category, that is such that $\mathrm{mor}(G)$ is a set. The family of objects of G is denoted by $\mathrm{ob}(G)$; we will often identify an object in G with its associated identity morphism. The family of morphisms in G is denoted by $\mathrm{mor}(G)$; by abuse of notation, we will often write $s \in G$ when we mean $s \in \mathrm{mor}(G)$. The domain and codomain of a morphism s in G is denoted by $d(s)$ and $c(s)$ respectively. We let $G^{(2)}$ denote the collection of composable pairs of morphisms in G, that is all (s,t) in $\mathrm{mor}(G) \times \mathrm{mor}(G)$ satisfying $d(s) = c(t)$. For $e,f \in \mathrm{ob}(G)$, we let $G_{f,e}$ denote the collection of $s \in G$ with $c(s) = f$ and $d(s) = e$ and G_e denotes the monoid $G_{e,e}$. A category is called *cancellative* (a *groupoid*) if all its morphisms are both monomorphisms and epimorphisms (isomorphisms). A subcategory of a groupoid is said to be a *subgroupoid* if it is closed under inverses. For more details concerning categories in general and groupoids in particular, see e.g. [11] and [5] respectively. Let R be a ring. We say that a set of additive subgroups, R_s, for $s \in G$, of R is a *G-filter* in R if for all $s,t \in G$, we have $R_s R_t \subseteq R_{st}$ if $(s,t) \in G^{(2)}$ and $R_s R_t = \{0\}$ otherwise. We say that a G-filter is *strong* if $R_s R_t = R_{st}$ for $(s,t) \in G^{(2)}$. Furthermore, we say that the ring R is *graded* by the category G if there is a G-filter, R_s, for $s \in G$, in R

such that $R = \bigoplus_{s\in G} R_s$. If R is graded by a strong G-filter, then we say that it is *strongly graded*. Analogously to the group graded situation, if H is a subcategory of G, then we let R_H denote the subring $\bigoplus_{s\in H} R_s$ of R. We say that R is *locally unital* if for each $e \in \mathrm{ob}(G)$ the ring R_e is unital, making every R_s, for $s \in G$, a unital $R_{c(s)}$-$R_{d(s)}$-bimodule. For more details concerning category graded rings, see e.g. [8], [9], [10] and [15].

In Section 3, we show that if R is a ring which is strongly graded by a groupoid G, then for each $s \in G$ there is a ring isomorphism σ_s from $C_{R_{G_{d(s)}}}(R_{d(s)})$ to $C_{R_{G_{c(s)}}}(R_{c(s)})$ (see Definition 3.3) with properties similar to the ones in the group case above (see Proposition 3.4). In the end of Section 3, we use this fact to show the following result.

Theorem 1.2. *Let R be a locally unital ring strongly graded by the groupoid G, and let H be a subgroupoid of G. If F denotes the set of elements in R of the form $\sum_{e\in \mathrm{ob}(H)} x_e$, where $x_e \in C_{R_{G_e}}(R_e)$, for $e \in \mathrm{ob}(H)$, and $\sigma_s(x_{d(s)}) = x_{c(s)}$ for all $s \in H$, and T denotes $\bigoplus_{s\in G\ominus H} R_s$, where $G \ominus H$ is the set of morphisms $s \in G$ satisfying $d(s), c(s) \notin H$, then $C_R(R_H) = F \oplus T$.*

There is a well-developed theory for invertible bimodules of *unital* rings (see [1], [7] and [12]). However, in order to be able to generalize this theory to locally unital groupoid graded rings, and in particular in order to show Theorem 1.2, we need to extend the theory slightly (see Section 2). Namely, given unital subrings A and B of a (not necessarily unital) ring R we say that a unital A-B-submodule X of R is invertible if there is a unital B-A-submodule X^{-1} of R such that $XX^{-1} = A$ and $X^{-1}X = B$. The collection of invertible submodules of R forms a groupoid (see Definition 2.1 for the details).

In Section 4, we illustrate Theorem 1.1 and Theorem 1.2 in two cases (see Example 4.2). To this end, we make an explicit construction (see Proposition 4.1) of graded rings, which is inspired by [3]. A particular case of our construction implies the following result.

Theorem 1.3. *Given a finite groupoid G, equipped with a nonidentity morphism $t : d(t) \to c(t)$, there is a unital strongly G-graded ring R with the properties that each R_s, for $s \in G$, is nonzero and R_t is nonfree as a left $R_{c(t)}$-module.*

We find that Theorem 1.3 is interesting in its own right since, in general, every component R_s, for $s \in G$, of a strongly groupoid graded and locally unital ring R, is finitely generated and projective as a left $R_{c(s)}$-module (see Proposition 3.1(b)).

2. Miyashita Action

Throughout this section, let A, B, C, R and S be rings such that A, B and C are unital subrings of R. Furthermore, let M, N and P be R-S-bimodules; we let $\mathrm{Hom}_{R,S}(M,N)$ denote the collection of simultaneously left R-linear and right S-linear maps $M \to N$.

Definition 2.1. We say that a unital A-B-submodule X of R is *invertible* in R if there is a unital B-A-submodule X^{-1} of R such that $XX^{-1} = A$ and $X^{-1}X = B$. Let $\mathrm{Grd}(R)$ denote the groupoid having subrings of R as objects and invertible A-B-submodules X of R as morphisms, for subrings A and B of R; in that case we will write $X : B \to A$. If $Y : C \to B$ is an invertible B-C-submodule of R, then the composition of X and Y is defined as the A-C-submodule XY of R. The identity morphism $A \to A$ is A itself.

