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A REIDEMEISTER-SCHREIER THEOREM FOR FINITELY L-PRESENTED GROUPS

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Abstract. We prove a variant of the well-known Reidemeister-Schreier Theorem for finitely L-presented groups. More precisely, we prove that each finite index subgroup of a finitely L-presented group is itself finitely L-presented. Our proof is constructive and it yields a finite L-presentation for the subgroup. We further study conditions on a finite index subgroup of an invariantly finitely L-presented group to be invariantly L-presented itself.

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1. Introduction

Group presentations play an important role in computational group theory. In particular finite presentations have been subject to extensive research in computational group theory dating back to the early days of computer-algebra-systems [21, 9,25,32]. Group presentations, on the one hand, provide an effective description of the group. On the other hand, a description of a group by its generators and relations leads to various decision problems which are known to be unsolvable in general [23]. For instance, the word problem of a finitely presented group is unsolvable [29,7]. However, various total and partial algorithms for finitely presented groups are known [32]. For instance, the coset-enumeration process introduced by Todd and Coxeter [33] enumerates the cosets of a subgroup in a finitely presented group. If the subgroup has finite index, coset-enumeration terminates and it computes a permutation representation for the group's action on the cosets. Coset-enumeration is a partial algorithm as the process will not terminate if the subgroup has infinite index. However, finite presentations often allow total algorithms that compute factor groups of special type (including abelian quotients, nilpotent quotients [27] and, in general, solvable quotients [22]).

Beside quotient and subgroup methods, the well-known theorem by Reidemeister [30] and Schreier [31] allows one to compute a presentation for a subgroup. The Reidemeister-Schreier Theorem explicitly shows that a finite index subgroup of a finitely presented group is itself finitely presented. A similar result can be shown for finite index ideals in finitely presented semi-groups [8]. In practice, a permutation representation for the group's action on the cosets allows one to compute the Schreier generators of the subgroup and the Reidemeister rewriting. The Reidemeister rewriting can be used to rewrite the relations of the group to relations of the subgroup [17,32,23]. A method for computing a finite presentation for a finite index subgroup can be applied in the investigation of the structure of a group by its finite index subgroups [18].

Even though finitely presented groups have been studied for a long time, most groups are not finitely presented because there are uncountably many two-generator groups [26] but only countably many finite presentations [1]. A generalization of finite presentations are *finite L-presentations* which were introduced in [1]; however, there are still only countably many finite L-presentations. It is known that various examples of self-similar or branch groups (including the Grigorchuk group [11] and its twisted twin [4]) are finitely L-presented but not finitely presented [1]. Finite L-presentations are possibly infinite presentations with finitely many generators whose relations (up to finitely many exceptions) are obtained by iteratively applying finitely many substitutions to a finite set of relations; see [1] or Section 2 for a definition. A finite L-presentation is *invariant* if the substitutions which generate the relations induce endomorphisms of the group. In fact, invariant finite L-presentations are finite presentations in the universe of *groups with operators* [20,28] in the sense that the operator domain of the group generates the possibly infinitely many relations out of a finite set of relations.

Finite L-presentations allow computer algorithms to be applied in the investigation of the groups they define. For instance, they allow to compute the lower central series quotients [2], the Dwyer quotients of the group's Schur multiplier [15], and even a coset-enumeration process exists for finitely L-presented groups [16]. It is the aim of this paper to prove the following variant of the well-known Reidemeister-Schreier Theorem:

Theorem 1.1. *Each finite index subgroup of a finitely L-presented group is finitely L-presented.*

If the finite index subgroup in Theorem 1.1 is normal and invariant under the substitutions (i.e., a normal and admissible subgroup in the notion of Krull &

Noether [20,28]), an easy argument gives a finite L-presentation for the subgroup; furthermore, if the group is invariantly finitely L-presented, so is the subgroup. However, more work is needed if the subgroup is not invariant under the substitutions. Under either of two extra conditions (the subgroup is *leaf-invariant*, see Definition 5.7; or it is normal and *weakly leaf-invariant*, see Definition 7.2), we show that the subgroup is invariantly finitely L-presented as soon as the group is. We have not been able to get rid of these extra assumptions. In particular, it is not clear whether a finite index subgroup of an invariantly finitely L-presented group is always invariantly finitely L-presented. We show that the methods presented in this paper will (in general) fail to compute invariant L-presentations for the subgroup even if the group is invariantly L-presented. However, we are not aware of a method to prove that a given subgroup does not admit an invariant finite L-presentation at all.

Our proof of Theorem 1.1 is constructive and it yields a finite L-presentation for the subgroup. These finite L-presentations can be applied in the investigation of the underlying groups as the methods in [18] suggest for finitely presented groups. Notice that Theorem 1.1 was already posed in Proposition 2.9 of [1]. The proof we explain in this paper follows the sketch given in [1], but fixes a gap as the L-presentation of the group in Theorem 1.1 is possibly non-invariant. Even if the L-presentation is assumed to be invariant, the considered subgroup cannot be assumed to be invariant under the substitutions.

This paper is organized as follows: In Section 2, we recall the notion of a finite L-presentation and we recall basic group theoretic constructions which preserve the property of being (invariantly) finitely L-presented. Then, in Section 3, we recall the well-known Reidemeister-Schreier process. Before we prove Theorem 1.1 in Section 6, we construct, in Section 4, a counter-example to the original proof of Theorem 1.1 in [1]. Then, in Section 5, we introduce the stabilizing subgroups which are the main tools in our proof of Theorem 1.1. In Section 7, we study conditions on the finite index subgroup of an invariantly L-presented group to be invariantly L-presented itself. We conclude this paper by considering two examples of subgroup L-presentations in Section 8 including the normal closure of a generator d of the Grigorchuk group $\mathfrak{G}$ as in [3,10]. We fix a mistake in the generating set of the normal closure $D = \langle d \rangle^{\mathfrak{G}}$ using our Reidemeister-Schreier Theorem for finitely L-presented groups. In particular we show, in the style of [18], how these computational methods can be applied in the investigation of self-similar groups.

2. Preliminaries

In the following, we briefly recall the notion of a finite L-presentation and the notion a finitely L-presented group as introduced in [1]. Moreover, we recall some basic constructions for finite L-presentations.

A *finite L-presentation* is a group presentation of the form

$$\Big\langle \mathcal{X} \,\Big|\, \mathcal{Q} \cup \bigcup_{\sigma\in\Phi^*} \mathcal{R}^\sigma \Big\rangle, \tag{1}$$

where $\mathcal{X}$ is a finite alphabet, $\mathcal{Q}$ and $\mathcal{R}$ are finite subsets of the free group F over $\mathcal{X}$, and $\Phi^* \subseteq \mathrm{End}(F)$ denotes the free monoid of endomorphisms which is finitely generated by Φ. We also write $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R}\rangle$ for the finite L-presentation in Eq. (1) and $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R}\rangle$ for the group it defines.

A group which admits a finite L-presentation is *finitely L-presented*. An L-presentation of the form $\langle \mathcal{X} \mid \emptyset \mid \Phi \mid \mathcal{R}\rangle$ is *ascending* and an L-presentation $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R}\rangle$ is *invariant* (and the group it presents is *invariantly L-presented*), if each endomorphism $\varphi \in \Phi$ induces an endomorphism of the group; that is, if the normal subgroup $\langle \mathcal{Q} \cup \bigcup_{\sigma\in\Phi^*} \mathcal{R}^\sigma\rangle^F \leq F$ is φ-invariant. Each ascending L-presentation is invariant and each invariant L-presentation $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R}\rangle$ admits an ascending L-presentation $\langle \mathcal{X} \mid \emptyset \mid \Phi \mid \mathcal{Q}\cup\mathcal{R}\rangle$ which defines the same group. On the other hand, we have the following

Proposition 2.1. *There are finite L-presentations that are not invariant.*

Proof. The group $B = \langle \{a,b,t\} \mid \{a^t\, a^{-4}, b^{t^{-1}}\, b^{-2}, [a, b^{t^i}] \mid i \in \mathbb{Z}\}\rangle$ is a metabelian, infinitely related group with trivial Schur multiplier [6]. By introducing a *stable letter* u, this group admits the finite L-presentation

$$\langle \{a,b,t,u\} \mid \{ub^{-1}\} \mid \{\sigma,\delta\} \mid \{a^t a^{-4}, b^{t^{-1}} b^{-2}, [a,u]\}\rangle,$$

where σ is the free group homomorphism induced by the map $\sigma\colon a \mapsto a$, $b \mapsto b$, $t \mapsto t$, and $u \mapsto u^t$, while δ is the free group homomorphism induced by the map $\delta\colon a \mapsto a$, $b \mapsto b$, $t \mapsto t$, and $u \mapsto u^{t^{-1}}$. This finite L-presentation is not invariant [14]. □

Another non-invariant L-presentation can be given for the free product $\mathbb{Z}_2 * \mathbb{Z}_2 = \langle \{a,b\} \mid \{a^2,b^2\}\rangle$: it is finitely L-presented by $\langle \{a,b\} \mid \{a^2\} \mid \{\sigma\} \mid \{b^2\}\rangle$ where σ is induced by the map $a \mapsto ab$ and $b \mapsto b^2$. If this latter L-presentation were invariant, the ascending finite L-presentation $\langle \{a,b\} \mid \emptyset \mid \{\sigma\} \mid \{a^2,b^2\}\rangle$ would also define $\mathbb{Z}_2 * \mathbb{Z}_2$. In this case $(a^2)^\sigma = abab$ is a relation and, since $a^2 = b^2 = 1$ in the group, the generators a and b commute. Thus the latter ascending finite

L-presentation defines a quotient of the 2-elementary abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2$. In particular, it defines a finite group. Hence, $\langle \{a,b\} \mid \emptyset \mid \{\sigma\} \mid \{a^2,b^2\}\rangle$ is not a finite L-presentation for $\mathbb{Z}_2 * \mathbb{Z}_2$ and so $\langle \{a,b\} \mid \{a^2\} \mid \{\sigma^2\} \mid \{b^2\}\rangle$ is not an invariant L-presentation.

