International Electronic Journal of Algebra
Volume 11 (2012) 64-81

ON CONATURAL CLASSES AND COTYPE SUBMODULES

Alejandro Alvarado-García, Hugo Alberto Rincón-Mejía, José Ríos-Montes and Bertha Tomé-Arreola

Received: 02 February 2011; Revised: 10 November 2011
Communicated by John Clark

Abstract. The concepts of cotype submodule and cotype dimension of a module are introduced, their basic properties are studied, and a characterization of the amply supplemented modules with finite cotype dimension is given.

Mathematics Subject Classification (2010): 16D80
Keywords: conatural class, cotype submodule, cotype dimension

1. Introduction

In the last decade, the study of lattices of classes of modules has played an important role in the developement of the theory of rings and modules. Of particular relevance is the study of the lattices of natural and conatural classes. In a series of papers that culminates with the publication of the book [7], the concept of type submodule with respect to a natural class is studied among several other topics. Type submodules have been used to describe the structure of several classes of modules, in particular, the class of nonsingular modules.

The concept of conatural class is introduced in [1]. The collection of all conatural classes turns out to be a Boolean lattice. This lattice has been used by different authors to classify certain classes of rings and modules. For instance, it is shown in [1] that a ring R is right MAX if and only if each conatural class in mod-R is generated by a family of simple modules. It is also shown that a ring R is right perfect if and only if R is semilocal and $\mid R - conat \mid = 2^k$ where $k \in \mathbb{N}$ is the number of isomorphism classes of simple R-modules. More examples can be found in [5,6,8,9,10,11,12].

Our aim in this paper is to introduce the concept of cotype submodule, namely, a submodule N of a module M is a cotype submodule if there exists a conatural class $\mathcal{C}$ such that, among all the submodules of M, N is minimal with respect to the property that $M/N \in \mathcal{C}$. After showing some of the basic properties of cotype submodules, we define the concept of cotype dimension, and characterize the amply supplemented modules with finite cotype dimension as those modules having DCC on their cotype submodules.

This paper is divided into five sections. The second one contains some preliminary notions and results which are either needed to understand the rest of the paper or used throughout it. In section 3, we introduce the concept of cotype submodule of a module and prove some of its basic properties. In section 4, we define the notions of supplement interior, cotype interior and cotype supplement of a submodule of a module M and show their existence under the condition that M be amply supplemented. Finally, in the last section we define the concept of cotype dimension of a module, prove some of its basic properties and characterize the amply supplemented modules with finite cotype dimension.

2. Preliminaries

In this section we recall some concepts and results which appear in [1] and [2] and will be used throughout the following sections. We start with the following definition taken from [1].

Definition 2.1. A class of R-modules is called a *cohereditary class* if it is closed under quotients. The class of all cohereditary classes is denoted by R-quot.

Examples of cohereditary classes are pretorsion, TTF and Serre classes in mod-R.

Let $\mathcal{L}$ be a big lattice with universal bounds $\mathbf{0}$ and $\mathbf{1}$. Then $b \in \mathcal{L}$ is a pseudocomplement of $a \in \mathcal{L}$ if b is maximal with respect to $a \wedge b = \mathbf{0}$. If every element $a \in \mathcal{L}$ has a pseudocomplement in $\mathcal{L}$, then $\mathcal{L}$ is called pseudocomplemented, and the class of all pseudocomplements in $\mathcal{L}$ is called the skeleton of $\mathcal{L}$.

It is shown in [1] that $(R - quot, \leq, \wedge, \vee)$ is a pseudocomplemented, complete, big lattice, where the partial order, lattice operations, and universal bounds are given by:

(1) For $\mathcal{C}_1, \mathcal{C}_2 \in R - quot$, $\mathcal{C}_1 \leq \mathcal{C}_2 \Leftrightarrow \mathcal{C}_1 \subseteq \mathcal{C}_2$.
(2) For any family $\{\mathcal{C}_i\}_{i \in I} \subseteq R - quot$, $\bigwedge\limits_{i \in I} \mathcal{C}_i = \bigcap\limits_{i \in I} \mathcal{C}_i$.
(3) For any family $\{\mathcal{C}_i\}_{i \in I} \subseteq R - quot$, $\bigvee\limits_{i \in I} \mathcal{C}_i = \bigcup\limits_{i \in I} \mathcal{C}_i$.
(4) $\mathbf{0} = \{0\}$.
(5) $\mathbf{1} = mod - R$.

Moreover, for $\mathcal{C} \in R$-quot, its unique pseudocomplement in R-quot is given by

$$\mathcal{C}^{\perp_{R-quot}} = \{N \in mod - R \mid N \text{ has no nonzero quotients in } \mathcal{C}\}.$$

The following definitions can be found in [1] and [2].

Definition 2.2. 1) The skeleton of $R - quot$ is denoted by $R - conat$ and its elements are called conatural classes.

2) Two $R-$modules M and N share nonzero quotients if there exists $0 \neq K \in mod - R$ such that K is a quotient of both M and N.

3) A class $\mathcal{C}$ of $R-$modules satisfies condition (CN) if, whenever every nonzero quotient of M shares nonzero quotients with some module in $\mathcal{C}$, it follows that $M \in \mathcal{C}$.

In [1] it is shown that condition (CN) characterizes conatural classes, and that any conatural class is closed under quotients, extensions and superfluous epimorphisms.

Examples of conatural classes are the following:

(1) The class of all projective semisimple R-modules.
(2) The class of all $M \in mod - R$ such that $MI = M$, where I is an ideal of R.

For a class $\mathcal{C}$ of $R-$modules, the conatural class generated by $\mathcal{C}$ is given by

$$\xi_{conat}(\mathcal{C}) = \left\{ M \in mod - R \mid \begin{array}{c} \text{every nonzero quotient of } M \text{ shares a} \\ \text{nonzero quotient with some module in } \mathcal{C} \end{array} \right\}$$

For example, if $\mathcal{C}$ is the class of all simple R-modules, then $\xi_{conat}(\mathcal{C})$ consists of all $M \in mod - R$ such that each of its proper submodules is included in a maximal submodule of M.

When $\mathcal{C}$ consists of a single R-module M, we denote $\xi_{conat}(\{M\})$ simply by $\xi_{conat}(M)$.

It is also shown in [1] that $(R - conat, \leq, \wedge, \vee)$ is a complemented, distributive, complete lattice, where the partial order, the lattice operations and universal bounds are given by:

(1) For $\mathcal{C}_1, \mathcal{C}_2 \in R - conat$, $\mathcal{C}_1 \leq \mathcal{C}_2 \Longleftrightarrow \mathcal{C}_1 \subseteq \mathcal{C}_2$.
(2) For any family $\{\mathcal{C}_i\}_{i \in I} \subseteq R - conat$, $\bigwedge\limits_{i \in I} \{\mathcal{C}_i\} = \bigcap\limits_{i \in I} \{\mathcal{C}_i\}$.
(3) For any family $\{\mathcal{C}_i\}_{i \in I} \subseteq R - conat$, $\bigvee\limits_{i \in I} \{\mathcal{C}_i\} = \xi_{conat}\left(\bigcup\limits_{i \in I} \{\mathcal{C}_i\}\right)$.
(4) $\mathbf{0} = \{0\}$.
(5) $\mathbf{1} = mod - R$.