Proposition 2.2. *Every $X : B \to A$ in $\mathrm{Grd}(R)$ is finitely generated and projective both as a left A-module and a right B-module.*

Proof. By the assumptions $A = XX^{-1}$ and hence there is a positive integer n and $x_i \in X$ and $y_i \in X^{-1}$, for $i \in \{1,\ldots,n\}$, such that $1_A = \sum_{i=1}^{n} x_i y_i$. For each $i \in \{1,\ldots,n\}$ define a right B-linear $f_i : X \to B$ by $f_i(x) = y_i x$, for $x \in X$. If $x \in X$, then $x = 1_A x = \sum_{i=1}^{n} x_i y_i x = \sum_{i=1}^{n} x_i f_i(x)$. Hence, by the dual basis lemma (see e.g [6, p. 23]), we get that X is a projective right B-module generated by $x_1,\ldots,x_n$. Analogously, one can prove that X is a finitely generated projective left A-module. □

Proposition 2.3. *If $X : B \to A$ is in $\mathrm{Grd}(R)$ and $f \in \mathrm{Hom}_{B,S}(BM, BN)$, then there is a unique $f^X \in \mathrm{Hom}_{A,S}(AM, AN)$ satisfying*

$$f^X(xm) = xf(1_B m) \tag{1}$$

for all $x \in X$ and all $m \in M$. Moreover, the following properties hold:

(a) $0^X = 0$ *and* $\mathrm{id}_{BM}^X = \mathrm{id}_{AM}$*;*
(b) *if $g \in \mathrm{Hom}_{B,S}(BM, BN)$, then $(f+g)^X = f^X + g^X$;*
(c) *if $g \in \mathrm{Hom}_{B,S}(BN, BP)$, then $(g \circ f)^X = g^X \circ f^X$;*
(d) *if $g \in \mathrm{Hom}_{A,S}(AM, AN)$, then $g^A = g$;*
(e) *if $Y : C \to B$ in $\mathrm{Grd}(R)$ and $g \in \mathrm{Hom}_{C,S}(CM, CN)$, then $(g^Y)^X = g^{XY}$.*

Proof. Fix $X : B \to A$ in $\mathrm{Grd}(R)$ and $f \in \mathrm{Hom}_{B,S}(BM, BN)$. Since $1_A \in A = XX^{-1}$, there is a positive integer n and $x_i \in X$, $y_i \in X^{-1}$, for $i \in \{1,\ldots,n\}$, such that $\sum_{i=1}^n x_i y_i = 1_A$. If a map $f^X \in \mathrm{Hom}_{A,S}(AM, AN)$ satisfying (1) exists, then it is unique, since

$$f^X(am) = f^X(a1_A m) = f^X\left(a\sum_{i=1}^n x_i y_i m\right) =$$

$$= \sum_{i=1}^n f^X(a x_i y_i m) = a\sum_{i=1}^n x_i f(y_i m) \tag{2}$$

for all $a \in A$ and all $m \in M$; define $f^X(am)$ by the last part of (2). We must show that f^X does not depend on the choice of the x_i's and y_i's. To this end, suppose that p is a positive integer and $x_j' \in X$ and $y_j' \in X^{-1}$, for $j \in \{1, \ldots, p\}$, are chosen so that $\sum_{j=1}^p x_j' y_j' = 1_A$. Take $a \in A$ and $m \in M$. Then, since $y_i x_j' \in B$, we get that

$$a\sum_{j=1}^p x_j' f(y_j' m) = a\sum_{j=1}^p 1_A x_j' f(y_j' m) = a\sum_{j=1}^p\sum_{i=1}^n x_i y_i x_j' f(y_j' m) =$$

$$= a\sum_{j=1}^p\sum_{i=1}^n x_i f(y_i x_j' y_j' m) = a\sum_{i=1}^n\sum_{j=1}^p x_i f(y_i x_j' y_j' m) =$$

$$= a\sum_{i=1}^n x_i f\left(y_i \sum_{j=1}^p x_j' y_j' m\right) = a\sum_{i=1}^n x_i f(y_i 1_A m) = a\sum_{i=1}^n x_i f(y_i m).$$

Now we show that (1) holds. If $x \in X$ and $m \in M$, then, since $y_i x \in X^{-1}X = B$ for $i \in \{1, \ldots, n\}$, we get that

$$f^X(xm) = \sum_{i=1}^n x_i f(y_i x m) = \sum_{i=1}^n x_i f(y_i x 1_B m) =$$

$$= \sum_{i=1}^n x_i y_i x f(1_B m) = 1_A x f(1_B m) = x f(1_B m).$$

Next we show that $f^X \in \mathrm{Hom}_{A,S}(AM, AN)$. It is clear that f^X respects addition and right S-multiplication. Now we show that f^X respects left A-multiplication. To this end, suppose that $m \in M$ and $a, a' \in A$. Since $ax_i \in X$, for $i \in \{1, \ldots, n\}$, we get, by (1), that

$$f^X(aa'm) = f^X(a1_A a'm) = f^X\left(a\sum_{i=1}^n x_i y_i a'm\right) = \sum_{i=1}^n f^X(a x_i y_i a'm) =$$

$$= \sum_{i=1}^n a x_i f(1_B y_i a'm) = a\left(\sum_{i=1}^n x_i f(y_i a'm)\right) = af^X(1_A a'm) = af^X(a'm).$$

(a) and (b) follow immediately.

(c) It is clear that both $(g \circ f)^X$ and $g^X \circ f^X$ belong to $\mathrm{Hom}_{A,S}(AM, AP)$. Moreover, if $x \in X$ and $m \in M$, then we get that

$$(g^X \circ f^X)(xm) = g^X(f^X(xm)) = g^X(xf(1_B m)) =$$

$$= xg(f(1_B m)) = x(g \circ f)(1_B m).$$

By uniqueness of the map h^X in $\mathrm{Hom}_{A,S}(AM, AP)$ satisfying $h^X(xm) = xh(1_B m)$, for $x \in X$ and $m \in M$, it follows that $(g \circ f)^X = g^X \circ f^X$.

(d) This follows if we let $x_1 = y_1 = 1_A$ and $x_i = y_i = 0$, for $i \in \{2, \ldots, n\}$.

(e) Suppose that $g \in \mathrm{Hom}_{C,S}(CM, CN)$ and that $Y : C \to B$. Take a positive integer p and $x'_j \in Y$, $y'_j \in Y^{-1}$, for $j \in \{1, \ldots, p\}$, such that $\sum_{j=1}^{p} x'_j y'_j = 1_B$. If $a \in A$ and $m \in M$, then

$$(g^Y)^X(am) = a \sum_{i=1}^{n} x_i g^Y(y_i m) = a \sum_{i=1}^{n} \sum_{j=1}^{p} x_i x'_j g(y'_j y_i m) = g^{XY}(am)$$

since for each i and j we have $x_i x'_j \in XY$, $y'_j y_i \in Y^{-1} X^{-1} = (XY)^{-1}$ and

$$\sum_{i=1}^{n} \sum_{j=1}^{p} x_i x'_j y'_j y_i = \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{p} x'_j y'_j \right) y_i = \sum_{i=1}^{n} x_i 1_B y_i = \sum_{i=1}^{n} x_i y_i = 1_A.$$

$\square$

Definition 2.4. Suppose that G and H are categories. Recall that an action of G on H is a functor $\widehat{\ }: G \to H$. If H is a category of abelian categories, then we say that an action $\widehat{\ }$ of G on H is *additive* if for each morphism g in G, the functor $\widehat{g}$ respects the additive structures on the hom-sets.