We are not aware of a method to decide whether or not a given (non-ascending) finite L-presentation is invariant. In particular, we have no answer to the following

Question 1. *Is there a finitely L-presented group so that each of its finite L-presentation is not invariant?*

The class of finitely L-presented groups contains all finitely presented groups:

Proposition 2.2. *Each finitely presented group $\langle \mathcal{X} \mid \mathcal{R}\rangle$ is finitely L-presented by the invariant (ascending) finite L-presentation $\langle \mathcal{X} \mid \emptyset \mid \emptyset \mid \mathcal{R}\rangle$.*

Therefore, (invariant or ascending) finite L-presentations generalize the concept of finite presentations. Examples of finitely L-presented, but not finitely presented, groups are various self-similar or branch groups [1] including the Grigorchuk group [11,24,12] and its twisted twin [4]. However, the concept of a finite L-presentation is quite general so that other examples of infinitely presented groups are finitely L-presented [6,19].

Various group theoretic constructions that preserve the property of being finitely L-presented have been studied in [1]. For completeness, we recall some of these constructions in the remainder of this section.

Proposition 2.3 (Bartholdi [1, Proposition 2.7])**.** *Let $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R}\rangle$ be a finitely L-presented group and let $H = \langle \mathcal{Y} \mid \mathcal{S}\rangle$ be finitely presented. The group K which satisfies the short exact sequence $1 \to G \to K \to H \to 1$ is finitely L-presented.*

Proof. We recall the constructions from [1]: Let $\delta\colon H \to K$ be a section of H to K and identify G with its image in K. Each relation $r \in \mathcal{S}$ of the finitely presented group H lifts, through the section δ, to an element $g_r \in G$. As the group G is normal in K, each generator $t \in \mathcal{Y}$ of the finitely presented group H acts, via δ, on the subgroup G. Thus we have $x^{\sigma(t)} = g_{x,t} \in G$ for each $x \in \mathcal{X}$ and $t \in \mathcal{Y}$. If $\mathcal{X} \cap \mathcal{Y} = \emptyset$, we consider the finite L-presentation

$$\langle \mathcal{X} \cup \mathcal{Y} \mid \mathcal{Q} \cup \{r\, g_r^{-1} \mid r \in \mathcal{S}\} \cup \{x^t g_{x,t}^{-1} \mid x \in \mathcal{X}, t \in \mathcal{Y}\} \mid \widehat{\Phi} \mid \mathcal{R}\rangle, \tag{2}$$

where the endomorphisms Φ of G's L-presentation are extended to endomorphisms $\widehat{\Phi} = \{\widehat{\sigma} \mid \sigma \in \Phi\}$ of the free group $F(\mathcal{X} \cup \mathcal{Y})$ by

$$\widehat{\sigma}\colon F(\mathcal{X} \cup \mathcal{Y}) \to F(\mathcal{X} \cup \mathcal{Y}), \begin{cases} x \mapsto x^{\sigma}, & \text{for each } x \in \mathcal{X} \\ y \mapsto y, & \text{for each } y \in \mathcal{Y}. \end{cases}$$

Then the finite L-presentation in Eq. (2) is a presentation for K; see [1]. □

As each finite group is finitely presented, Proposition 2.3 yields the immediate

Corollary 2.4. *Each finite extension of a finitely L-presented group is finitely L-presented.*

The construction in the proof of Proposition 2.3 gives a finite L-presentation for K which is not ascending – even if the group G we started with has an ascending L-presentation. We therefore ask the following

Question 2. *Is every finite extension of an invariantly (finitely) L-presented group invariantly (finitely) L-presented?*

We do not have an answer to this question in general; though we suspect its answer is negative, see Remark 7.8. Given endomorphisms Φ of the normal subgroup G in Proposition 2.3, one problem is to construct endomorphisms of the finite extension K which restrict to Φ. This does not seem to be possible in general.

A finite L-presentation for a free product of two finitely L-presented groups is given by the following improved version of [1, Proposition 2.6].

Proposition 2.5. *The free product of two finitely L-presented groups is finitely L-presented. If both finitely L-presented groups are invariantly L-presented, so is their free product.*

Proof. Although a proof of the first statement can be found in [1], we summarize its construction for our proof of the second statement. For this purpose, let $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ and $H = \langle \mathcal{Y} \mid \mathcal{S} \mid \Psi \mid \mathcal{T} \rangle$ be finitely L-presented groups. Suppose that $\mathcal{X} \cap \mathcal{Y} = \emptyset$ holds. Then, by [1], the free product $G * H$ is finitely L-presented by $\langle\, \mathcal{X} \cup \mathcal{Y} \mid \mathcal{Q} \cup \mathcal{S} \mid \widehat{\Phi} \cup \widehat{\Psi} \mid R \cup \mathcal{T}\, \rangle$ where the endomorphisms in Φ and Ψ are extended to endomorphisms in $\widehat{\Phi}$ and $\widehat{\Psi}$ of the free group $F(\mathcal{X} \cup \mathcal{Y})$ over $\mathcal{X} \cup \mathcal{Y}$ as follows: for each $\sigma \in \Phi$, we let

$$\widehat{\sigma}\colon F(\mathcal{X} \cup \mathcal{Y}) \to F(\mathcal{X} \cup \mathcal{Y}), \begin{cases} x \mapsto x^{\sigma}, & \text{for each } x \in \mathcal{X} \\ y \mapsto y, & \text{for each } y \in \mathcal{Y}; \end{cases}$$

and, accordingly, for each $\delta \in \Psi$.

Suppose that the L-presentations of G and H are invariant. As an invariant L-presentation $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R}\rangle$ can be considered as an ascending L-presentation $\langle \mathcal{X} \mid \emptyset \mid \Phi \mid \mathcal{Q}\cup\mathcal{R}\rangle$, we can consider $\mathcal{Q}$ and $\mathcal{S}$ to be empty. Then the construction above shows that the free product $G * H$ is ascendingly finitely L-presented and thus it is invariantly finitely L-presented. □

We further have the following improved version of [1, Proposition 2.9]:

Proposition 2.6. *Let $N \trianglelefteq G$ be a normal subgroup of a finitely L-presented group $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R}\rangle$. If N is finitely generated as a normal subgroup, the factor group G/N is finitely L-presented. If, furthermore, G is invariantly L-presented and the normal subgroup N is invariant under the induced endomorphisms Φ, G/N is invariantly L-presented.*

Proof. Let $N = \langle g_1, \ldots, g_n\rangle^G$ be a finite normal generating set of the normal subgroup N. We consider the normal generators $g_1, \ldots, g_n$ as elements of the free group F over $\mathcal{X}$. By [1], the L-presentation $\langle \mathcal{X} \mid \mathcal{Q}\cup\{g_1,\ldots,g_n\} \mid \Phi \mid \mathcal{R}\rangle$ is a finite L-presentation for the factor group G/N.