Moreover, for $\mathcal{C} \in R - conat$, its unique complement in $R - conat$ is given by

$$\mathcal{C}^c = \{M \in mod - R \mid M \text{ has no nonzero quotients in } \mathcal{C}\} \ .$$

In [2], it is shown that R-conat is a set, and is therefore a Boolean lattice (see Theo. 8 and Cor. 9). It is also shown there that, when R is a right perfect ring, then $\mathcal{C} \in R-conat$ if and only if $\mathcal{C}$ is closed under quotients, projective covers and direct sums of simple modules (see Theo. 17).

Finally, it is worth noting that conatural classes have been used to characterize certain kinds of rings. Recall that a ring R is right MAX if each nonzero right R-module has maximal submodules. It is shown in [1] that a ring R is right MAX if and only if each conatural class in mod-R is generated by a family of simple modules. Since every right perfect ring R is right MAX, the same thing happens to each conatural class in mod-R, except that in this case the family is finite (see Theo. 42 and Cor. 43 in [1]).

3. Cotype submodules

In this section we introduce the concept of cotype submodule of a module and prove some of its basic properties.

Definition 3.1. A submodule L of a module M is called a *cotype submodule* of M if there exists a conatural class $\mathcal{C}$ such that, among all the submodules of M, L is minimal with respect to the property that ${}^{M}/_{L} \in \mathcal{C}$.

In this case, we shall also say that ${}^{M}/_{L}$ is a cotype quotient of M of cotype $\mathcal{C}$.

In order to give a more intrinsic characterization of this concept we need the following definitions.

Definition 3.2. Two modules M and N are called *coorthogonal*, denoted as $M \perp_c N$, if they do not share nonzero quotients. They are called *coparallel*, denoted as $M \parallel_c N$, if every nonzero quotient of each one of them shares nonzero quotients with the other.

Definition 3.3. Let L be a submodule of a module M. A submodule K of M is called a *supplement of L in M* if K is minimal with respect to the property that $K + L = M$.

It is known that K is a supplement of L in M if and only if $K + L = M$ and $K \cap L$ is superfluous in K (denoted by $K \cap L \ll K$).

Definition 3.4. A module M is called *supplemented* if every submodule of M has a supplement in M.

Finally, we have:

Theorem 3.5. *Let M be a supplemented module. For a submodule L of M, the following statements are equivalent:*

1) *L is a cotype submodule of M.*
2) *If $K \leq L$ and $M/K \parallel_c M/L$, then $L = K$.*
3) *If $K \lneqq L$, then $M/L \perp_c M/N$ for some $K \leq N \lneqq M$.*
4) *There exists $K \leq M$ such that L is a supplement of K in M and $M/L \perp_c M/K$.*

Proof. ($1 \Rightarrow 2$) Assume that $K \leq L$ and $M/K \parallel_c M/L$. Let $\mathcal{C}$ be as in 1). Since $\mathcal{C}$ satisfies condition (CN), $M/K \in \mathcal{C}$. Hence by 1), $K = L$.

($2 \Rightarrow 1$) Let $\mathcal{C} = \xi_{conat}(M/L)$. Assume that $K \leq L$ and $M/K \in \mathcal{C}$. Then $M/K \parallel_c M/L$, and by 2) $K = L$.

($2 \Rightarrow 3$) Assume that $K \lneqq L$. Then by 2), M/K is not coparallel to M/L. Thus there exists a nonzero quotient M/N of M/K such that $M/N \perp_c M/L$.

($3 \Rightarrow 2$) Assume that $K \leq L$ and $M/K \parallel_c M/L$. Then by 3), $K = L$.

($4 \Rightarrow 3$) By 4), there exists $N \leq M$ such that L is a supplement of N in M and $M/L \perp_c M/N$. Assume that $K \lneqq L$. Then $K \leq K + N \lneqq M$ and $M/L \perp_c M/K{+}N$.

($3 \Rightarrow 4$) Let K be a supplement of L in M, and assume that there exists $J \lneqq L$ such that $K + J = M$. By 3), M/J has a nonzero quotient M/N such that $M/L \perp_c M/N$. Consider the following diagram with exact row

$$\begin{array}{ccccccccc} 0 & \to & L/J & \rightarrowtail & M/J & \twoheadrightarrow & M/L & \to & 0 \\ & & & & {\scriptstyle\pi}\downarrow & & & & \\ & & & & M/N & & & & \end{array}$$

Note that $\pi\left(L/J\right) \neq 0$, for otherwise there is an epimorphism $M/L \twoheadrightarrow M/N$. Then we can take the quotient $\frac{M/N}{\pi(L/J)}$, and obtain an epimorphism $M/L \twoheadrightarrow \frac{M/N}{\pi(L/J)}$. It follows that $\pi\left(L/J\right) = M/N$, and therefore $L/J + N/J = M/J$, which gives $L + N = M$. We now observe that

$$L + (K \cap N) = (L + J) + (K \cap N) = L + (J + (K \cap N)) =$$

$$= L + (N \cap (K + J)) = L + (N \cap M) = L + N = M.$$

Since K is a supplement of L, we get that $K \cap N = K$, that is, $K \leq N$. Hence $M = K + J \leq N$, which contradicts that $M/N \neq 0$. This contradiction shows that L is a supplement of K in M.

Let $\mathcal{C} = \xi_{conat}(M/L)$. By ($2 \Rightarrow 1$), we know that among all the submodules of M, L is minimal with respect to the property that $M/L \in \mathcal{C}$. Then $L \in \mathcal{C}^c$, for otherwise there is a nonzero quotient $L/N \in \mathcal{C}$ for some $N \leq L$. From the exact sequence

$$0 \to L/N \rightarrowtail M/N \twoheadrightarrow M/L \to 0$$

with first and last terms in $\mathcal{C}$, we get that $M/N \in \mathcal{C}$, contradicting the minimality of L. Therefore

$$M/K = L+K/K \cong L/L\cap K \in \mathcal{C}^c$$

and so $M/L \perp_c M/K$. $\square$

Some basic facts about cotype submodules are contained in the following lemma.

Lemma 3.6. *Let K and L be submodules of a supplemented module M. Then the following statements hold:*

1) If $K \leq L$ and L is a cotype submodule of M, then L/K is a cotype submodule of M/K.

2) If L is a cotype submodule of M, then $L+K = M$ if and only if $M/L \perp_c M/K$.

3) If K is a cotype submodule of L and L is a cotype submodule of M, then K is a cotype submodule of M.

Proof. 1) By Theorem 3.5, L is a supplement of L' in M for some $L' \leq M$, and $M/L \perp_c M/L'$. Then $L/K + L'+K/K = M/K$. Let $X/K \leq L/K$ be such that $X/K + L'+K/K = M/K$. Then $X+L'/K = M/K$ and therefore $X + L' = M$. It follows that $X = L$. Furthermore, since $M/L'+K$ is a quotient of M/L', it follows that

$$\frac{M/K}{L/K} \cong M/L \perp_c M/L'+K \cong \frac{M/K}{L'+K/K}.$$

2) $\Rightarrow$) If L is a cotype submodule of M, then as seen in the proof of Theorem 3.5, among all the submodules of M, L is minimal with respect to $M/L \in \xi_{conat}(M/L) = \mathcal{C}$, and $L \in \mathcal{C}^c$. Therefore $M/K = L+K/K \cong L/L\cap K \in \mathcal{C}^c$ and $M/L \perp_c M/K$.