Remark 2.5. *For each subring A of R, we let* $\mathrm{Hom}_{A,S}$ *denote the abelian category having A-S-bimodules AM as objects, for R-S-bimodules M, and A-S-bimodule maps $f : AM \to AN$ as morphisms, for R-S-bimodules M and N. Furthermore, we let* Hom_S *denote the category having* $\mathrm{Hom}_{A,S}$ *as objects, for subrings A of R, and functors* $\mathrm{Hom}_{B,S} \to \mathrm{Hom}_{A,S}$ *as morphisms, for subrings A and B of R. Then* Proposition 2.3 *can be formulated by saying that there is a unique additive action $\widehat{\ }$ of* $\mathrm{Grd}(R)$ *on* Hom_S *subject to the condition that for any $X : B \to A$ in* $\mathrm{Grd}(R)$*, any R-S-bimodules M and N, and any $f \in \mathrm{Hom}_{B,S}(BM, BN)$, we have that $\widehat{X}(f)(xm) = xf(1_B m)$ for all $x \in X$ and all $m \in M$.*

Proposition 2.6. *For any $X : B \to A$ in* $\mathrm{Grd}(R)$ *there is a unique ring isomorphism $\sigma^X : C_{BR}(B) \to C_{AR}(A)$ with the property that $\sigma^X(r)x = xr$, for $r \in C_{BR}(B)$ and $x \in X$. If we choose a positive integer n and $x_i \in X$ and $y_i \in X^{-1}$, for $i \in \{1, \ldots, n\}$, satisfying $\sum_{i=1}^{n} x_i y_i = 1_A$, then $\sigma^X(r) = \sum_{i=1}^{n} x_i r y_i$,* *for $r \in C_{BR}(B)$. Moreover, $\sigma^A = \mathrm{id}_{C_{AR}(A)}$ and if $X : B \to A$ and $Y : C \to B$ belong to* Grd(R)*, then $\sigma^{XY} = \sigma^X \circ \sigma^Y$.*

Proof. For each subring A of R, define maps $h^A : \mathrm{End}_{A,R}(AR) \to C_{AR}(A)$ and $h_A : C_{AR}(A) \to \mathrm{End}_{A,R}(AR)$ by $h^A(f) = f(1_A)$, for $f \in \mathrm{End}_{A,R}(AR)$, respectively $h_A(c)(ar) = car$, for $c \in C_{AR}(A)$, $a \in A$ and $r \in R$. It is clear that h^A and h_A are well-defined ring homomorphisms satisfying $h^A \circ h_A = \mathrm{id}_{C_{AR}(A)}$ and $h_A \circ h^A = \mathrm{id}_{\mathrm{End}_{A,R}(AR)}$. Suppose that $X : B \to A$ is in $\mathrm{Grd}(R)$ and that there is a ring isomorphism $\sigma^X : C_{BR}(B) \to C_{AR}(A)$ with the property that $\sigma^X(r)x = xr$, for $r \in C_{BR}(B)$ and $x \in X$. By the above, it follows that for each $f \in \mathrm{End}_{B,R}(BR)$ the map $(h_A \circ \sigma^X \circ h^B)(f) \in \mathrm{End}_{A,R}(AR)$ satisfies

$$(h_A \circ \sigma^X \circ h^B)(f)(xr) = h_A(\sigma^X(h^B(f)))(xr) =$$

$$= \sigma^X(h^B(f))xr = xh^B(f)r = xf(1_B)r$$

for all $x \in X$ and all $r \in R$; by uniqueness, we get that $(h_A \circ \sigma^X \circ h^B)(f) = f^X$. Hence, if $r \in C_{BR}(B)$, then we get that

$$\sigma^X(r) = (h^A \circ h_A \circ \sigma^X \circ h^B \circ h_B)(r) = (h^A \circ (\cdot)^X \circ h_B)(r) =$$

$$= h^A(h_B(r)^X) = h_B(r)^X(1_A) = \sum_{i=1}^{n} x_i r y_i.$$

By Proposition 2.3(a)-(e), it follows that σ^X is a ring isomorphism satisfying $\sigma^A = \mathrm{id}_{C_{AR}(A)}$ and $\sigma^{XY} = \sigma^X \circ \sigma^Y$. □

Remark 2.7. *If we for each subring A of R, consider the ring $C_{AR}(A)$ to be an abelian category with one object AR, then the disjoint union $C(R) := \biguplus C_{AR}(A)$, where the union runs over all subrings A of R, has an induced structure of an abelian category. Therefore,* Proposition 2.6 *can be formulated by saying that the action of* Grd(R) *on* Hom_S *defined in* Remark 2.5 *induces a unique additive action $\widehat{\ }$ of* Grd(R) *on $C(R)$ subject to the condition that for each $X : B \to A$ in* Grd(R)*, the equality $\widehat{X}(r)x = xr$ holds for all $r \in C_{BR}(B)$ and all $x \in X$.*

The commutant $C_A(A)$ is called the *center* of A and is denoted by $Z(A)$.