Suppose that G is given by an invariant L-presentation $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R}\rangle$. Then $G = \langle \mathcal{X} \mid \emptyset \mid \Phi \mid \mathcal{Q}\cup\mathcal{R}\rangle$. As $N^\sigma \subseteq N$, each $\sigma \in \Phi^*$ induces an endomorphism of the L-presented factor group G/N. Thus the images $g_1^\sigma, \ldots, g_n^\sigma$ are consequences of the relations of G/N's finite L-presentation above. Hence, we have that $G/N \cong \langle \mathcal{X} \mid \{g_1,\ldots,g_n\} \mid \Phi \mid \mathcal{R}\cup\mathcal{Q}\rangle = \langle \mathcal{X} \mid \emptyset \mid \Phi \mid \mathcal{Q}\cup\mathcal{R}\cup\{g_1,\ldots,g_n\}\rangle$. □

If G is invariantly L-presented and N is a normal Φ-invariant subgroup, then, in the notion of Krull & Noether [20,28], the group G is a *group with operator domain* Φ and the normal subgroup N is an *admissible subgroup.* Proposition 2.5 and Proposition 2.6 yield the following straightforward

Corollary 2.7. *Let G and H be finitely L-presented groups and let F be a finitely generated group with embeddings $\psi\colon F \to G$ and $\phi\colon F \to H$. Then the amalgamated free product $G *_F H$ is finitely L-presented.*

For further group theoretic constructions which preserve the property of being finitely L-presented were refer to [1].

3. The Reidemeister-Schreier Process

In the following, we briefly recall the Reidemeister-Schreier process as, for instance, outlined in [23,32]. For this purpose, let G be a group given by a group presentation $\langle \mathcal{X} \mid \mathcal{K}\rangle$ where $\mathcal{X}$ is a (finite) alphabet which defines the free group F

and $\mathcal{K} \subseteq F$ is a (possibly infinite) set of relations. Denote the normal closure of $\mathcal{K}$ in F by $K = \langle \mathcal{K} \rangle^F$. Then $G = F/K$.

Let $H \leq G$ be a subgroup with finite index that is given by its generators $g_1, \ldots, g_n$ and let $\mathcal{T} \subseteq F$ be a Schreier transversal for H in G (i.e., a transversal for H in G so that every initial segment of an element of $\mathcal{T}$ itself belongs to $\mathcal{T}$, see [23]; note that we always act by multiplication from the right). We consider the generators of H as words over the alphabet $\mathcal{X}$ and thus as elements of the free group F. Then the subgroup $U = \langle g_1, \ldots, g_n \rangle$ satisfies that $H \cong UK/K$. In the style of [23], we define the *Schreier map* $\gamma \colon \mathcal{T} \times \mathcal{X} \to F$ by $\gamma(t,x) = tx\,(\overline{tx})^{-1}$ where $\overline{tx}$ denotes the unique element $s \in \mathcal{T}$ from the Schreier transversal so that $UK\,s = UK\,tx$ holds. The Schreier Theorem (as, for instance, in [23, Proposition I.3.7]) shows that the subgroup $UK \leq F$ is freely generated by the *Schreier generating set*

$$\mathcal{Y} = \{\gamma(t,x) \neq 1 \mid t \in \mathcal{T}, x \in \mathcal{X}\}.$$

In particular, it shows that each finite index subgroup of a finitely generated group is finitely generated. We consider the set $\mathcal{Y}$ as an alphabet and we denote by $F(\mathcal{Y})$ the free group over $\mathcal{Y}$. The *Reidemeister rewriting* τ is a map $\tau \colon F \to F(\mathcal{Y})$ given by

$$\tau(y_1 \cdots y_n) = \gamma(1, y_1) \cdot \gamma(\overline{y_1}, y_2) \cdots \gamma(\overline{y_1 \cdots y_{n-1}}, y_n)$$

where each $y_i \in \mathcal{X} \cup \mathcal{X}^-$. In general, the Reidemeister rewriting τ is not a group homomorphism. However, we have the following

Lemma 3.1. *For $V \leq UK$, the restriction $\tau \colon V \to F(\mathcal{Y})$ is a homomorphism.*

Proof. Let $g, h \in V$ be given. Write $g = g_1 \cdots g_n$ and $h = h_1 \cdots h_m$ with each $h_i, g_j \in \mathcal{X} \cup \mathcal{X}^-$. Then, as $\overline{g_1 \cdots g_n} = \overline{g} = 1$ holds, we obtain that

$$\tau(gh) = \gamma(1, g_1) \cdots \gamma(\overline{g_1 \cdots g_{n-1}}, g_n) \cdot \gamma(1, h_1) \cdots \gamma(\overline{h_1 \cdots h_{m-1}}, h_m) = \tau(g)\,\tau(h)$$

while we already have $\tau(1) = 1$ by definition. $\square$

By Schreier's theorem, the Reidemeister rewriting $\tau \colon UK \to F(\mathcal{Y})$ gives an isomorphism of free groups. A group presentation for the subgroup $H \cong UK/K$ is given by the following well-known theorem of Reidemeister [30] and Schreier [31]; see also [23, Section II.4].

Theorem 3.2 (Reidemeister-Schreier Theorem)**.** *Let H be a subgroup of G. If τ denotes the Reidemeister-Schreier rewriting, $\mathcal{T}$ denotes a Schreier transversal for H in G, and if $\langle \mathcal{X} \mid \mathcal{K} \rangle$ is a presentation for G, the subgroup H is presented by*

$$H \cong \left\langle \mathcal{Y} \mid \{\tau(trt^{-1}) \mid r \in \mathcal{K}, t \in \mathcal{T}\} \right\rangle. \tag{3}$$

Proof. We recall the proof for completeness: Note that $H \cong UK/K \cong \tau(UK)/\tau(K)$ holds. By Schreier's theorem, we have $\tau(UK) = F(\mathcal{Y})$. It therefore suffices to determine a normal generating set for $\tau(K)$. As $\mathcal{K}$ is a normal generating set for $K \trianglelefteq F$, a generating set for $\tau(K)$ is given by $\tau(K) = \langle\{\tau(grg^{-1}) \mid r \in \mathcal{K}, g \in F\}\rangle$. Let $r \in \mathcal{K}$ and $g \in F$ be given. Consider the relation $\tau(grg^{-1})$. Since $\mathcal{T}$ is a transversal for UK in F, $g \in F$ can be written as $g = u\,t$ with $t \in \mathcal{T}$ and $u \in UK$. This yields $\tau(grg^{-1}) = \tau(utrt^{-1}u^{-1})$. For $r \in \mathcal{K}$, we have that $trt^{-1} \in UK$. By Lemma 3.1, we obtain that $\tau(grg^{-1}) = \tau(utrt^{-1}u^{-1}) = \tau(u)\,\tau(trt^{-1})\,\tau(u)^{-1}$. Therefore the relation $\tau(grg^{-1})$ is a consequence of $\tau(trt^{-1})$. Hence, it suffices to consider the normal generating set $\tau(K) = \left\langle\{\tau(trt^{-1}) \mid r \in \mathcal{K}, t \in \mathcal{T}\}\right\rangle^{F(\mathcal{Y})}$ for $\tau(K)$. □

If H is a finite index subgroup of a finitely presented group G, there exist a finite set of relations $\mathcal{K}$ and a finite Schreier transversal $\mathcal{T}$ so that the subgroup H is finitely presented by Theorem 3.2. This latter result for finitely presented groups is well-known and it is often simply referred to the Reidemeister-Schreier Theorem for finitely presented groups. In this paper, we prove a variant of the Reidemeister-Schreier Theorem for finitely L-presented groups using the ideas of Theorem 3.2.

4. A Typical Example of a Subgroup L-Presentation

Before proving Theorem 1.1, we first consider an example of a finite L-presentation for a finite index subgroup of a finitely L-presented group. For this purpose we consider a subgroup of the Basilica group [13]. The Basilica group satisfies the following

Proposition 4.1 (Bartholdi & Virág [5]). *The Basilica group G is invariantly finitely L-presented by $G \cong \langle\{a,b\} \mid \emptyset \mid \{\sigma\} \mid \{[a,a^b]\}\rangle$ where σ is the free group homomorphism that is induced by the map $a \mapsto b^2$ and $b \mapsto a$.*

The substitution σ in Proposition 4.1 induces an endomorphism of G. The group G will often provide an exclusive (counter-) example throughout this paper.