$\Leftarrow$) It follows from the fact that $M/L+K$ is a quotient of both M/L and M/K.

3) By Theorem 3.5, K is a supplement of K' in L for some $K' \leq M$, and $L/K \perp_c L/K'$. Also, L is a supplement of L' in M for some $L' \leq M$, and $M/L \perp_c M/L'$. Then $K + K' + L' = M$. We show first that K is minimal with this property. Let $X < K$. Then as K is a supplement in L, $X + K' < L$, and as L is a supplement in M, $X + K' + L' < M$. Now, let $N = K' + L'$. Then $L = L\cap M = L\cap(K+N) = K+(L\cap N)$. By (2), $L/K \perp_c L/L\cap N \cong L+N/N = M/N$. Also $M = L+N$ gives $M/L \perp_c M/N$. Hence L/K and M/L belong to $\xi_{conat}(M/N)^c$. Then from the exact sequence $0 \longrightarrow L/K \longrightarrow M/K \longrightarrow M/L \longrightarrow 0$ we get that $M/K \perp_c M/N$. $\square$

In order to show another interesting fact about cotype submodules (or rather cotype quotients) of a module M, we need the following definitions.

Definition 3.7. A submodule L of a module M has *ample supplements in* M if, for every submodule K of M such that $K+L=M$, there is a supplement K' of L in M with $K' \leq K$.

Definition 3.8. If every submodule of a module M has ample supplements in M, then M is called *amply supplemented.*

Recall that a ring R is right perfect if and only if every right R-module M has a projective cover $P(M)$, if and only if every right R-module M is amply supplemented (see [14, 43.9]).

Now we are able to state the following result.

Proposition 3.9. *Let R be a right perfect ring and $\mathcal{C}$ a conatural class in mod-R. Assume that M/K_1 and M/K_2 are cotype quotients of M of cotype $\mathcal{C}$. Then the following statements hold:*

1) $P\left(M/K_1\right) \cong P\left(M/K_2\right)$.

2) *There exist superfluous epimorphisms $M/K_1 \twoheadrightarrow M/L_1$, $M/K_2 \twoheadrightarrow M/L_2$ with $M/L_1 \cong M/L_2$.*

3) $P\left(K_1\right) \cong P\left(K_2\right)$.

Proof. 1) Let L_2 be a supplement of K_2 in M. Since M/K_2 is a cotype quotient, $M/K_2 \perp_c M/L_2$. Moreover, $M/L_2 \in \mathcal{C}^c$. Indeed, otherwise there is an epimorphism $M/L_2 \twoheadrightarrow M/N$ with $0 \neq M/N \in \mathcal{C}$. Consequently, $M/K_2 \perp_c M/N$ and so $K_2+N=M$. Then $M/K_2 = K_2+N/K_2 \cong N/K_2\cap N \in \mathcal{C}$, and from the exact sequence

$$0 \rightarrow N/K_2\cap N \rightarrow M/K_2\cap N \rightarrow M/N \rightarrow 0$$

it follows that $M/K_2\cap N \in \mathcal{C}$. Since $K_2 \cap N \leq K_2$ and K_2 is minimal with respect to $M/K_2 \in \mathcal{C}$, $K_2 \cap N = K_2$ and $K_2 \leq N$. But then M/K_2 and M/L_2 share the nonzero quotient M/N, contradicting that $M/K_2 \perp_c M/L_2$. Next, we show that the cotype quotient M/L_2 is of cotype $\mathcal{C}^c$. Assume that there exists $L \leq L_2$ such that $M/L \in \mathcal{C}^c$. Then $M/L+K_2 \in \mathcal{C} \cap \mathcal{C}^c$ and so $L+K_2=M$. Since L_2 is a supplement of K_2, $L=L_2$.

Now, as $M/K_1 \perp_c M/L_2$, $K_1+L_2=M$, and we can take a supplement L_1 of K_1 in M such that $L_1 \leq L_2$. Since M/K_1 is a cotype quotient, $M/K_1 \perp_c M/L_1$ and, as above, $M/L_1 \in \mathcal{C}^c$. It follows that $L_1 = L_2$.

Finally, note that $M/K_1 = K_1+L_2/K_1 \cong L_2/K_1\cap L_2$, $M/K_2 = K_2+L_2/K_2 \cong L_2/K_2\cap L_2$, and there are superfluous epimorphisms $P(L_2) \overset{\ll}{\twoheadrightarrow} L_2 \overset{\ll}{\twoheadrightarrow} L_2/K_1\cap L_2$ and $P(L_2) \overset{\ll}{\twoheadrightarrow}$

$L_2 \overset{\ll}{\twoheadrightarrow} {}^{L_2}/_{K_2 \cap L_2}$. Therefore

$$P\left({}^{M}/_{K_1}\right) \cong P\left({}^{L_2}/_{K_1 \cap L_2}\right) \cong P\left(L_2\right) \cong P\left({}^{L_2}/_{K_2 \cap L_2}\right) \cong P\left({}^{M}/_{K_2}\right).$$

2) Let $f : P({}^{M}/_{K_1}) \xrightarrow{\cong} P({}^{M}/_{K_2})$ be an isomorphism, and for $i = 1, 2$, let $\pi_i : P\left({}^{M}/_{K_i}\right) \twoheadrightarrow {}^{M}/_{K_i}$ be superfluous epimorphisms. Since the image of a superfluous submodule and the sum of two superfluous submodules are superfluous, $Ker\pi_1 + f^{-1}\left(Ker\pi_2\right) \ll P\left({}^{M}/_{K_1}\right)$ and $f\left(Ker\pi_1\right) + Ker\pi_2 \ll P\left({}^{M}/_{K_2}\right)$. It follows that

$${}^{Ker\pi_1 + f^{-1}(Ker\pi_2)}/_{Ker\pi_1} \ll {}^{P({}^{M}/_{K_1})}/_{Ker\pi_1} \cong {}^{M}/_{K_1}$$

and

$${}^{f(Ker\pi_1)+Ker\pi_2}/_{Ker\pi_2} \ll {}^{P({}^{M}/_{K_2})}/_{Ker\pi_2} \cong {}^{M}/_{K_2},$$

that is, there are superfluous epimorphisms

$${}^{M}/_{K_1} \cong {}^{P({}^{M}/_{K_1})}/_{Ker\pi_1} \twoheadrightarrow {}^{P({}^{M}/_{K_1})}/_{Ker\pi_1 + f^{-1}(Ker\pi_2)}$$

and

$${}^{M}/_{K_2} \cong {}^{P({}^{M}/_{K_2})}/_{Ker\pi_2} \twoheadrightarrow {}^{P({}^{M}/_{K_2})}/_{f(Ker\pi_1)+Ker\pi_2}.$$

Finally, $Ker\pi_1 + f^{-1}\left(Ker\pi_2\right) \cong f\left(Ker\pi_1\right) + Ker\pi_2$, gives

$${}^{P({}^{M}/_{K_1})}/_{Ker\pi_1 + f^{-1}(Ker\pi_2)} \cong {}^{P({}^{M}/_{K_2})}/_{f(Ker\pi_1)+Ker\pi_2}.$$

3) Let L_i be a supplement of K_i in M for $i = 1, 2$. Then ${}^{M}/_{L_1}$ and ${}^{M}/_{L_2}$ are cotype quotients of cotype $\mathcal{C}^c$, and by 1), $P\left({}^{M}/_{L_1}\right) \cong P\left({}^{M}/_{L_2}\right)$. Furthermore, by [8, 5.2.4], there are superfluous epimorphisms

$$P\left(K_i\right) \overset{\ll}{\twoheadrightarrow} K_i \overset{\ll}{\twoheadrightarrow} {}^{K_i}/_{K_i \cap L_i} \cong {}^{K_i + L_i}/_{L_i} = {}^{M}/_{L_i}$$

for $i = 1, 2$. Consequently, $P\left(K_1\right) \cong P\left({}^{M}/_{L_1}\right) \cong P\left({}^{M}/_{L_2}\right) \cong P\left(K_2\right)$. □

We end this section with some examples.