Proposition 2.8. *For any $X : B \to A$ in* Grd(R) *there is a unique ring isomorphism $\sigma^X : Z(B) \to Z(A)$ with the property that $\sigma^X(r)x = xr$, for $r \in Z(B)$ and $x \in X$. If we choose a positive integer n and $x_i \in X$ and $y_i \in X^{-1}$, for $i \in \{1, \ldots, n\}$, satisfying $\sum_{i=1}^n x_i y_i = 1_A$, then $\sigma^X(r) = \sum_{i=1}^n x_i r y_i$, for $r \in Z(B)$. Moreover, $\sigma^A = \mathrm{id}_{Z(A)}$ and if $X : B \to A$ and $Y : C \to B$ belong to* Grd(R)*, then $\sigma^{XY} = \sigma^X \circ \sigma^Y$.*

Proof. This follows immediately from Proposition 2.6. □

Remark 2.9. *If we for each subring A of R, consider the ring $Z(A)$ to be an abelian category with one object A, then the disjoint union $Z_R := \biguplus Z(A)$, where the union runs over all subrings A of R, has an induced structure of an abelian category. Therefore,* Proposition 2.8 *can be formulated by saying that the action of* $\mathrm{Grd}(R)$ *on* Hom_S *defined in* Remark 2.5 *induces a unique additive action* $\widehat{}$ *of* $\mathrm{Grd}(R)$ *on Z_R subject to the condition that for each $X : B \to A$ in* $\mathrm{Grd}(R)$*, the equality $\widehat{X}(r)x = xr$ holds for all $r \in Z(B)$ and all $x \in X$.*

3. Graded Rings

At the end of this section, we prove Theorem 1.2. To achieve this, we first show three propositions concerning rings graded by categories and, in particular, groupoids.

Proposition 3.1. *Let R be a locally unital ring graded by a category G.*

(a) *If $s \in G$ is an isomorphism, then $R_s R_{s^{-1}} = R_{c(s)}$ if and only if $R_s R_t = R_{st}$ for all $t \in G$ with $d(s) = c(t)$. In particular, if G is a groupoid (or group), then R is strongly graded if and only if $R_s R_{s^{-1}} = R_{c(s)}$, for all $s \in G$.*

(b) *Suppose that R is strongly graded. If $s \in G$ is an isomorphism, then R_s is finitely generated and projective, both as a left $R_{c(s)}$-module and a right $R_{d(s)}$-module. In particular, if G is a groupoid then the same conclusion holds for each $s \in G$.*

(c) *The ring R is unital if and only if $R = R_H = \bigoplus_{s \in H} R_s$ for a subcategory H of G with finitely many objects. The subcategory H may be chosen so that 1_{R_e} is nonzero for all $e \in \mathrm{ob}(H)$.*

Proof. (a) The "if" statement is clear. Now we show the "only if" statement. Take $(s,t) \in G^{(2)}$ and suppose that $R_s R_{s^{-1}} = R_{c(s)}$. Then, by the assumptions we get that $R_s R_t \subseteq R_{st} = R_{c(s)} R_{st} = R_s R_{s^{-1}} R_{st} \subseteq R_s R_{s^{-1}st} = R_s R_t$. Therefore, $R_s R_t = R_{st}$. The last part follows immediately.

(b) This follows from Proposition 2.2.

(c) The "if" statement is clear since if $\mathrm{ob}(H)$ is finite, then $\sum_{e \in \mathrm{ob}(H)} 1_{R_e}$ is an identity element of R. Now we show the "only if" statement of the claim. Suppose that R has an identity element $1_R = \sum_{s \in G} r_s$ for some $r_s \in R_s$, for $s \in G$, such that $r_s = 0$ for all but finitely many $s \in G$. Take $e, f \in \mathrm{ob}(G)$. If $e \neq f$, then $0 = 1_{R_e} 1_{R_f} = 1_{R_e} 1_R 1_{R_f} = \sum_{s \in G} 1_{R_e} r_s 1_{R_f} = \sum_{s \in G_{e,f}} r_s$. This implies that $r_s = 0$ for all $s \in G$ with $d(s) \neq c(s)$. Also $1_{R_e} = 1_{R_e} 1_{R_e} = 1_{R_e} 1_R 1_{R_e} = \sum_{s \in G} 1_{R_e} r_s 1_{R_e} =$

$\sum_{s\in G_e} r_s$. This implies that $r_e = 1_{R_e}$ and that $r_s = 0$ for all nonidentity $s \in G$ with $d(s) = c(s)$. Therefore $1_R = \sum_{e\in \mathrm{ob}(G)} 1_{R_e}$ which in turn implies that $1_{R_e} = 0$ for all but finitely many $e \in \mathrm{ob}(G)$. Put $H = \{s \in G \mid 1_{R_{d(s)}} = 1_{R_{c(s)}} \neq 0\}$. Then H is a finite object subcategory of G satisfying $R = R_H$. □

In general there is not any obvious connection between local unitality and unitality of a graded ring. This is illustrated by the following remark.

Remark 3.2. (a) *If R is a unital ring graded by a cancellative category, then R is also a locally unital ring. Indeed, let us write $1_R = \sum_{s\in G} 1_s$ where $1_s \in R_s$ for $s \in G$. If $t \in G$, then $1_t = 1_R 1_t = \sum_{s\in G} 1_s 1_t$. Since G is cancellative, this implies that $1_s 1_t = 0$ whenever $s \in G \setminus \mathrm{ob}(G)$. Therefore, if $s \in G \setminus \mathrm{ob}(G)$, then $1_s = 1_s 1_R = \sum_{t\in G} 1_s 1_t = 0$. It is clear that $\{1_e\}_{e\in \mathrm{ob}(G)}$ is a set of local units for R.*

(b) *The conclusion in* (a) *does not hold if G is not cancellative. Indeed, let $G = \{e, s\}$ be the monoid with $e^2 = e$, $s^2 = s$ and $es = se = s$. Define*

$$R = \begin{pmatrix} \mathbb{C} & \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{C} & 0 \\ 0 & 0 & \mathbb{C} \end{pmatrix} \quad R_e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbb{C} \end{pmatrix} \quad R_s = \begin{pmatrix} \mathbb{C} & \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{C} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $R = R_e \bigoplus R_s$ is a unital G-graded ring which is not a locally unital ring.