Consider the subgroup $H = \langle a, bab^{-1}, b^3\rangle$ of the Basilica group. Then coset-enumeration for finitely L-presented groups [16] shows that H is a normal subgroup of G with index 3. A Schreier generating set for the subgroup H is given by $\{a, bab^{-1}, b^2ab^{-2}, b^3\}$. Write $x_1 = a$, $x_2 = bab^{-1}$, $x_3 = b^2ab^{-2}$, and $x_4 = b^3$. Denote the free group over $\{a,b\}$ by F and let E denote the free group over $\{x_1, x_2, x_3, x_4\}$. For each $n \in \mathbb{N}_0$, we define $a_n = (2^n + 2)/3$ and $b_n = (2^n + 1)/3$. Then the σ-images of the relation $r = [a, a^b]$ can be rewritten with the Reidemeister rewriting

$\tau\colon F \to E$. Their images have the form

$$\tau(r^{\sigma^{2n}}) = \begin{cases} \left[x_1^{2^n},\, x_4^{-a_n}\, x_3^{2^n}\, x_4^{a_n}\right], & \text{if } n \text{ is even,}\\ \left[x_1^{2^n},\, x_4^{-b_n}\, x_2^{2^n}\, x_4^{b_n}\right], & \text{if } n \text{ is odd,}\end{cases}$$

and

$$\tau(r^{\sigma^{2n+1}}) = \begin{cases} x_4^{-b_{n+1}} x_2^{-2^n} x_4^{-b_{n+1}} x_3^{2^n}\, x_4^{b_{n+1}}\, x_2^{-2^n}\, x_4^{b_{n+1}}\, x_1^{2^n}, & \text{if } n \text{ is even,}\\ x_4^{-a_{n+1}} x_3^{-2^n}\, x_4^{-a_{n+1}+1} x_2^{2^n}\, x_4^{a_{n+1}-1}\, x_3^{-2^n}\, x_4^{a_{n+1}}\, x_1^{2^n}, & \text{if } n \text{ is odd.}\end{cases}$$

Note that $\tau(r^{\sigma^{2n}}) \in [E,E]$ while $\tau(r^{\sigma^{2n+1}}) \notin [E,E]$. Therefore, the images $\tau(r^{\sigma^i})$ split into two classes which are recursive images of the endomorphism

$$\widehat{\sigma}\colon E \to E, \begin{cases} x_1 & \mapsto & x_1^2,\\ x_2 & \mapsto & x_3^2,\\ x_3 & \mapsto & x_4\, x_2^2\, x_4^{-1},\\ x_4 & \mapsto & x_4^2;\end{cases}$$

in the sense that $\widehat{\sigma}$ satisfies

$$\tau(r^{\sigma^{2n}}) = [x_1, x_4^{-1} x_3\, x_4]^{\widehat{\sigma}^n} \text{ and } \tau(r^{\sigma^{2n+1}}) = (x_4^{-1} x_2^{-1} x_4^{-1} x_3\, x_4\, x_2^{-1} x_4\, x_1)^{\widehat{\sigma}^n},$$

for each $n \in \mathbb{N}_0$. In Section 8, we show that a finite L-presentation for the subgroup H is given by

$$H \cong \left\langle \{x_1,\ldots,x_4\} \,\middle|\, \emptyset \,\middle|\, \{\widehat{\sigma},\delta\} \,\middle|\, \{[x_1, x_4^{-1} x_3\, x_4], x_4^{-1} x_2^{-1} x_4^{-1} x_3\, x_4\, x_2^{-1} x_4\, x_1\}\right\rangle$$

where the endomorphism δ is induced by the map

$$\delta\colon E \to E, \begin{cases} x_1 & \mapsto & x_2,\\ x_2 & \mapsto & x_3,\\ x_3 & \mapsto & x_4\, x_1\, x_4^{-1},\\ x_4 & \mapsto & x_4.\end{cases}$$

These subgroup L-presentations are typical for finite index subgroups of a finitely L-presented group. Besides, the subgroup H and its subgroup L-presentation provide a counter-example to the original proof of Theorem 1.1 in [1] as there is no endomorphism ε of the free group E such that $\tau(r^{\sigma^{n+1}}) = (\tau(r^{\sigma^n}))^{\varepsilon}$ holds for each $n \in \mathbb{N}_0$. A reason for the failure of the proof in [1] is that the subgroup H is not σ-invariant but σ^2-invariant. Therefore, the method suggested in the proof of [1, Proposition 2.9] will fail to compute a finite L-presentation for H.

5. Stabilizing Subgroups

In this section, we introduce the stabilizing subgroups. These subgroups will be central to what follows.

Let $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R}\rangle$ be a finitely L-presented group and let $H \leq G$ be a finite index subgroup which is generated by $g_1, \ldots, g_n$. Denote the free group over $\mathcal{X}$ by F and let $K = \langle \mathcal{Q} \cup \bigcup_{\sigma\in\Phi^*} \mathcal{R}^\sigma\rangle^F$. We consider the generators $g_1, \ldots, g_n$ of the subgroup H as words over the alphabet $\mathcal{X} \cup \mathcal{X}^-$. Then the subgroup $U = \langle g_1, \ldots, g_n\rangle \leq F$ satisfies $H \cong UK/K$. The group F acts on the right-cosets $UK\backslash F$ by multiplication from the right. Let $\pi\colon F \to \mathrm{Sym}(UK\backslash F)$ be a permutation representation for the group's action on $UK\backslash F$. Such a permutation representation can be computed with the coset-enumeration process from [16]. The kernel of the permutation representation π is the normal core, $\mathrm{Core}_F(UK)$, of UK in F; i.e., it is the largest normal subgroup of F that is contained in UK.

The following definition introduces the stabilizing subgroups of H. These subgroups will be central to our proof of Theorem 1.1 in Section 6.

Definition 5.1. Let $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R}\rangle$ be a finitely L-presented group and let $H \leq G$ be a finite index subgroup which admits the permutation representation $\pi\colon F \to \mathrm{Sym}(UK\backslash F)$ as above. The *stabilizing subgroup* of H is

$$\tilde{L} = \bigcap_{\sigma\in\Phi^*} (\sigma\pi)^{-1}(\mathrm{Stab}_{\mathrm{Sym}(UK\backslash F)}(UK\,1)) = \bigcap_{\sigma\in\Phi^*} \sigma^{-1}(UK). \tag{4}$$

The *stabilizing core* of H is

$$L = \bigcap_{\sigma\in\Phi^*} \ker(\sigma\pi). \tag{5}$$

For $\sigma \in \Phi^*$, we denote by $\|\sigma\|$ the usual word-length in the generating set Φ of the free monoid Φ^*. The free monoid Φ^* has the structure of a $|\Phi|$-regular tree with its root being the identity map $\mathrm{id}\colon F \to F$. We can further endow the monoid Φ^* with a length-plus-(from the right)-lexicographic ordering $\prec$ by choosing an arbitrary ordering on the finite generating set Φ. We then define $\sigma \prec \delta$ if $\|\sigma\| < \|\delta\|$ or, otherwise, if $\sigma = \sigma_1 \cdots \sigma_n$ and $\delta = \delta_1 \cdots \delta_n$, with each $\sigma_i, \delta_j \in \Phi$, and there exists a positive integer $1 \leq k \leq n$ such that $\sigma_i = \delta_i$ for each $k < i \leq n$, and $\sigma_k \prec \delta_k$. Since Φ is finite, the constructed ordering $\prec$ is a well-ordering on Φ^* [32]. Thus, there is no infinite descending sequences $\sigma_1 \succ \sigma_2 \succ \ldots$ in Φ^*.

We consider a variation of the algorithm IsValidPermRep from [16] in Algorithm 1 below. If $\pi\colon F \to \mathrm{Sym}(UK\backslash F)$ denotes a permutation representation as

ITERATINGENDOMORPHISMS($\mathcal{X}$, $\mathcal{Q}$, Φ, $\mathcal{R}$, H, π)
  Choose an ordering on $\Phi = \{\phi_1, \ldots, \phi_n\}$ with $\phi_i \prec \phi_{i+1}$.
  Initialize $\mathcal{S} := [\phi_1, \ldots, \phi_n]$ and $\mathcal{V} := [\mathrm{id}\colon F \to F]$.
  while $\mathcal{S} \neq [\ ]$ **do**
    Remove the first entry δ from the list $\mathcal{S}$.
    if not $(\exists\, \sigma \in \mathcal{V}\colon\ \delta\pi = \sigma\pi)$ **then**
      Append $\phi_1\delta, \ldots, \phi_n\delta$ to the list $\mathcal{S}$.
      Add δ to the list $\mathcal{V}$.
  return($\mathcal{V}$)

Algorithm 1: Computing a finite set of endomorphisms $\mathcal{V} \subseteq \Phi^*$.

in Definition 5.1, the algorithm ITERATINGENDOMORPHISMS returns a finite image of a section of the map $\Phi^* \to \mathrm{Hom}(F, \mathrm{Sym}(UK\backslash F))$ defined by $\sigma \mapsto \sigma\pi$; see Lemma 5.2 and Lemma 5.4 below. More precisely, we have the following

Lemma 5.2. *The algorithm* ITERATINGENDOMORPHISMS *terminates and it returns a finite set of endomorphisms* $\mathcal{V} \subseteq \Phi^*$ *satisfying the following property: For each* $\sigma \in \Phi^*$ *there exists a unique* $\tau \in \mathcal{V}$ *so that* $\sigma\pi = \tau\pi$*. The element* $\tau \in \mathcal{V}$ *is minimal with respect to the ordering* $\prec$ *constructed above.*

Proof. Let $\mathcal{X}$ be a basis of the free group F. A homomorphism $\psi\colon F \to \mathrm{Sym}(UK\backslash F)$ is uniquely defined by the image of this basis. Since $UK\backslash F$ is finite, the symmetric group $\mathrm{Sym}(UK\backslash F)$ is finite. Moreover, as F is finitely generated, the set of homomorphisms $\mathrm{Hom}(F, \mathrm{Sym}(UK\backslash F))$ is finite. Therefore, the algorithm ITERATINGENDOMORPHISMS can add only finitely many elements to $\mathcal{V}$ and the stack $\mathcal{S}$ will eventually be reduced. Thus the algorithm terminates.