Example 3.10. 1) *If p is a prime number, the only cotype submodules of $\mathbb{Z}_{p^\infty}$ are 0 and $\mathbb{Z}_{p^\infty}$.*

2) *From the decomposition ${}^{\mathbb{Q}}/_{\mathbb{Z}} = \bigoplus_{p \in \mathcal{P}} \mathbb{Z}_{p^\infty}$, where $\mathcal{P}$ denotes the set of prime numbers, it follows that each direct summand of ${}^{\mathbb{Q}}/_{\mathbb{Z}}$ is a cotype submodule.*

4. Cotype supplement and cotype interior

Throughout this section, we will assume that the module M is amply supplemented. Given a submodule L of M, we define three submodules of L, the supplement interior, the cotype supplement and the cotype interior of L in M, as follows.

Definition 4.1. Let L be a submodule of M. A *supplement interior* of L in M is a submodule N of M minimal with respect to the property that ${}^{L}/_{N} \ll {}^{M}/_{N}$.

We note that "supplement interior" is the same notion as "coclosure" as defined in [4, 3.10, 20.25] (and was also earlier called "s-closure" by Keskin [13]).

Theorem 4.2. *Let L be a submodule of M. Then:*

1) *There is a submodule K of M such that K is minimal with respect to the property that ${}^{M}/_{K} \perp_c {}^{M}/_{L}$. Moreover, K is a cotype submodule of M.*

2) *There is a submodule J of M such that J is minimal with respect to the properties that $J \leq L$ and ${}^{M}/_{J} \parallel_c {}^{M}/_{L}$. Moreover, J is a cotype submodule of M, which is also a supplement interior of L in M.*

Proof. 1) Let $\mathcal{C} = \xi_{conat}\,({}^{M}/_{L})$ and $\mathcal{D} = \xi_{conat}\,(M)$. Since R-conat is a Boolean lattice, there exists $\mathcal{C}' \in R$-conat such that $\mathcal{D} = \mathcal{C} \vee \mathcal{C}'$ and $\mathcal{C} \wedge \mathcal{C}' = \mathbf{0}$. If $\mathcal{C}' \neq \mathbf{0}$, there is a nonzero module $M^{'} \in \mathcal{C}'$ which shares a nonzero quotient ${}^{M}/_{K}$ with M. Then ${}^{M}/_{K} \in \mathcal{C}'$ and therefore ${}^{M}/_{K} \perp_c {}^{M}/_{L}$. Let $\bar{K}$ be a supplement of L in M with $\bar{K} \leq K$. By [8, 5.2.4c], ${}^{K}/_{\bar{K}} \ll {}^{M}/_{\bar{K}}$, that is, there is a superfluous epimorphism ${}^{M}/_{\bar{K}} \twoheadrightarrow {}^{M}/_{K}$. Hence ${}^{M}/_{\bar{K}} \in \mathcal{C}'$ and ${}^{M}/_{\bar{K}} \perp_c {}^{M}/_{L}$. This shows that $\bar{K}$ is a cotype submodule of M, and by Lemma 3.6(2), $\bar{K}$ is minimal with respect to ${}^{M}/_{\bar{K}} \perp_c {}^{M}/_{L}$.

2) Let $\mathcal{C} = \xi_{conat}({}^{M}/_{L})$. As in the proof of a), there is a supplement K of L in M such that K is minimal with respect to ${}^{M}/_{K} \perp_c {}^{M}/_{L}$. Let J be a supplement of K in M with $J \leq L$. Again, by [8, 5.2.4c], ${}^{L}/_{J} \ll {}^{M}/_{J}$, that is, the epimorphism ${}^{M}/_{J} \twoheadrightarrow {}^{M}/_{L}$ is superfluous. Then ${}^{M}/_{J} \in \mathcal{C}$ and therefore ${}^{M}/_{J} \parallel_c {}^{M}/_{L}$. Since ${}^{M}/_{L} \perp_c {}^{M}/_{K}$, also ${}^{M}/_{J} \perp_c {}^{M}/_{K}$. Hence ${}^{M}/_{J}$ is a cotype quotient of cotype $\xi_{conat}({}^{M}/_{J}) = \mathcal{C}$, that is, J is minimal with respect to ${}^{M}/_{J} \parallel_c {}^{M}/_{L}$. Moreover, J is also minimal with respect to ${}^{L}/_{J} \ll {}^{M}/_{J}$. Indeed, if $J^{'} \leq J \leq L$ and ${}^{L}/_{J'} \ll {}^{M}/_{J'}$, then, as above, ${}^{M}/_{J'} \in \mathcal{C}$ and ${}^{M}/_{J'} \parallel_c {}^{M}/_{L}$. Therefore $J^{'} = J$ and J is a supplement interior of L in M. $\square$

We observe that if $\mathcal{C}' = \mathbf{0}$ in the proof above, that is, $\mathcal{D} = \mathcal{C}$, then $K = M$ and $J = 0$.

Definition 4.3. Let L be a submodule of M. The submodules K and J of M satisfying respectively 1) and 2) of the above theorem are called, respectively, a *cotype supplement* and a *cotype interior* of L.

Proposition 4.4. *Let L be a submodule of M. Then:*

1) *There exists a cotype submodule J of M maximal with respect to $J \leq L$.*
2) *There exists a cotype submodule K of M minimal with respect to $K + L = M$.*
3) *J and K are supplements of each other and $J \cap K \ll M$.*

Proof. Let J and K be, respectively, a cotype interior and a cotype supplement of L in M.

1) Let J' be a cotype submodule of M such that $J \leq J' \leq L$. Assume that M/J' is of cotype $\mathcal{C}$. Then, as M/L is a quotient of M/J' and the epimorphism $M/J \twoheadrightarrow M/L$ is superfluous, M/L and M/J belong to $\mathcal{C}$. But J' is minimal with respect to $M/J \in \mathcal{C}$, hence $J = J'$.

2) It follows from 1) of the above theorem and Lemma 3.6.

3) It follows from the proof of 2) of the above theorem and [8, 5.2.4b]. $\square$

5. Cotype dimension

In this section we define the notion of cotype dimension of a module. We prove its basic properties for amply supplemented modules.

We start with a more general result.