(c) *There are examples of G-graded rings R which are non-unital, but locally unital. Indeed, suppose that G is a category with $\mathrm{ob}(G)$ infinite and that K is a nontrivial ring which is unital. Let $R = KG$ be the category algebra of G over K (this is sometimes called a quiver algebra of G over K, see e.g.* [4]*). Recall that KG is the set of formal sums $\sum_{s\in G} k_s u_s$ where $k_s \in K$, for $s \in G$, and $k_s = 0$ for all but finitely many $s \in G$. The addition on KG is defined by $\sum_{s\in G} k_s u_s + \sum_{s\in G} k_s' u_s = \sum_{s\in G}(k_s + k_s')u_s$ and the multiplication is defined as the bilinear extension of the rule $(k_s u_s)(k_t' u_t) = k_s k_t' u_{st}$ for $s, t \in G$ and $k_s, k_t' \in K$ if $c(t) = d(s)$ and $(k_s u_s)(k_t' u_t) = 0$ otherwise. If we put $R_s = K u_s$, for $s \in G$, then $R = \bigoplus_{s\in G} R_s$ is a (strongly) G-graded ring. For each $e \in \mathrm{ob}(G)$, it is clear that R_e is a unital ring with identity $1_K u_e$. This makes R a locally unital ring. However, from* Proposition 3.1(c) *and the fact that* $\mathrm{ob}(G)$ *is infinite, it follows that R is non-unital.*

Definition 3.3. Suppose that R is a locally unital ring strongly graded by a groupoid G. By Proposition 2.6 we can use the invertible $R_{c(s)}$-$R_{d(s)}$-bimodules R_s, for $s \in G$, to define a subgroupoid $C(R, G)$ of $\mathrm{Grd}(R)$ with $C_{R_{G_e}}(R_e)$, for $e \in \mathrm{ob}(G)$, as objects, and the ring isomorphisms $C_{R_{G_{d(s)}}}(R_{d(s)}) \to C_{R_{G_{c(s)}}}(R_{c(s)})$, for $s \in G$, as morphisms. In the sequel, these will be denoted by σ_s.

Proposition 3.4. *Suppose that R is a locally unital ring strongly graded by a groupoid G. Then the association of each $e \in \mathrm{ob}(G)$ and each $s \in G$ to the ring $C_{R_{G_e}}(R_e)$ and the function $\sigma_s : C_{R_{G_{d(s)}}}(R_{d(s)}) \to C_{R_{G_{c(s)}}}(R_{c(s)})$, respectively, defines a functor of groupoids $\sigma : G \to C(R,G)$. Moreover, σ is uniquely defined on morphisms given that the relation $\sigma_s(x)r_s = r_s x$ holds for all $s \in G$, all $x \in C_R(R_{d(s)})$ and all $r_s \in R_s$.*

Proof. This follows immediately from Proposition 2.6 (or Remark 2.7). $\square$

Remark 3.5. *Suppose that R is a locally unital ring strongly graded by a groupoid G. Take $s \in G$. By* Proposition 3.1(a) *and the equalities $R_{c(s)} = R_s R_{s^{-1}}$ and $R_{d(s)} = R_{s^{-1}} R_s$ it follows that $R_{d(s)} = 0$ if and only if $R_{c(s)} = 0$; in that case σ_s is of course the zero map. If one wants to avoid such maps one may, by* Proposition 3.1(c)*, assume that all components of R are nonzero and in particular that each ring R_e, for $e \in \mathrm{ob}(G)$, has a nonzero identity element.*

Definition 3.6. Suppose that R is a locally unital ring strongly graded by a groupoid G. By abuse of notation, we let $Z(R,G)$ denote the subcategory of $C(R,G)$ having $Z(R_e)$, for $e \in \mathrm{ob}(G)$, as objects, and the ring isomorphisms $Z(R_{d(s)}) \to Z(R_{c(s)})$, for $s \in G$, as morphisms.

Proposition 3.7. *Suppose that R is a locally unital ring strongly graded by a groupoid G. Then the association of each $e \in \mathrm{ob}(G)$ and each $s \in G$ to the ring $Z(R_e)$ and the function $\sigma_s : Z(R_{d(s)}) \to Z(R_{c(s)})$, respectively, defines a functor of groupoids $\sigma : G \to Z(R,G)$. Moreover, σ is uniquely defined on morphisms given that the relation $\sigma_s(x)r_s = r_s x$ holds for all $s \in G$, all $x \in Z(R_{d(s)})$ and all $r_s \in R_s$.*

Proof. This follows immediately from Proposition 2.8 (or Remark 2.9). $\square$

Proof of Theorem 1.2. First we show that $C_R(R_H) \supseteq F \oplus T$. From Proposition 3.4 it follows that $C_R(R_H) \supseteq F$. By the definition of T it follows that $R_H T = T R_H = \{0\}$. In particular this implies that $C_R(R_H) \supseteq T$.

Now we show that $C_R(R_H) \subseteq F \oplus T$. Suppose that $y = \sum_{s \in G} y_s \in C_R(R_H)$ where $y_s \in R_s$, for $s \in G$, and $y_s = 0$ for all but finitely many $s \in G$. Note that, for an arbitrary $s \in G$ exactly one of the following three cases occur; (1): $c(s) \in \mathrm{ob}(H)$ or $d(s) \in \mathrm{ob}(H)$, and $c(s) \neq d(s)$, (2): $s \in G \ominus H$, (3): $s \in G_e$ for some $e \in \mathrm{ob}(H)$. Since $1_e y = y 1_e$, for $e \in \mathrm{ob}(H)$, we get that $y_s = 0$ whenever case (1) holds for s. By case (2) and case (3) there is some $t \in T$ such that $y = t + \sum_{e \in \mathrm{ob}(H)} x_e$ where $x_e := \sum_{s \in G_e} y_s \in R_{G_e}$, for $e \in \mathrm{ob}(G)$. Since $y - t \in C_R(R_H) \subseteq C_R(R_e)$, for $e \in \mathrm{ob}(H)$, we get that $x_e \in C_{R_{G_e}}(R_e)$, for $e \in \mathrm{ob}(H)$. Take an arbitrary $s \in H$.

By the last part of Proposition 3.4 and the fact that the equality $r_s y = y r_s$ holds for all $r_s \in R_s$, we get that $\sigma_s(x_{d(s)}) = x_{c(s)}$. Therefore $C_R(R_H) \subseteq F \oplus T$. □

4. Examples

In this section, we show Theorem 1.3 and illustrate it in two cases (see Example 4.2). Our method will be to generalize, to category graded rings (see Proposition 4.1), the construction given in [3] for the group graded situation. In order to do this, we first need to introduce some additional notation. Let K be a commutative ring with $1_K \neq 0$ and suppose that G is a category. Fix a positive integer n and choose $s_i \in G$, for $1 \leq i \leq n$. Put $S = \{s_i \mid 1 \leq i \leq n\}$. If $1 \leq i,j \leq n$, then let $e_{ij} \in M_n(K)$ be the matrix with 1_K in the ij:th position and 0 elsewhere. For $s \in G$, we let R_s be the left K-submodule of $M_n(K)$ spanned by the set $\{e_{ij} \mid 1 \leq i,j \leq n,\ (s_i,s) \in G^{(2)},\ s_i s = s_j\}$. With the above notation, the following result holds.