The ordering $\prec$ on Φ extends to a total well-ordering on the free monoid Φ^* as described above. The elements in the stack $\mathcal{S}$ are always ordered with respect to $\prec$. They further always succeed those elements in $\mathcal{V}$. In particular, the elements $\sigma \in \mathcal{V} \subseteq \Phi^*$ are $\prec$-minimal representatives of the composed homomorphism $\sigma\pi\colon F \to \mathrm{Sym}(UK\backslash F)$.

Let $\sigma \in \Phi^*$ be given and write $\sigma_1 = \sigma$. There exists $w \in \Phi^*$ minimal subject to the existence of $\delta \in \mathcal{V}$ so that $\sigma_1 = w\delta$. If $\|w\| = 0$ holds, then $\sigma_1 \in \mathcal{V}$ and the claim is proved. Otherwise, there exists $\psi \in \Phi$ so that $\sigma_1 = v\psi\delta$ for some $v \in \Phi^*$ and $\psi\delta \notin \mathcal{V}$. Our algorithm yields the existence of $\varepsilon \in \mathcal{V}$ so that $\varepsilon \prec \psi\delta$ and $\psi\delta\pi = \varepsilon\pi$. We also have that $\sigma_2 = v\varepsilon \prec v\psi\delta = \sigma_1$. This rewriting process yields a descending sequence $\sigma_1 \succ \sigma_2 \succ \ldots$ in Φ^*. As $\prec$ is a well-ordering, there exists $\sigma_n \in \mathcal{V}$ so that $\sigma_1 \succ \sigma_2 \succ \ldots \succ \sigma_n$ and $\sigma\pi = \sigma_1\pi = \sigma_n\pi$. The element $\tau = \sigma_n$ is unique. □

If $\pi\colon F \to \mathrm{Sym}(UK\backslash F)$ is a permutation representation for an infinite index subgroup $UK \leq F$, we cannot ensure finiteness of the set $\mathcal{V}$ and termination of the algorithm. In the remainder, we always consider finite index subgroups $UK \leq F$ only.

For finite L-presentations $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ with $\Phi = \{\sigma\}$, finiteness of the set $\{\sigma^\ell\pi \mid \ell \in \mathbb{N}_0\} \subseteq \mathrm{Hom}(F, \mathrm{Sym}(UK\backslash F))$ yields the following

Corollary 5.3. *If $\Phi = \{\sigma\}$, there exist integers $0 \leq i < j$ with $\sigma^j\pi = \sigma^i\pi$.*

The set $\mathcal{V} \subseteq \Phi^*$ returned by Algorithm 1 satisfies the following

Lemma 5.4. *The set $\mathcal{V}$ can be considered as a subtree of Φ^*. The image of the finite set $\mathcal{V}$ and the image of the monoid Φ^* in $\mathrm{Hom}(F, \mathrm{Sym}(UK\backslash F))$ coincide.*

Proof. The identity mapping $\mathrm{id}\colon F \to F$ is contained in $\mathcal{V}$ and it represents the root of $\mathcal{V}$ and Φ^*. Let $\sigma \in \mathcal{V}$ be given. Then either $\sigma \in \Phi$ or there exists $\psi \in \Phi$ and $\delta \in \Phi^*$ so that $\sigma = \psi\delta$. In the first case, $\mathrm{id}\colon F \to F$ is a unique parent of $\sigma \in \Phi$. Otherwise, if $\sigma = \psi\delta$, we need to show that $\delta \in \mathcal{V}$ holds. Our algorithm IteratingEndomorphisms only adds elements from the stack $\mathcal{S}$ to $\mathcal{V}$. At some stage of the algorithm we had $\sigma = \psi\delta \in \mathcal{S}$. The latter element is added to the stack $\mathcal{S}$ as a child of the element δ and thus $\delta \in \mathcal{V}$. The second statement follows immediately from Algorithm 1 and Lemma 5.2. □

We define a binary relation $\sim$ on the free monoid Φ^* by defining $\sigma \sim \delta$ if and only if the unique element $\sigma_n \in \Phi^*$ in Lemma 5.2 coincides for both σ and δ. Thus $\sigma \sim \delta$ if and only if $\sigma\pi = \delta\pi$. This definition yields the immediate

Lemma 5.5. *The relation $\sigma \sim \delta$ is an equivalence relation. Each equivalence class is represented by a unique element in $\mathcal{V}$ which is minimal with respect to the total and well-ordering $\prec$.*

Recall that $\pi\colon F \to \mathrm{Sym}(UK\backslash F)$ is a permutation representation for the group's action on the right-cosets $UK\backslash F$. If $\mathcal{T}$ is a transversal for UK in F, $\sigma \sim \delta$ implies that $UK\,t \cdot g^\sigma = UK\,t \cdot g^\delta$ for each $t \in \mathcal{T}$ and $g \in F$. We therefore obtain the following

Lemma 5.6. *If $\sigma \in \Phi^*$ satisfies $\sigma\pi = \pi$, the subgroup UK is σ-invariant. There are σ-invariant subgroups UK that do not satisfy $\sigma\pi = \pi$.*

Proof. The first statement holds in general for a group acting on a set: As $\sigma\pi = \pi$, we have $UK\,t \cdot g^\sigma = UK\,t\,g$ for each $t \in \mathcal{T}$ and $g \in F$. If $g \in UK$, then $UK\,1 \cdot g^\sigma = UK\,1 \cdot g = UK\,1$ and so $g^\sigma \in UK$. The index-2 subgroup $H = \langle a, b^2, bab^{-1}\rangle$ of the Basilica group satisfies $(UK)^\sigma \subseteq UK$ and $\sigma\pi \neq \pi$. This (and similar results in the remainder of this paper) can be easily verified with a computer algebra system such as GAP. □

The latter observation motivates the following

Definition 5.7. Let $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R}\rangle$ be a finitely L-presented group and let $H \leq G$ be a finite index subgroup with permutation representation π as above. The π-*leafs* $\Psi \subseteq \Phi^* \setminus \mathcal{V}$ of $\mathcal{V}$ are

$$\Psi = \{\psi\delta \mid \psi \in \Phi,\ \delta \in \mathcal{V},\ \psi\delta \notin \mathcal{V},\ \psi\delta\pi = \pi\}. \tag{6}$$

The subgroup H is *leaf-invariant* if $\Psi = \{\psi\delta \mid \psi \in \Phi, \delta \in \mathcal{V}, \psi\delta \notin \mathcal{V}\}$ holds.

For a finitely L-presented group $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R}\rangle$, the generating set Φ of Φ^* is finite. Moreover, the equivalence $\sim$ yields finitely many equivalence classes. Hence, the set of π-leafs Ψ of $\mathcal{V}$ is finite. We obtain the following

Lemma 5.8. *If H is a leaf-invariant subgroup of G, each π-leaf $\psi\delta \in \Psi$ induces an endomorphism of UK. Moreover, each $\sigma \in \Phi^*$ can be written as $\sigma = v\,\sigma$ with $v \in \mathcal{V}$ and $\sigma \in \Psi^*$.*

Proof. We again follow the ideas of Algorithm 1. By Lemma 5.6, the condition $\psi\sigma\pi = \pi$ implies $\psi\sigma$-invariance of UK and hence $\Psi^* \subseteq \mathrm{End}(UK)$. Write $W = \{\psi\delta \mid \psi \in \Phi, \delta \in \mathcal{V}, \psi\delta \notin \mathcal{V}\}$ and let $\sigma \in \Phi^*$ be given. Write $\sigma_1 = \sigma$. There exists $w \in \Phi^*$ minimal subject to the existence of $\delta \in \mathcal{V}$ so that $\sigma_1 = w\delta$. If $\|w\| = 0$, then $\sigma_1 = \delta\,\mathrm{id}$ with $\delta \in \mathcal{V}$ and $\mathrm{id} \in \Psi^*$. Otherwise, there exists $\psi \in \Phi$ and $\sigma_2 \in \Phi^*$ so that $\sigma_1 = \sigma_2\psi\delta$ and $\psi\delta \notin \mathcal{V}$. Then $\psi\delta \in W$. Since H is leaf-invariant, we have $W = \Psi$ and hence $\psi\delta \in \Psi$. Therefore $\psi\delta$ induces an endomorphism of UK. Clearly $\sigma_2 \prec \sigma_1$. Rewriting the prefix σ_2 as above yields a descending sequence $\sigma_1 \succ \sigma_2 \ldots$ in Φ^*. As $\prec$ is a well-ordering, we eventually have $\sigma_1 \succ \sigma_2 \succ \ldots \succ \sigma_n$ with $\sigma_n \in \mathcal{V}$ and $\sigma = \sigma_1 = \sigma_n\delta$ for some $\delta \in \Psi^*$. □