Proposition 5.1. *Let $\{M_i\}_{i\in I}$ be a family of R-modules and assume that N is a nonzero quotient of $P\left(\bigoplus_{i\in I} M_i\right)$. Then N shares a nonzero quotient with M_j for some $j \in I$.*

Proof. Since $P\left(\bigoplus_{i\in I} M_i\right) \cong \bigoplus_{i\in I} (P(M_i))$, we may assume that the M_i, $i \in I$, are all projective, and that there is an epimorphism $f : \bigoplus_{i\in I} M_i \twoheadrightarrow N$. Let $K = ker(f)$ and, for $i \in I$, let $\rho_i : \bigoplus_{i\in I} M_i \twoheadrightarrow M_i$ be the projection and g_i the composition $K \hookrightarrow \bigoplus_{i\in I} M_i \twoheadrightarrow M_i$. Note that if g_i is surjective then, as M_i is projective, M_i is a summand of K. Hence the g_i, $i \in I$, cannot be all surjective, for otherwise $K = \bigoplus_{i\in I} M_i$, which contradicts that $N \neq 0$. Then we may choose $j \in I$ such that g_j is not surjective. If $g_j = 0$, there is an epimorphism $N \twoheadrightarrow M_j$. If not, we may consider the nonzero quotient $M_j/Im(g_j)$ and obtain again an epimorphism $N \twoheadrightarrow M_j/Im(g_j)$. $\square$

If $\mathcal{L}$ is a lattice, an element $a \in \mathcal{L}$ is called an *atom* if $a \neq \mathbf{0}$ and, whenever $c \leq a$ with $\mathbf{0} \neq c \in \mathcal{L}$, then $c = a$. The lattice $\mathcal{L}$ is called *atomic* if, for every $\mathbf{0} \neq c \in \mathcal{L}$, there exists an atom $a \in \mathcal{L}$ such that $a \leq c$.

The following definition appears in [2].

Definition 5.2. A module M is called *q-atomic* if $\xi_{conat}(M)$ is an atom in R-conat.

Example 5.3. 1) *If M is simple, then M is q-atomic.*

2) *More generally, if M is hollow, that is, if every proper submodule of M is superfluous in M, then M is q-atomic.*

3) *If R is right perfect, then every indecomposable projective R-module is hollow, and so q-atomic.*

4) *If M is amply supplemented, then M is q-atomic if and only if its only cotype submodules are* 0 *and M. If M is q-atomic and L is a cotype submodule of M with ${}^{M}/_{L}$ of cotype $\xi_{conat}({}^{M}/_{L})$, then $L = M$ or $L = 0$. If $\mathcal{C} \in R - conat$ is such that $\mathbf{0} \neq \mathcal{C} \subseteq \xi_{conat}(M)$, then $\mathcal{C}$ contains a nonzero quotient ${}^{M}/_{K}$ of M. Let $L \leq M$ be a cotype interior of K, then $\mathbf{0} \neq \xi_{conat}({}^{M}/_{L}) = \xi_{conat}({}^{M}/_{K}) \subseteq \mathcal{C}$ yields $L = 0$, and so $\xi_{conat}(M) = \mathcal{C}$.*

5) *If R is right MAX (in particular, if R is right perfect), then R-conat is an atomic lattice.*

Lemma 5.4. *Assume that $A_1, ..., A_k$ are pairwise coorthogonal q-atomic quotients of M, M with a projective cover, such that there is a superfluous epimorphism $M \twoheadrightarrow \bigoplus_{i=1}^{k} A_i$. If $B_1, ..., B_m$ are pairwise coorthogonal nonzero quotients of M, then $m \leq k$.*

Proof. Since the projective cover of M is that of $\bigoplus_{i=1}^{k} A_i$, by the above proposition, each B_i shares a nonzero quotient Q with some A_j. Now, as $\xi_{conat}(Q) \subseteq \xi_{conat}(A_j)$ and A_j is q-atomic, it follows that Q is also q-atomic. Hence, without loss of generality, we may assume that the B_i are q-atomic. Again, by the above proposition, and after renumbering the A_i, we may assume that $B_1 \parallel_c A_1$. Repeating this process, we obtain that $B_2 \parallel_c A_i$ for some i. Since $B_2 \perp_c B_1$, $i \neq 1$, and after renumbering the A_i, we may take $i = 2$, if $2 \leq k$. Suppose that $k + 1 \leq m$. Then we are able to repeat this process n times obtaining $B_i \parallel_c A_i$ for $i = 1, ..., k$. Repeating the process once more, we obtain $B_{k+1} \parallel_c A_i$ for some $i \leq k$ and so $B_{k+1} \parallel_c B_i$. This contradiction shows that $m \leq k$. $\square$

Definition 5.5. A module M has *finite cotype dimension* n, denoted by $ct.dim\,(M) = n$, if $\xi_{conat}\,(M)$ is the join of n atoms in R-conat. If no such n exists, we say that the cotype dimension of M is infinite. If $M = 0$, then $ct.dim\,(M) = 0$.

Note that $ct.dim\,(M) = ct.dim\,(P(M))$.

Example 5.6. 1) *If M is q-atomic, then $ct.dim\,(M) = 1$ (in particular, $ct.dim\,(\mathbb{Z}_{p^\infty}) = 1$).*

2)] *If R is right perfect, then for all $M \in mod - R$, $ct.dim\,(M) \leq n$ where n is the number of isomorphism classes of simple R-modules.*

3) *From the decomposition $\mathbb{Q}/\mathbb{Z} = \bigoplus\limits_{p\in\mathcal{P}} \mathbb{Z}_{p^\infty}$, where $\mathcal{P}$ denotes the set of prime numbers, it follows that $ct.dim\,(\mathbb{Q}/\mathbb{Z}) = \infty$.*

The following lemmas contain characterizations of both finite and infinite cotype dimensions, respectively, for amply supplemented modules. Given an R-homomorphism $f : M \longrightarrow N$, we say that f is *superfluous* if its kernel is superfluous in M.

Lemma 5.7. *For an amply supplemented module M, the following statements are equivalent:*

1) $ct.dim\,(M) = n$.

2) *There exist n pairwise coorthogonal q-atomic quotients of M, $A_1, ..., A_n$, such that the morphism $M \longrightarrow \bigoplus\limits_{i=1}^{n} A_i$ induced by the canonical epimorphisms is superfluous.*

Proof. 1) $\Rightarrow$ 2) Assume that $\mathcal{C} = \xi_{conat}\,(M) = \bigvee\limits_{i=1}^{n} \mathcal{C}_i$, where $\mathcal{C}_1, ..., \mathcal{C}_n$ are atoms in R-conat. For $i = 1, ..., n$, let $0 \neq N_i \in \mathcal{C}_i$. Then $\mathcal{C}_i = \xi_{conat}\,(N_i)$ and N_i is q-atomic. Since $N_i \in \mathcal{C}$, N_i and M share a nonzero quotient $A_i = {}^{M}/_{K_i}$ which is also q-atomic, and $\mathcal{C}_i = \xi_{conat}\,(A_i)$. Note that by Theorem 4.2, $K_1, ..., K_n$ can be chosen as cotype submodules of M. Let $K = \bigcap\limits_{i=1}^{n} K_i$ be the kernel of the morphism $M \longrightarrow \bigoplus\limits_{i=1}^{n} {}^{M}/_{K_i}$ induced by the canonical epimorphisms. If K is not superfluous in M, we can take a supplement L of K in M. Then $L + K = M$, and so $K_i + L = M$ for all $i = 1, ..., n$. Hence ${}^{M}/_{K_i} \perp_c {}^{M}/_{L}$ for all $i = 1, ..., n$. Since $\mathcal{C}' = \xi_{conat}\,({}^{M}/_{L}) \leq \mathcal{C}$ and R-conat is Boolean, ${}^{M}/_{L}$ has finite cotype dimension and there exists $i \in \{1, ...n\}$ such that ${}^{M}/_{K_i}$ and ${}^{M}/_{L}$ share a q-atomic quotient. This contradicts the fact that ${}^{M}/_{K_i} \perp_c {}^{M}/_{L}$.