Proposition 4.1. *If we put $R := \sum_{s\in G} R_s$, then*

(a) *the collection of left K-modules R_s, for $s \in G$, of R is a G-filter in R;*

(b) *if $s_i s \in S$, for all $(s_i, s) \in (S \times G) \cap G^{(2)}$, then R_s, for $s \in G$, is a strong G-filter in R;*

(c) *if $G = S$, then $R_s \neq \{0\}$ for $s \in G$;*

(d) *if $d(s_i) \in S$, for $i \in \{1,\ldots,n\}$, then R has an identity element given by $\sum_{f\in \mathrm{ob}(G)} 1_f$, where for each $f \in \mathrm{ob}(G)$, the element $1_f \in R_f$ is the sum of all e_{ii} satisfying $d(s_i) = f$;*

(e) *if G is cancellative, then the collection of left K-modules R_s, for $s \in G$, of R makes R a graded ring;*

(f) *if G is a groupoid and $G = S$, then R_s, for $s \in G$, makes R a unital strongly graded ring with $R_s \neq \{0\}$, for $s \in G$.*

Proof. (a) Suppose that $(s,t) \in G^{(2)}$. Take $e_{ij} \in R_s$ and $e_{lk} \in R_t$. If $j \neq l$, then $e_{ij}e_{lk} = 0 \in R_{st}$. Now let $j = l$. Then, since $s_i s = s_j$ and $s_j t = s_k$, we get that $s_i st = s_j t = s_k$. Hence, $e_{ij}e_{jk} = e_{ik} \in R_{st}$.

(b) Take $(s,t) \in G^{(2)}$ and $e_{ik} \in R_{st}$. Then $s_i st = s_k$. Since $s_i s \in S$ there is $s_j \in S$ with $s_i s = s_j$. This means that $e_{ij} \in R_s$. Moreover, $s_j t = s_i st = s_k$ which yields $e_{jk} \in R_t$. Hence $e_{ik} = e_{ij}e_{jk} \in R_s R_t$.

(c) Take $s \in G$. Since $G = S$, there is $s_i, s_j \in S$ with $s_i = c(s)$ and $s_j = s$. Therefore $s_i s = c(s)s = s = s_j$. Hence $e_{ij} \in S$ which, in turn, implies that $R_s \neq \{0\}$.

(d) Take $s \in G$ and suppose that $e_{jk} \in R_s$ for some $j, k \in \{1, \ldots, n\}$. By the assumptions we get that $d(s_j) \in S$. Therefore $e_{jj} \in R_{d(s_j)}$ and hence $\sum_{f \in \mathrm{ob}(G)} 1_f e_{jk} = e_{jj} e_{jk} = e_{jk}$. In the same way $\sum_{f \in \mathrm{ob}(G)} e_{jk} 1_f = e_{jk}$.

(e) Let X_s denote the collection of pairs (i,j), where $1 \leq i, j \leq n$, such that $(s_i, s) \in G^{(2)}$ and $s_i s = s_j$. Suppose that $s \neq t$. Seeking a contradiction suppose that $X_s \cap X_t \neq \emptyset$. Then there are integers k and l, with $1 \leq k, l \leq n$, such that $s_k s = s_l = s_k t$. By the cancellability of G this implies that $s = t$ which is a contradiction. Therefore, the sets X_s, for $s \in G$, are pairwise disjoint. The claim now follows from (a) and the fact that $R_s = \sum_{(i,j) \in X_s} K e_{ij}$ for all $s \in G$.

(f) This follows immediately from (a), (b), (c), (d) and (e). If we use Proposition 3.1(a) the strongness condition can be proven directly in the following way. Take $s \in G$ and $s_i \in S$. Since $G = S$ there is $s_j \in S$ with $s_i s = s_j$. This means that $e_{ij} \in R_s$. Since G is a groupoid we get that $s_j s^{-1} = s_i$, i.e. $e_{ji} \in R_{s^{-1}}$. Therefore $e_{ii} = e_{ij} e_{ji} \in R_s R_{s^{-1}}$. □

Proof of Theorem 1.3. We first consider the case when G is connected. If G only has one object, then it is a group in which case it has already been treated in [3]. Therefore, from now on, we assume that we can choose two different objects e and f from G. We denote the morphisms of G by $t_1, t_2, \ldots, t_n$. For technical reasons, we suppose that $d(t_1) = f$, $c(t_1) = e$ and $t_n = e$. Let us now choose $n+1$ morphisms $s_1, s_2, \ldots, s_{n+1}$ from G in the following way; $s_i = t_i$, when $1 \leq i \leq n$, and $s_{n+1} = t_n$. Now we define R according to the beginning of this section. By Proposition 4.1(f), the ring R is strongly G-graded and each R_s, for $s \in G$, is nonzero.

We shall now show that the morphism $t := t_1$ has the desired property. Let m denote the cardinality of the set of $s \in G$ with $d(s) = e$. The component R_e is the left K-module spanned by the collection of e_{ij} with $s_i e = s_j$, that is, such that $s_i = s_j$ and $d(s_j) = e$. By the construction of S it follows that the K-dimension of R_e equals $m+3$. Analogously, the component R_{t_1} is the left K-module spanned by the collection of e_{ij} with $s_i t_1 = s_j$. Since $d(t_1) = f \neq e$, this implies that the K-dimension of R_{t_1} equals $m+1$. Seeking a contradiction, suppose that R_{t_1} is free on some generators u_l, $1 \leq l \leq d$, as a left R_e-module. Then the map $\theta : R_e^d \to R_{t_1}$, defined by $\theta(x_1, \ldots, x_d) = \sum_{l=1}^{d} x_l u_l$, for $x_l \in R_e$, for $l \in \{1, \ldots, d\}$, is, in particular, an isomorphism of left K-modules. Since $\dim_K(R_e^d) = d(m+3) > m+1 = \dim_K(R_{t_1})$, this is impossible.