If the finite L-presentation $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R}\rangle$ satisfies $\Phi = \{\sigma\}$ and if there exists a minimal positive integer $0 < j$ so that $\sigma^j\pi = \pi$, the set

$$W = \{\psi\delta \mid \psi \in \Phi, \delta \in \mathcal{V}, \psi\delta \notin \mathcal{V}\}$$

in the proof of Lemma 5.8 above becomes $W = \{\sigma^j\}$. Note the following

Remark 5.9. *The condition $\sigma^j\pi = \sigma^0\pi$ is essential for the σ^{j-0}-invariance of the subgroup. For instance, the subgroup $H = \langle a, bab^{-1}, b^{-1}a^2b, b^4, b^2ab^{-2}\rangle$ of the Basilica group satisfies $\sigma^4\pi = \sigma^3\pi$ but it is not σ-invariant.*

The stabilizing subgroup $\tilde{L}$ introduced in Definition 5.1 satisfies the following

Proposition 5.10. *Let $\mathcal{V} \subseteq \Phi^*$ be the finite set returned by Algorithm 1. The stabilizing subgroup $\tilde{L}$ satisfies that*

$$\tilde{L} = \bigcap_{\sigma\in\mathcal{V}} (\sigma\pi)^{-1}\left(\mathrm{Stab}_{\mathrm{Sym}(UK\backslash F)}(UK\,1)\right) = \bigcap_{\sigma\in\mathcal{V}} \sigma^{-1}(UK).$$

The stabilizing subgroup $\tilde{L}$ is Φ-invariant (i.e., we have $\tilde{L}^\psi \subseteq \tilde{L}$ for each $\psi \in \Phi$). It is contained in the subgroup UK and it has finite index in F. The stabilizing subgroup $\tilde{L}$ is the largest Φ^*-invariant subgroup of UK. It is not necessarily normal in F.

Proof. By Lemma 5.4, the sets $\{\sigma\pi \mid \sigma \in \Phi^*\}$ and $\{\sigma\pi \mid \sigma \in \mathcal{V}\}$ coincide and thus we have

$$\tilde{L} = \bigcap_{\sigma\in\Phi^*} (\sigma\pi)^{-1}\left(\mathrm{Stab}_{\mathrm{Sym}(UK\backslash F)}(UK\,1)\right) = \bigcap_{\sigma\in\mathcal{V}} (\sigma\pi)^{-1}\left(\mathrm{Stab}_{\mathrm{Sym}(UK\backslash F)}(UK\,1)\right).$$

Since $(\sigma\pi)^{-1}\left(\mathrm{Stab}_{\mathrm{Sym}(UK\backslash F)}(UK\,1)\right) = \sigma^{-1}(UK)$, we have $\tilde{L} = \bigcap_{\sigma\in\mathcal{V}} \sigma^{-1}(UK)$. For $\psi \in \Phi$, we have

$$\psi^{-1}(\tilde{L}) = \textstyle\bigcap_{\sigma\in\Phi^*} (\sigma\psi)^{-1}(UK) \supseteq \bigcap_{\sigma\in\Phi^*} \sigma^{-1}(UK) = \tilde{L}$$

since the first intersection is over a smaller set than the second one. Thus $\psi(\tilde{L}) \subseteq \tilde{L}$. Since $\sigma = \mathrm{id} \in \Phi^*$, we have $\tilde{L} \subseteq UK$. Because the stabilizing subgroup $\tilde{L}$ is the intersection of finitely many finite index subgroups $(\sigma\pi)^{-1}(\mathrm{Stab}_{\mathrm{Sym}(UK\backslash F)}(UK\,1)$ of F, it has finite index in F. If $N \leq UK$ is Φ^* invariant, we have $N \subseteq \sigma^{-1}(N) \subseteq \sigma^{-1}(UK)$ for each $\sigma \in \Phi^*$. Hence $N \subseteq \bigcap_{\sigma\in\Phi^*} \sigma^{-1}(UK) = \tilde{L}$.

The stabilizing subgroup $\tilde{L} = \langle a, bab^{-1}, b^{-1}a^2b, b^2ab^{-2}, b^3a^{-1}b, b^{-1}ab^3\rangle$ of the subgroup $H = \langle a, bab^{-1}, b^{-1}a^{-2}b, b^2ab^{-2}, b^3a^{-1}b, b^{-1}ab^3\rangle$ of the Basilica group is not normal in F. □

The stabilizing subgroup $\tilde{L}$ always satisfies that $\tilde{L} \subseteq UK$. Conditions for equality are given by the following

Lemma 5.11. *The following conditions are equivalent:*

(i) $\tilde{L} = UK$,
(ii) $(UK)^\psi \subseteq UK$ *for all* $\psi \in \mathcal{V}$, *and*
(iii) $(UK)^\delta \subseteq UK$ *for all* $\delta \in \Phi^*$.

Proof. We have that $\tilde{L} = \bigcap_{\sigma\in\Phi^*} \sigma^{-1}(UK) = \bigcap_{\sigma\in\mathcal{V}} \sigma^{-1}(UK)$. Therefore $\tilde{L} = \bigcap_{\sigma\in\Phi^*} \sigma^{-1}(UK) = UK$ if and only if $UK \subseteq \tilde{L} \subseteq \sigma^{-1}(UK)$ and so $(UK)^\sigma \subseteq UK$ for all $\sigma \in \Phi^*$. Similarly, we have $(UK)^\psi \subseteq UK$, for all $\psi \in \mathcal{V}$, if and only if $(UK)^\sigma \subseteq UK$, for all $\sigma \in \Phi^*$. $\square$

In the style of [16], we define a binary relation $\rightsquigarrow_\pi$ on the free monoid Φ^* as follows: For $\sigma, \delta \in \Phi^*$ we define $\sigma \rightsquigarrow_\pi \delta$ if and only if there exists a homomorphism $\gamma\colon \mathrm{im}(\delta\pi) \to \mathrm{im}(\sigma\pi)$ so that $\sigma\pi = \delta\pi\gamma$ holds. It is known [16] that it is decidable whether or not $\sigma \rightsquigarrow_\pi \delta$ holds. This yields that

Lemma 5.12. *Let $\mathcal{V} \subseteq \Phi^*$ be the finite set returned by Algorithm 1. Then there exists a subset $\tilde{\mathcal{V}} \subseteq \mathcal{V}$ with the following property: For each $\sigma \in \Phi^*$ there exists a unique element $\delta \in \tilde{\mathcal{V}}$ so that $\sigma \rightsquigarrow_\pi \delta$ and δ is minimal with respect to the ordering $\prec$ in Lemma 5.2.*

Proof. This is straightforward as the set $\mathcal{V}$ returned by Algorithm 1 is an upper bound on $\tilde{\mathcal{V}}$ because $\sigma \sim \delta$ implies both $\sigma \rightsquigarrow_\pi \delta$ or $\delta \rightsquigarrow_\pi \sigma$. $\square$

Again, the set $\tilde{\mathcal{V}}$ in Lemma 5.12 can be considered a subtree of Φ^* or even as a subtree of $\mathcal{V}$. The binary relation $\rightsquigarrow_\pi$ is reflexive and transitive but not necessarily symmetric. The equivalence relation $\sim$ and the relation $\rightsquigarrow_\pi$ are related by the following

Lemma 5.13. *Let $\pi\colon F \to \mathrm{Sym}(UK\backslash F)$ be a permutation representation as above. For $\sigma, \delta \in \Phi^*$, we have the following*

(i) *We have $\sigma \rightsquigarrow_\pi \delta$ and $\delta \rightsquigarrow_\pi \sigma$ if and only if the homomorphism $\gamma\colon \mathrm{im}(\delta\pi) \to \mathrm{im}(\sigma\pi)$ with $\sigma\pi = \delta\pi\gamma$ is bijective.*

(ii) *If $\sigma \sim \delta$, then $\sigma \rightsquigarrow_\pi \delta$ and $\delta \rightsquigarrow_\pi \sigma$. The converse is not necessarily true.*

(iii) *If $k > 0$ is minimal so that $\sigma^k \sim \mathrm{id}$, there exists a minimal positive integer ℓ so that $\ell \mid k$ and $\sigma^\ell \rightsquigarrow_\pi \mathrm{id}$. If $\Phi = \{\sigma\}$, the set $\tilde{\mathcal{V}}$ from Lemma 5.12 becomes $\tilde{\mathcal{V}} = \{\mathrm{id}, \sigma, \ldots, \sigma^{\ell-1}\}$.*