2) $\Rightarrow$ 1) For $1 \leq i \leq n$, let $A_i = {}^{M}/_{K_i}$, and let $K = \bigcap\limits_{i=1}^{n} K_i$ be the kernel of the superfluous morphism $M \longrightarrow \bigoplus\limits_{i=1}^{n} {}^{M}/_{K_i}$ induced by the canonical epimorphisms.

Let $\mathcal{C} = \xi_{conat}(M)$ and, for $1 \leq i \leq n$, let $\mathcal{C}_i = \xi_{conat}({}^{M}/{}_{K_i})$. Clearly, $\bigvee_{i=1}^{n} \mathcal{C}_i \leq \mathcal{C}$. Since R-conat is Boolean, if $\bigvee_{i=1}^{n} \mathcal{C}_i \neq \mathcal{C}$, there exists $\mathbf{0} \neq \mathcal{C}' \in R$-conat such that $\mathcal{C} = \bigvee_{i=1}^{n} \mathcal{C}_i \vee \mathcal{C}'$ and $\bigvee_{i=1}^{n} \mathcal{C}_i \wedge \mathcal{C}' = \mathbf{0}$. As $\mathbf{0} \neq \mathcal{C}' \subseteq \mathcal{C}$, there exists a nonzero module $M' \in \mathcal{C}'$ which shares a nonzero quotient ${}^{M}/{}_{L}$ with M. Therefore ${}^{M}/{}_{L} \perp_c {}^{M}/{}_{K_i}$ for all $i = 1, ..., n$.

Now we shall prove by induction on n that any module N which is coorthogonal to ${}^{M}/{}_{K_i}$ for all $i = 1, ..., n$, is coorthogonal to ${}^{M}/{}_{\bigcap_{i=1}^{n} K_i}$. If $n = 1$, the assertion is clear. Assume it is true for $n = r$. Then $N \perp_c {}^{M}/{}_{\bigcap_{i=1}^{r} K_i}$, and since the ${}^{M}/{}_{K_i}$ are pairwise coorthogonal, also ${}^{M}/{}_{K_{r+1}} \perp_c {}^{M}/{}_{\bigcap_{i=1}^{r} K_i}$. It follows that $\bigcap_{i=1}^{r} K_i + K_{r+1} = M$ and thus ${}^{M}/{}_{K_{r+1}} = {}^{\bigcap_{i=1}^{r} K_i + K_{r+1}}/{}_{K_{r+1}} \cong {}^{\bigcap_{i=1}^{r} K_i}/{}_{\bigcap_{i=1}^{r+1} K_i}$. Then from the exact sequence

$$0 \longrightarrow {}^{\bigcap_{i=1}^{r} K_i}/{}_{\bigcap_{i=1}^{r+1} K_i} \longrightarrow {}^{M}/{}_{\bigcap_{i=1}^{r+1} K_i} \longrightarrow {}^{M}/{}_{\bigcap_{i=1}^{r} K_i} \longrightarrow 0$$

it follows that $N \perp_c {}^{M}/{}_{\bigcap_{i=1}^{r+1} K_i}$.

From the above paragraph we obtain that ${}^{M}/{}_{L} \perp_c {}^{M}/{}_{K}$, which gives $K + L = M$. But $K \ll M$ gives $M = L$, contradicting the fact that ${}^{M}/{}_{L} \neq 0$. $\square$

Lemma 5.8. *For an amply supplemented module M, ct.dim$(M) = \infty$ if and only if there exist an infinite number of pairwise coorthogonal nonzero quotients of M.*

Proof. $\Rightarrow$) We may assume that there is only a finite number of pairwise coorthogonal q-atomic quotients of M. Let these be ${}^{M}/{}_{K_1}, ..., {}^{M}/{}_{K_n}$. By Theorem 4.2, they can be chosen so that they are all cotype quotients of M. Since *ct.dim*$(M) = \infty$, the morphism $M \longrightarrow \bigoplus_{i=1}^{n} {}^{M}/{}_{K_i}$ induced by the canonical epimorphisms is not superfluous. Let $K = \bigcap_{i=1}^{n} K_i$ and L be a supplement of K in M. Then $K + L = M$ implies $K_i + L = M$ for all $i = 1, ..., n$, and therefore ${}^{M}/{}_{K_i} \perp_c {}^{M}/{}_{L}$ for all $i = 1, ..., n$.

Now, since $\xi_{conat}({}^{M}/{}_{L})$ is not an atom in R-conat, it contains properly a conatural class $\mathcal{C} \neq 0$. In fact, since R-conat is Boolean, $\xi_{conat}({}^{M}/{}_{L}) = \mathcal{C} \vee \mathcal{C}'$, where $\mathbf{0} \neq \mathcal{C}' \in R$-conat is such that $\mathcal{C} \wedge \mathcal{C}' = \mathbf{0}$. Let ${}^{M}/{}_{N_1}$ and ${}^{M}/{}_{N_2}$ be cotype quotients of ${}^{M}/{}_{L}$ of cotype $\mathcal{C}$ and $\mathcal{C}'$, respectively. Then ${}^{M}/{}_{N_1} \perp_c {}^{M}/{}_{N_2}$ and ${}^{M}/{}_{N_j} \perp_c {}^{M}/{}_{K_i}$ for $j = 1, 2$ and $i = 1, ..., n$. A similar argument can now be applied to $\xi_{conat}({}^{M}/{}_{N_j})$ for $j = 1, 2$.

$\Leftarrow$) Let $\{{}^{M}/{}_{N_j}\}_{j \in J}$ be an infinite family of pairwise coorthogonal nonzero quotients of M, and assume that $\xi_{conat}(M) = \bigvee_{i=1}^{n} \mathcal{C}_i$, where each $\mathcal{C}_i$ is an atom in

R-conat. Note that, for $1 \leq i \leq n$, $\mathcal{C}_i = \xi_{conat}(A_i)$ with A_i q-atomic. Now, since R-conat is Boolean, each ${}^{M}/{}_{N_j}$ has finite cotype dimension, and therefore shares a q-atomic quotient with A_i for some $i \in \{1, ..., n\}$. Since there is only a finite number of indexes i and an infinite number of indexes $j \in J$, it necessarily occurs that for some $k \neq j \in J$, ${}^{M}/{}_{N_k}$ and ${}^{M}/{}_{N_j}$ share a q-atomic quotient. This contradiction shows that $ct.dim\,(M) = \infty$. □

The next lemma contains several basic properties of the cotype dimension, which is allowed to be ∞, in this case $n + \infty = \infty$ for every $n \geq 0$.