We shall now show that our groupoid G, in the general case, is the disjoint union of connected groupoids. Define an equivalence relation $\sim$ on $\mathrm{ob}(G)$ by saying that

$e \sim f$, for $e, f \in \text{ob}(G)$, if there is a morphism in G from e to f. Choose a set E of representatives for the different equivalence classes defined by $\sim$. For each $e \in E$, let $[e]$ denote the equivalence class to which e belongs. Let $G_{[e]}$ denote the subgroupoid of G with $[e]$ as set of objects and morphisms $s \in G$ with the property that $c(s), d(s) \in [e]$. Then each $G_{[e]}$, for $e \in E$, is a connected groupoid and $G = \biguplus_{e \in E} G_{[e]}$.

For each $e \in E$, we now wish to define a strongly $G_{[e]}$-graded ring $R_{[e]}$. We consider three cases. If $G_{[e]} = \{e\}$, then let $R_{[e]} = K$. If $[e] = \{e\}$ but the group $G_{[e]}$ contains a nonidentity morphism t, then let $R_{[e]}$ be any strongly $G_{[e]}$-graded ring with the desired property (following [3]). If $[e]$ has more than one element, let $R_{[e]}$ denote the strongly $G_{[e]}$-graded ring constructed in the first part of the proof. We may define a new ring to be the direct sum $\bigoplus_{e \in E} R_{[e]}$ which is strongly graded by G and has the desired property. □

Example 4.2. *We have chosen nontrivial examples of graded rings R in the sense that not all graded components R_s are free left $R_{c(s)}$-modules. In the free case the groupoid action is defined by a single conjugation which makes the analysis easier; in the general case the action is a sum of such maps.*

(a) *Suppose that G is the cyclic additive group $\mathbb{Z}_4 = \{0, 1, 2, 3\}$. Using the notation from the proof of* Theorem 1.3 *above, we put*

$$s_1 = 0 \quad s_2 = 1 \quad s_3 = 2 \quad s_4 = s_5 = 3$$

Then $R := M_5(K)$ is a strongly $\mathbb{Z}_4$-graded ring with components defined by

$$\begin{aligned}
R_0 &= Ke_{11} + Ke_{22} + Ke_{33} + Ke_{44} + Ke_{45} + Ke_{54} + Ke_{55} \\
R_1 &= Ke_{12} + Ke_{23} + Ke_{34} + Ke_{35} + Ke_{41} + Ke_{51} \\
R_2 &= Ke_{13} + Ke_{24} + Ke_{25} + Ke_{31} + Ke_{42} + Ke_{52} \\
R_3 &= Ke_{14} + Ke_{15} + Ke_{21} + Ke_{32} + Ke_{43} + Ke_{53}
\end{aligned}$$

By a straightforward calculation, we get that

$$C_R(R_0) = Ke_{11} + Ke_{22} + Ke_{33} + K(e_{44} + e_{55}).$$

It is easy to see that

$$\sigma_1(x) = e_{12}xe_{21} + e_{23}xe_{32} + e_{34}xe_{43} + e_{41}xe_{14} + e_{51}xe_{15}$$

and hence that

$$\sigma_2(x) = \sigma_1^2(x) = e_{13}xe_{31} + e_{24}xe_{42} + e_{31}xe_{13} + e_{42}xe_{24} + e_{52}xe_{25}$$

for all $x \in C_R(R_0)$. *If we put* $H = \{0,2\}$, *then, by* Theorem 1.1, *we get that*

$$C_R(R_H) = C_R(R_0)^H = C_R(R_0)^{\{2\}} = K(e_{11} + e_{33}) + K(e_{22} + e_{44} + e_{55})$$

and

$$Z(R) = C_R(R) = C_R(R_0)^{\mathbb{Z}_4} = C_R(R_0)^{\{1\}} = K1_R.$$

(b) *Now suppose that* G *is the groupoid with two objects* e *and* f *and nonidentity morphisms* $\alpha : e \to e$, $\beta : f \to f$, $u_0 : f \to e$, $u_1 : f \to e$, $t_0 : e \to f$ *and* $t_1 : e \to f$ *with composition given by the following relations*

$$\alpha^2 = e \quad \alpha u_0 = u_1 \quad \alpha u_1 = u_0 \quad u_0\beta = u_1 \quad u_1\beta = u_0$$

$$\beta^2 = f \quad \beta t_0 = t_1 \quad \beta t_1 = t_0 \quad t_0\alpha = t_1 \quad t_1\alpha = t_0$$

$$u_0t_0 = e \quad u_1t_0 = \alpha \quad u_0t_1 = \alpha \quad u_1t_1 = e$$

$$t_0u_0 = f \quad t_0u_1 = \beta \quad t_1u_0 = \beta \quad t_1u_1 = f$$

Using the notation from the proof of Theorem 1.3 *above, we put*

$$s_1 = f \quad s_2 = \beta \quad s_3 = u_0 \quad s_4 = u_1$$

$$s_5 = t_0 \quad s_6 = t_1 \quad s_7 = \alpha \quad s_8 = s_9 = e$$

Now we define the strongly G*-graded subring* R *of* $M_9(K)$ *according to the beginning of this section. A straightforward calculation shows that*

$$\begin{aligned}
R_e &= Ke_{55} + Ke_{66} + Ke_{77} + Ke_{88} + Ke_{89} + Ke_{98} + Ke_{99} \\
R_\alpha &= Ke_{56} + Ke_{65} + Ke_{78} + Ke_{79} + Ke_{87} + Ke_{97} \\
R_{t_0} &= Ke_{15} + Ke_{26} + Ke_{38} + Ke_{39} + Ke_{47} \\
R_{t_1} &= Ke_{16} + Ke_{25} + Ke_{37} + Ke_{48} + Ke_{49} \\
R_f &= Ke_{11} + Ke_{22} + Ke_{33} + Ke_{44} \\
R_\beta &= Ke_{12} + Ke_{21} + Ke_{34} + Ke_{43} \\
R_{u_0} &= Ke_{51} + Ke_{62} + Ke_{74} + Ke_{83} + Ke_{93} \\
R_{u_1} &= Ke_{52} + Ke_{61} + Ke_{73} + Ke_{84} + Ke_{94}
\end{aligned}$$