(iv) *If ℓ is a minimal positive integer such that $\mathrm{id} \rightsquigarrow_\pi \sigma^\ell$, there exists a minimal integer $k \geq \ell$ so that $\sigma^k \sim \mathrm{id}$. If $\Phi = \{\sigma\}$, the set $\mathcal{V}$ returned by Algorithm 1 becomes $\mathcal{V} = \{\mathrm{id}, \sigma, \ldots, \sigma^{k-1}\}$ while $\tilde{\mathcal{V}} = \{\mathrm{id}, \sigma, \ldots, \sigma^{\ell-1}\}$.*

(v) *The subgroup $H = \langle a, b^2, bab^{-1}\rangle$ of the Basilica group satisfies $\sigma \rightsquigarrow_\pi \mathrm{id}$ but there is no positive integer $\ell > 0$ so that $\sigma^\ell \sim \mathrm{id}$ holds.*

Proof. If the homomorphism $\gamma\colon \mathrm{im}(\delta\pi) \to \mathrm{im}(\sigma\pi)$ with $\sigma\pi = \delta\pi\gamma$ is bijective, we obtain $\sigma\pi\gamma^{-1} = \delta\pi$ and thus $\delta \rightsquigarrow_\pi \sigma$. On the other hand, suppose that both $\delta \rightsquigarrow_\pi \sigma$ and $\sigma \rightsquigarrow_\pi \delta$ hold. Then there are homomorphisms $\gamma\colon \mathrm{im}(\sigma\pi) \to \mathrm{im}(\delta\pi)$

and $\tau\colon \mathrm{im}(\delta\pi) \to \mathrm{im}(\sigma\pi)$ so that $\delta\pi = \sigma\pi\gamma$ and $\sigma\pi = \delta\pi\tau$. This yields $\delta\pi = \sigma\pi\gamma = \delta\pi\tau\gamma$ and $\sigma\pi = \delta\pi\tau = \sigma\pi\gamma\tau$. Hence γ and τ are isomorphisms.

Since $\sigma \sim \delta$ implies $\sigma\pi = \delta\pi$, we immediately obtain both $\sigma \rightsquigarrow_\pi \delta$ and $\delta \rightsquigarrow_\pi \sigma$. The subgroup $H = \langle a, bab^{-1}, b^3\rangle$ of the Basilica group admits the permutation representation $\pi\colon a \mapsto (\,)$, $b \mapsto (1,2,3)$. We have $\sigma^2\pi\colon a \mapsto (\,)$, $b \mapsto (1,3,2)$ and therefore $\sigma^2 \rightsquigarrow_\pi \mathrm{id}$ and $\mathrm{id} \rightsquigarrow_\pi \sigma^2$. Though $\sigma^2\pi \neq \pi$.

Suppose that $k \in \mathbb{N}$ is minimal so that $\sigma^k \sim \mathrm{id}$ and so $\sigma^k\pi = \pi$. Then $\mathrm{im}(\pi) \supseteq \mathrm{im}(\sigma\pi) \supseteq \ldots \supseteq \mathrm{im}(\sigma^k\pi) = \mathrm{im}(\pi)$. There exists a minimal integer $0 < \ell \leq k$ such that $\sigma^\ell \rightsquigarrow_\pi \mathrm{id}$. Hence, there exists a homomorphism $\gamma\colon \mathrm{im}(\pi) \to \mathrm{im}(\sigma^\ell\pi)$ with $\sigma^\ell\pi = \pi\gamma$. The homomorphism γ is onto and, since $\mathrm{im}(\pi) = \mathrm{im}(\sigma^\ell\pi)$ is finite, γ is bijective. As $\ell \leq k$ holds, we can write $k = s\,\ell + t$ for some $0 \leq t < \ell$ and $s \in \mathbb{N}$. This yields that $\pi = \sigma^k\pi = \sigma^t\,\sigma^{s\,\ell}\,\pi = \sigma^t\pi\gamma^s$ and so $\pi\,\gamma^{-s} = \sigma^t\pi$. If $t > 0$, the latter yields that $\sigma^t \rightsquigarrow_\pi \mathrm{id}$ which contradicts the minimality of ℓ. Thus $t = 0$ and $\ell \mid k$. If $\Phi = \{\sigma\}$, the set $\{\mathrm{id}, \sigma, \ldots, \sigma^{\ell-1}\}$ is an upper bound on the set $\tilde{\mathcal{V}}$ from Lemma 5.12 because $\sigma^\ell \rightsquigarrow_\pi \mathrm{id}$ holds. By the minimal choice of ℓ, we obtain that $\tilde{\mathcal{V}} = \{\mathrm{id}, \sigma, \ldots, \sigma^{\ell-1}\}$.

Suppose that $\mathrm{id} \rightsquigarrow_\pi \sigma^\ell$ holds. There exists a homomorphism $\gamma\colon \mathrm{im}(\sigma^\ell\pi) \to \mathrm{im}(\pi)$ with $\sigma^\ell\pi\gamma = \pi$. Since γ is a surjective map from a subgroup $\mathrm{im}(\sigma^\ell\pi) \leq \mathrm{im}(\pi)$ onto $\mathrm{im}(\pi)$, γ is bijective and hence, we also have that $\sigma^\ell \rightsquigarrow_\pi \mathrm{id}$. Suppose that the automorphism γ of the finite group $\mathrm{im}(\pi)$, has finite order n. Write $k = n\ell$. Then $\sigma^k\pi = \sigma^{n\ell}\pi = \pi\gamma^n = \pi$ and so $\sigma^k \sim \mathrm{id}$ and k is minimal. If $\Phi = \{\sigma\}$ holds, then, by the minimal choice of k, we obtain $\mathcal{V} = \{\mathrm{id}, \sigma, \ldots, \sigma^{k-1}\}$ for the set $\mathcal{V}$ returned by Algorithm 1 while $\mathcal{V} = \{\mathrm{id}, \sigma, \ldots, \sigma^{\ell-1}\}$ by the minimality of ℓ.

The permutation representation $\pi\colon F \to \mathrm{Sym}(UK\backslash F)$ of the subgroup $H = \langle a, b^2, bab^{-1}\rangle$ is induced by the map $a \mapsto (\,)$ and $b \mapsto (1,2)$. Therefore, H satisfies that $\sigma \rightsquigarrow_\pi \mathrm{id}$, $|\mathrm{im}(\pi)| = 2$, and $|\mathrm{im}(\sigma\pi)| = 1$. In particular, for each $\ell \geq 1$, we have $|\mathrm{im}(\sigma^\ell\pi)| = 1$. Thus there is no integer ℓ so that $\sigma^\ell \sim \mathrm{id}$ holds. However, we have $\sigma^2\pi = \sigma\pi$ so that the set $\mathcal{V} = \{\mathrm{id}, \sigma, \sigma^2\}$ returned by Algorithm 1 is still finite. $\square$

The stabilizing core L introduced in Definition 5.1 satisfies the following

Proposition 5.14. *Let $\mathcal{V} \subseteq \Phi^*$ be the finite set returned by Algorithm 1. The stabilizing core L satisfies that*

$$L = \bigcap_{\sigma\in\mathcal{V}} \ker(\sigma\pi).$$

The stabilizing core L is the largest Φ-invariant subgroup of UK which is normal in F and thus $L = \mathrm{Core}_F(\tilde{L})$. It is finitely generated, has finite index in F, and it *contains all iterated relations $\mathcal{R}$ of G's L-presentation $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R}\rangle$. We have $L \subseteq \tilde{L} \subseteq UK \subseteq F$ and $L \subseteq \mathrm{Core}_F(UK) \subseteq UK \subseteq F$.*

Proof. By Lemma 5.4, the sets $\{\sigma\pi \mid \sigma \in \Phi^*\}$ and $\{\sigma\pi \mid \sigma \in \mathcal{V}\}$ coincide and thus we have

$$L = \bigcap_{\sigma\in\Phi^*} \ker(\sigma\pi) = \bigcap_{\sigma\in\mathcal{V}} \ker(\sigma\pi).$$

The stabilizing core L is normal in F because it is the intersection of normal subgroups. Since $L \subseteq \ker(\pi) = \mathrm{Core}_F(UK)$ holds, the stabilizing core L is contained in UK. Since $\sigma^{-1}(\ker(\pi)) = \ker(\sigma\pi)$, we have that $L = \bigcap_{\sigma\in\Phi^*} \sigma^{-1}(\ker(\pi))$. For any $\psi \in \Phi^*$, we obtain

$$\psi^{-1}(L) = \bigcap_{\sigma\in\Phi^*} (\sigma\psi)^{-1}(\ker(\pi)) \supseteq \bigcap_{\sigma\in\Phi^*} \sigma^{-1}(\ker(\pi)) = L$$

as the first intersection is over a small set of indices. Thus L is Φ^*-invariant. Let $N \leq UK$ be a Φ^*-invariant subgroup which is normal in F. Then $N \leq \mathrm{Core}_F(UK) = \ker(\pi)$ and so $N \subseteq \sigma^{-1}(N) \subseteq \sigma^{-1}(\ker(\pi))$ for each $\sigma \in \Phi^*$. Thus $N \subseteq \bigcap_{\sigma\in\Phi^*} \sigma^{-1}(\ker(\pi)) = L$. The stabilizing core L has finite index in F because it is the intersection of finitely many finite index subgroups $\ker(\sigma\pi)$ with $\sigma \in \mathcal{V}$.