Lemma 5.9. *Let L be a submodule of an amply supplemented module M. Then the following properties hold:*

1) *If $L \ll M$, then $ct.dim\,(M)$=$ct.dim\,({}^{M}/{}_{L})$.*

2) *$ct.dim\,(M) \leq ct.dim\,(L) + ct.dim\,({}^{M}/{}_{L})$.*

3) *If L is a cotype submodule of M, then $ct.dim\,(M) = ct.dim\,(L) + ct.dim\,({}^{M}/{}_{L})$.*

4) *If $M = \bigoplus_{i=1}^{n} M_i$, then $ct.dim\,(M) \leq \sum_{i=1}^{n} ct.dim\,(M_i)$.*

5) *Let $M = \bigoplus_{i=1}^{n} M_i$. If $M_i \perp_c M_j$ for all $i \neq j$ then $ct.dim\,(M) = \sum_{i=1}^{n} ct.dim\,(M_i)$. The converse holds if $ct.dim\,(M) < \infty$.*

6) *If $M \parallel_c {}^{M}/{}_{L}$ then $ct.dim\,(M) = ct.dim\,({}^{M}/{}_{L})$. In particular, if J is a cotype interior of L, then $ct.dim\,({}^{M}/{}_{L}) = ct.dim\,({}^{M}/{}_{J})$.*

7) *If $ct.dim\,(M) < \infty$ and $ct.dim\,(M) = ct.dim\,({}^{M}/{}_{L})$, then $M \parallel_c {}^{M}/{}_{L}$.*

Proof. 1) It is clear.

2) Let Q be a quotient of M and assume that $Q \perp_c {}^{M}/{}_{L}$. Consider the following diagram with exact row

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & {}^{M}/{}_{L} & \longrightarrow & 0 \\ & & & & \downarrow g & & & & \\ & & & & Q & & & & \end{array}$$

If $g(L) = Q$, then Q is a quotient of L. If not, we may consider the nonzero quotient ${}^{Q}/{}_{g(L)}$ and obtain an epimorphism ${}^{M}/{}_{L} \twoheadrightarrow {}^{Q}/{}_{g(L)}$ contradicting that $Q \perp_c {}^{M}/{}_{L}$. Since not every quotient of L is a quotient of M, $ct.dim\,(M) \leq ct.dim\,(L) + ct.dim\,({}^{M}/{}_{L})$.

3) If L is a cotype submodule of M with ${}^{M}/{}_{L}$ of cotype $\mathcal{C}$, then there exists $K \leq M$ such that L is a supplement of K in M and ${}^{M}/{}_{L} \perp_c {}^{M}/{}_{K}$; moreover, $L \in \mathcal{C}^c$. Now, since $L \cap K \ll L$, it follows that

$$\xi_{conat}(L) = \xi_{conat}({}^{L}/{}_{L\cap K}) = \xi_{conat}({}^{M}/{}_{K}) \subseteq \xi_{conat}(M) \wedge \mathcal{C}^c.$$

The result then follows by noting that $\xi_{conat}(L) = \xi_{conat}(M) \wedge \mathcal{C}^c$. Indeed, if $N \in \xi_{conat}(M) \wedge \mathcal{C}^c$, then any nonzero quotient Q of N shares a nonzero quotient ${}^{M}/_{J}$ with M, and ${}^{M}/_{J} \perp_c {}^{M}/_{L}$. Therefore $J + L = M$ and ${}^{M}/_{J} \cong {}^{L}/_{J\cap L}$. Hence Q shares a nonzero quotient with L, thereby proving that $N \in \xi_{conat}(L)$.

4) and 5) Let $M = \bigoplus\limits_{j=1}^{n} M_j$, and assume that, for $i \le j \le n$, $\xi_{conat}(M_j) = \bigvee\limits_{i=1}^{m_j} \xi_{conat}(A_{j_i})$, where the A_{j_i} are pairwise coorthogonal q-atomic quotients of M_j. We establish an equivalence relation on the A_{j_i}, namely, A_{j_i} is equivalent to A_{k_l} if and only if $A_{j_i} \parallel_c A_{k_l}$. Let r be the number of equivalence classes. Then $ct.dim(M) = r \le m_1 + ... + m_k$, and equality holds if and only if each equivalence class is a singleton if and only if $M_j \perp_c M_k$ whenever $j \ne k$.

6) Assume that $M \parallel_c N$, and that $\xi_{conat}(M) = \bigvee\limits_{i=1}^{n} \xi_{conat}(A_i)$ with the A_i pairwise coorthogonal q-atomic quotients of M. Then each A_i, with $1 \le i \le n$, shares a nonzero quotient B_i with N. Since the B_i are also pairwise coorthogonal and q-atomic, it follows that $ct.dim(N) \ge n = ct.dim(M)$. Similarly, $ct.dim(M) \ge ct.dim(N)$.

7) If M is not coparallel to ${}^{M}/_{L}$, there exists a nonzero quotient Q of M such that $Q \perp_c {}^{M}/_{L}$. Since M is of finite cotype dimension, so are both ${}^{M}/_{L}$ and Q. Thus we may assume that $\xi_{conat}({}^{M}/_{L}) = \bigvee\limits_{i=1}^{n} \xi_{conat}(A_i)$ and $\xi_{conat}(Q) = \bigvee\limits_{j=1}^{m} \xi_{conat}(B_j)$, where the A_i and the B_j are pairwise coorthogonal q-atomic quotients of ${}^{M}/_{L}$ and Q, respectively. Since $A_i \perp_c B_j$ for all $i \in \{1, ..., n\}$ and all $j \in \{1, ..., m\}$, it follows that $ct.dim(M) \ge n + m > n = ct.dim({}^{M}/_{L})$. This contradiction shows that $M \parallel_c {}^{M}/_{L}$. □

We end this section with the following result.

Theorem 5.10. *For any amply supplemented module M, the following are equivalent:*

1) $ct.dim(M) < \infty$.

2) *M has DCC on cotype submodules.*

Proof. 1) $\Rightarrow$ 2) Assume that $\mathcal{C} = \xi_{conat}(M)$ is the join of n atoms in R-conat.

Let $K_1 \ge ... \ge K_i \ge K_{i+1} \ge ...$ be a descending chain of cotype submodules of M. Then, for $i \ge 1$, there are epimorphisms ${}^{M}/_{K_{i+1}} \twoheadrightarrow {}^{M}/_{K_i}$, and therefore inclusions $\xi_{conat}({}^{M}/_{K_i}) \subseteq \xi_{conat}({}^{M}/_{K_{i+1}})$ of conatural subclasses of $\mathcal{C}$. As R-conat is Boolean, each element in the interval $[\mathbf{0}, \mathcal{C}]$ is the join of k atoms, where $0 \le k \le n$.

Hence $\mid [\mathbf{0},\mathcal{C}] \mid = 2^n$, and the ascending chain

$$\xi_{conat}\left({}^{M}/_{K_1}\right) \subseteq ... \subseteq \xi_{conat}\left({}^{M}/_{K_i}\right) \subseteq \xi_{conat}\left({}^{M}/_{K_{i+1}}\right) \subseteq ...$$

of conatural subclasses of $\mathcal{C}$ is stationary. Since for $i \geq 1$, K_i is minimal with respect to ${}^{M}/_{K_i} \in \xi_{conat}\left({}^{M}/_{K_i}\right)$, the descending chain $K_1 \geq ... \geq K_i \geq K_{i+1} \geq ...$ is stationary too.