By a straightforward calculation we get that

$$C_{R_{G_e}}(R_e) = Ke_{55} + Ke_{66} + Ke_{77} + K(e_{88} + e_{99})$$

and

$$C_{R_{G_f}}(R_f) = Ke_{11} + Ke_{22} + Ke_{33} + Ke_{44}.$$

It is easy to see that

$$\sigma_\alpha(x) = e_{56}xe_{65} + e_{65}xe_{56} + e_{78}xe_{87} + e_{87}xe_{78} + e_{97}xe_{79}$$

for all $x \in C_{R_{G_e}}(R_e)$ *and that*

$$\sigma_\beta(y) = e_{12}xe_{21} + e_{21}xe_{12} + e_{34}xe_{43} + e_{43}xe_{34}$$

for all $y \in C_{R_{G_f}}(R_f)$. *Now we use this and* Theorem 1.2 *to compute* $C_R(R_H)$ *for all eleven subgroupoids* H *of* G*:*

$$H_1 = \{e\} \quad H_2 = \{f\} \quad H_3 = \{e, f\} \quad H_4 = G_e \quad H_5 = G_f$$

$$H_6 = \{e, f, \alpha\} \quad H_7 = \{e, f, \beta\} \quad H_8 = \{e, f, \alpha, \beta\}$$

$$H_9 = \{e, f, t_0, u_0\} \quad H_{10} = \{e, f, t_1, u_1\} \quad H_{11} = G$$

We immediately get that

$$C_R(R_{H_1}) = C_R(R_e) = C_{R_{G_e}}(R_e) + R_{G_f} =$$

$$= Ke_{55} + Ke_{66} + Ke_{77} + K(e_{88} + e_{99}) + Ke_{11} + Ke_{22} + Ke_{33} + Ke_{44} +$$

$$+ Ke_{12} + Ke_{21} + Ke_{34} + Ke_{43}$$

and similarly that

$$C_R(R_{H_2}) = C_R(R_f) = C_{R_{G_f}}(R_f) + R_{G_e} =$$

$$= Ke_{11} + Ke_{22} + Ke_{33} + Ke_{44} + Ke_{55} + Ke_{66} + Ke_{77} + Ke_{88} +$$

$$+ Ke_{89} + Ke_{98} + Ke_{99} + Ke_{56} + Ke_{65} + Ke_{78} + Ke_{79} + Ke_{87} + Ke_{97}.$$

Furthermore, we get that

$$C_R(R_{H_3}) = C_{R_{G_f}}(R_f) + C_{R_{G_e}}(R_e) =$$

$$= Ke_{11} + Ke_{22} + Ke_{33} + Ke_{44} + Ke_{55} + Ke_{66} + Ke_{77} + K(e_{88} + e_{99}).$$

Next we get that

$$C_R(R_{H_4}) = C_{R_{G_e}}(R_e)^{G_e} + R_{G_f} = C_{R_{G_e}}(R_e)^{\{\alpha\}} + R_{G_f} =$$

$$= (Ke_{55} + Ke_{66} + Ke_{77} + K(e_{88} + e_{99}))^{\{\alpha\}} + R_{G_f} =$$

$$= K(e_{55} + e_{66}) + K(e_{77} + e_{88} + e_{99}) +$$

$$+ Ke_{11} + Ke_{22} + Ke_{33} + Ke_{44} + Ke_{12} + Ke_{21} + Ke_{34} + Ke_{43}$$

and

$$C_R(R_{H_5}) = C_{R_{G_f}}(R_f)^{G_f} + R_{G_e} =$$

$$= (Ke_{11} + Ke_{22} + Ke_{33} + Ke_{44})^{\{\beta\}} + R_{G_e} =$$

$$= K(e_{11} + e_{22}) + K(e_{33} + e_{44}) + Ke_{55} + Ke_{66} + Ke_{77} + Ke_{88} + Ke_{89} + Ke_{98} + Ke_{99}$$

$$+ Ke_{56} + Ke_{65} + Ke_{78} + Ke_{79} + Ke_{87} + Ke_{97}.$$

By the above calculations, we get that

$$C_R(R_{H_6}) = C_{R_{G_e}}(R_e)^{G_e} + C_{R_{G_f}}(R_f) =$$

$$= K(e_{55}+e_{66}) + K(e_{77}+e_{88}+e_{99}) + Ke_{11} + Ke_{22} + Ke_{33} + Ke_{44}$$

and

$$C_R(R_{H_7}) = C_{R_{G_f}}(R_f)^{G_f} + C_{R_{G_e}}(R_e) =$$

$$= K(e_{11}+e_{22}) + K(e_{33}+e_{44}) + Ke_{55} + Ke_{66} + Ke_{77} + K(e_{88}+e_{99})$$

and

$$C_R(R_{H_8}) = C_{R_{G_e}}(R_e)^{G_e} + C_{R_{G_f}}(R_f)^{G_f} =$$

$$= K(e_{55}+e_{66}) + K(e_{77}+e_{88}+e_{99}) + K(e_{11}+e_{22}) + K(e_{33}+e_{44}).$$

By a straightforward calculation, we get that

$$\sigma_{t_0}(x) = e_{15}xe_{51} + e_{26}xe_{62} + e_{39}xe_{93} + e_{47}xe_{74}$$

and

$$\sigma_{t_1}(x) = e_{16}xe_{61} + e_{25}xe_{52} + e_{37}xe_{73} + e_{48}xe_{84}$$

for all $x \in C_{R_{G_e}}(R_e)$. *By the above calculations, we get that*

$$C_R(R_{H_9}) = \{x + \sigma_{t_0}(x) \mid x \in C_{R_{G_e}}(R_e)\} =$$

$$= K(e_{11}+e_{55}) + K(e_{22}+e_{66}) + K(e_{44}+e_{77}) + K(e_{33}+e_{88}+e_{99})$$

and

$$C_R(R_{H_{10}}) = \{x + \sigma_{t_1}(x) \mid x \in C_{R_{G_e}}(R_e)\} =$$

$$= K(e_{22}+e_{55}) + K(e_{11}+e_{66}) + K(e_{33}+e_{77}) + K(e_{44}+e_{88}+e_{99})$$

and

$$C_R(R_{H_{11}}) = Z(R) = \{x + \sigma_{t_0}(x) \mid x \in C_{R_{G_e}}(R_e)^{G_e}\} =$$

$$= K(e_{11}+e_{22}+e_{55}+e_{66}) + K(e_{33}+e_{44}+e_{77}+e_{88}+e_{99}).$$

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Johan Öinert
Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
DK-2100 Copenhagen Ø, Denmark
e-mail: oinert@math.ku.dk

Patrik Lundström
University West
Department of Engineering Science
SE-46186 Trollhättan, Sweden
e-mail: patrik.lundstrom@hv.se