2) $\Rightarrow$ 1) First we show that $\mathcal{C} = \xi_{conat}(M)$ contains only a finite number of atoms. Assume on the contrary that $\{\mathcal{C}_i\}_{i\in\mathbb{N}}$ is an infinite family of disjoint atoms contained in $\mathcal{C}$. Then we have a strictly ascending chain $\mathcal{C}_1 \subset \mathcal{C}_1 \vee \mathcal{C}_2 \subset \mathcal{C}_1 \vee \mathcal{C}_2 \vee \mathcal{C}_3 \subset ...$ of conatural classes contained in $\mathcal{C}$. For $i \in \mathbb{N}$, let ${}^{M}/_{K_i}$ be of cotype $\mathcal{C}_i$. Then $\mathcal{C}_i = \xi_{conat}\left({}^{M}/_{K_i}\right)$ and ${}^{M}/_{K_i} \perp_c {}^{M}/_{K_j}$ for $i \neq j$.

We get the strictly descending chain of submodules of M:

$$K_1 > K_1 \cap K_2 > K_1 \cap K_2 \cap K_3 > ... \quad (1)$$

Consider the following exact sequence

$$0 \to {}^{K_2}/_{K_1 \cap K_2} \to {}^{M}/_{K_1 \cap K_2} \to {}^{M}/_{K_2} \to 0$$

Since ${}^{K_2}/_{K_1\cap K_2} \cong {}^{K_1+K_2}/_{K_1} = {}^{M}/_{K_1}$, ${}^{M}/_{K_1\cap K_2} \in \mathcal{C}_1 \vee \mathcal{C}_2$ and so $\xi_{conat}\left({}^{M}/_{K_1\cap K_2}\right) \subset \mathcal{C}_1 \vee \mathcal{C}_2$. As there are epimorphisms ${}^{M}/_{K_1\cap K_2} \twoheadrightarrow {}^{M}/_{K_i}$ for $i = 1, 2$, $\mathcal{C}_1$, $\mathcal{C}_2 \subset \xi_{conat}\left({}^{M}/_{K_1\cap K_2}\right)$ and thus $\xi_{conat}\left({}^{M}/_{K_1\cap K_2}\right) = \mathcal{C}_1 \vee \mathcal{C}_2$.

Let $J_1 = K_1$ and J_2 be a cotype interior of $K_1 \cap K_2$. Then ${}^{M}/_{J_2}$ is a cotype quotient of cotype $\mathcal{C}_1 \vee \mathcal{C}_2$. In the descending chain (1), we can then replace $K_1 \cap K_2$ by J_2 and consider the following exact sequence

$$0 \to {}^{J_2}/_{J_2 \cap K_3} \to {}^{M}/_{J_2 \cap K_3} \to {}^{M}/_{J_2} \to 0$$

Since ${}^{M}/_{J_2+K_3} \in (\mathcal{C}_1 \vee \mathcal{C}_2) \wedge \mathcal{C}_3 = \mathbf{0}$, we get that ${}^{J_2}/_{J_2\cap K_3} \cong {}^{J_2+K_3}/_{K_3} = {}^{M}/_{K_3}$. Then, as above, $\xi_{conat}\left({}^{M}/_{J_2\cap K_3}\right) = \mathcal{C}_1 \vee \mathcal{C}_2 \vee \mathcal{C}_3$. Now we let J_3 be a cotype interior of $J_2 \cap K_3$, and repeat the above argument.

In this way we obtain a strictly descending chain $J_1 > J_2 > ... > J_n > J_{n+1} > ...$ of cotype submodules of M, where ${}^{M}/_{J_n}$ is of cotype $\bigvee\limits_{i=1}^{n} \mathcal{C}_i$ for all $n \in \mathbb{N}$. This contradiction shows that there is only a finite number of disjoint atoms $\mathcal{C}_1, ..., \mathcal{C}_n$ contained in $\mathcal{C}$.

Now we show that $\mathcal{C} = \bigvee\limits_{i=1}^{n} \mathcal{C}_i$. If not, $\mathcal{C} = \bigvee\limits_{i=1}^{n} \mathcal{C}_i \vee \mathcal{D}$ for some conatural class, $\mathbf{0} \neq \mathcal{D}$ such that $\mathcal{D} \wedge \bigvee\limits_{i=1}^{n} \mathcal{C}_i = \mathbf{0}$. Since the interval $[\mathbf{0}, \mathcal{D}]$ contains neither atoms nor coatoms, we obtain a strictly ascending chain $\mathcal{D}_1 \subset \mathcal{D}_2 \subset ... \subset \mathcal{D}_n \subset ... \subset \mathcal{D}$ of nonzero conatural classes contained in $\mathcal{D}$. Note that M has nonzero quotients in each $\mathcal{D}_n$. Hence we can take $K_1 \leq M$ such that ${}^{M}/_{K_1}$ is of cotype $\mathcal{D}_1$. Now,

$\mathcal{D}_2 = \mathcal{D}_1 \vee (\mathcal{D}_2 \wedge \mathcal{D}_1^c)$, and as $\mathbf{0} \neq \mathcal{D}_2 \wedge \mathcal{D}_1^c \subset \mathcal{C}$, M has a nonzero quotient ${}^{M}/{}_{L} \in \mathcal{D}_2 \wedge \mathcal{D}_1^c$. Since ${}^{M}/{}_{L} \perp_c {}^{M}/{}_{K_1}$, ${}^{M}/{}_{L}$ turns out to be a quotient of K_1, and we can take $K_2 < K_1$ such that ${}^{K_1}/{}_{K_2}$ is of cotype $\mathcal{D}_2$. By Lemma 3.6, K_2 is a cotype submodule of M. The exact sequence

$$0 \quad \to \quad {}^{K_1}/{}_{K_2} \quad \to \quad {}^{M}/{}_{K_2} \quad \to \quad {}^{M}/{}_{K_1} \quad \to \quad 0$$

shows that ${}^{M}/{}_{K_2} \in \mathcal{D}_2$.

Now we can repeat the argument above in order to obtain $K_3 < K_2$ such that ${}^{K_2}/{}_{K_3}$ is of cotype $\mathcal{D}_3$, and Lemma 3.6 shows that K_3 is a cotype submodule of M. Proceeding this way, we obtain a strictly descending chain $K_1 > K_2 > ... > K_n > ...$ of cotype submodules of M, contradicting 2). $\square$

We note that statement 1) of the above theorem implies that M has *ACC* on cotype submodules (the proof is similar to that of 1) $\Rightarrow$ 2)).

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Alejandro Alvarado-García, **Hugo Alberto Rincón-Mejía**, **Bertha Tomé-Arreola**
Facultad de Ciencias
Universidad Nacional Autónoma de México
Circuito Exterior, C.U.
04510 México D.F.
e-mails: alejandroalvaradogarcia@gmail.com (A. Alvarado)
hurincon@gmail.com (H. Rincón)
bta@hp.fciencias.unam.mx (B. Tomé)

José Ríos-Montes
Instituto de Matemáticas
Universidad Nacional Autónoma de México
Área de la Investigación Científica
Circuito Exterior, C.U.
04510 México D.F.
e-mail: jrios@matem.unam.mx