The stabilizing core L is finitely generated as a finite index subgroup of a finitely generated free group F. Let $r \in \mathcal{R}$ be an iterated relator of G's L-presentation $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R}\rangle$. Then, for each $\sigma \in \mathcal{V}$, the image r^σ is a relator of G. Thus $r \in \ker(\sigma\pi)$ and $r \in L$.

As L is a Φ-invariant subgroup of UK, we have $L \subseteq \tilde{L}$ by Proposition 5.10. Moreover, $L \subseteq \ker(\pi) = \mathrm{Core}_F(UK)$. □

Since the stabilizing core L contains the iterated relations $\mathcal{R}$ of the L-presentation, it also contains the normal closure $\langle \bigcap_{\sigma\in\Phi^*} \mathcal{R}^\sigma\rangle^F$. We obtain the immediate

Corollary 5.15. *If $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R}\rangle = \langle \mathcal{X} \mid \emptyset \mid \Phi \mid \mathcal{Q} \cup \mathcal{R}\rangle$ is invariantly L-presented, we have $K \subseteq L \subseteq \tilde{L} \subseteq UK \subseteq F$. The subgroup $H \cong UK/K \leq F/K$ contains the Φ-invariant normal subgroup L/K. The index $[UK/K : L/K] = [UK : L]$ is finite.*

The subgroup H in Corollary 5.15 is a finite extension of L/K. Since the stabilizing core L is the largest Φ-invariant subgroup which is normal in F, the stabilizing subgroup $\tilde{L}$ is normal in F if and only if $L = \tilde{L}$ holds. More precisely, we have the following

Lemma 5.16. *We have $\tilde{L} = L$ if and only if $\tilde{L} \subseteq \mathrm{Core}_F(UK)$.*

Proof. We have $L \subseteq \tilde{L}$ and $\tilde{L}^\sigma \subseteq \tilde{L}$ for each $\sigma \in \Phi^*$. If $L = \tilde{L}$, then $\tilde{L} = L \subseteq \mathrm{Core}_F(UK)$. If $\tilde{L} \subseteq \mathrm{Core}_F(UK) = \ker(\pi)$, then $\tilde{L} \subseteq \sigma^{-1}(\tilde{L}) \subseteq \sigma^{-1}(\ker(\pi))$ for all $\sigma \in \Phi^*$. Thus $\tilde{L} \subseteq \bigcap_{\sigma\in\Phi^*} \sigma^{-1}(\ker(\pi)) = L$. □

If $UK \trianglelefteq F$ is a normal subgroup, then $\tilde{L} \subseteq UK = \mathrm{Core}_F(UK)$. Hence, we obtain the immediate

Corollary 5.17. *If $UK \trianglelefteq F$, then $L = \tilde{L}$.*

Note the following

Remark 5.18. *There are subgroups that satisfy* $\mathrm{Core}_F(UK) \subset \tilde{L}$. *For instance, the subgroup $H = \langle a, b^2, ba^2b^{-1}, bab^{-2}a^{-1}b^{-1}\rangle$ of the Basilica group is Φ-invariant (and hence $\tilde{L} = UK$ by Lemma 5.11) but not normal in G.*

There are subgroups that satisfy $\tilde{L} \subset \mathrm{Core}_F(UK)$. For instance, the subgroup $H = \langle a^2, b, aba^{-1}\rangle$ of the Basilica group has index 2 in G (and thus it is normal in G); though the subgroup H is not σ-invariant.

There are subgroups that neither satisfy $\tilde{L} \subseteq \mathrm{Core}_F(UK)$ nor $\mathrm{Core}_F(UK) \subseteq \tilde{L}$. For instance, the subgroup $H = \langle a, bab^{-1}, b^{-1}a^2b, b^2ab^2, b^3a^{-1}b\rangle$ of the Basilica group satisfies $[F : \tilde{L}] = [F : \mathrm{Core}_F(UK)]$ and $\tilde{L} \neq \mathrm{Core}_F(UK)$.

6. The Reidemeister-Schreier Theorem

In this section, we finally prove our variant of the Reidemeister-Schreier Theorem in Theorem 1.1. For this purpose, let $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R}\rangle$ be a finitely L-presented group and let $H \leq G$ be a finite index subgroup given by its generators $g_1, \ldots, g_n$. We consider the generators $g_1, \ldots, g_n$ as elements of the free group F over $\mathcal{X}$. Denote the normal closure of the relations of G by $K = \langle \mathcal{Q} \cup \bigcup_{\sigma\in\Phi^*} \mathcal{R}^\sigma\rangle^F$ and let $U = \langle g_1, \ldots, g_n\rangle \leq F$. Then $H \cong UK/K$. If $\mathcal{T} \subseteq F$ denotes a Schreier transversal for UK in F, the Reidemeister-Schreier Theorem in Section 3 shows that the subgroup H admits the group presentation

$$H \cong \left\langle \mathcal{Y} \,\middle|\, \{\tau(tqt^{-1}) \mid t \in \mathcal{T}, q \in \mathcal{Q}\} \cup \bigcup_{\sigma\in\Phi^*} \{\tau(tr^\sigma t^{-1}) \mid t \in \mathcal{T}, r \in \mathcal{R}\} \right\rangle, \quad (7)$$

where τ denotes the Reidemeister rewriting. We will construct a finite L-presentation from the group presentation in Eq. (7). First, we note the following

Theorem 6.1. *Let $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R}\rangle$ be invariantly finitely L-presented. Each Φ-invariant normal subgroup with finite index in G is invariantly finitely L-presented.*

Proof. Let $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ be an invariantly finitely L-presented group and let $H \trianglelefteq G$ be a Φ-invariant normal subgroup with finite index in G. Every invariantly L-presented group can be considered as an ascendingly L-presented group. Thus, we may consider $\mathcal{Q} = \emptyset$. Consider the notation introduced above. As G is invariantly L-presented, we have $K^\sigma \subseteq K$ for each $\sigma \in \Phi^*$. Since the subgroup H is Φ-invariant, we also $(UK)^\sigma \subseteq UK$ for each $\sigma \in \Phi^*$. Then Lemma 5.11 shows that $\tilde{L} = UK$. Moreover, as $UK \trianglelefteq F$, we have $L = \tilde{L}$ and thus $UK = \tilde{L} = L$. Let $t \in \mathcal{T}$ be given. As $UK \trianglelefteq F$, the map $\delta_t \colon UK \to UK$, $g \mapsto tgt^{-1}$ defines an automorphism of UK. The Reidemeister rewriting $\tau \colon UK \to F(\mathcal{Y})$ is an isomorphism of free groups and therefore the endomorphisms $\Phi \cup \{\delta_t \mid t \in \mathcal{T}\}$ of UK translate to endomorphisms $\widehat{\Phi} \cup \{\widehat{\delta}_t \mid t \in \mathcal{T}\}$ of the free group $F(\mathcal{Y})$. Consider the invariant finite L-presentation

$$\langle\, \mathcal{Y} \mid \emptyset \mid \widehat{\Phi} \cup \{\widehat{\delta}_t \mid t \in \mathcal{T}\} \mid \{\tau(r) \mid r \in \mathcal{R}\}\,\rangle. \tag{8}$$

In order to prove that the finite L-presentation in Eq. (8) defines the subgroup H, it suffices to prove that each relation of the presentation in Eq. (7) is a consequence of the relations of the L-presentation in Eq. (8) and vice versa. For $t \in \mathcal{T}$, $r \in \mathcal{R}$, and $\sigma \in \Phi^*$, we consider the relation $\tau(t\, r^\sigma\, t^{-1})$ of the group presentation in Eq. (7). Clearly, this relation is relation in the finite L-presentation in Eq. (8) because there exists $\widehat{\sigma} \in \widehat{\Phi}^*$ so that $(\tau(r))^{\widehat{\sigma}} = \tau(r^\sigma)$. Then $(\tau(r))^{\widehat{\sigma}\widehat{\delta}_t} = \tau(tr^\sigma t^{-1})$. On the other hand, consider the relation $\tau(r)^{\widehat{\sigma}}$ of the finite L-presentation in Eq. (8) where $r \in \mathcal{R}$ and $\widehat{\sigma} \in (\widehat{\Phi} \cup \{\widehat{\delta}_t \mid t \in \mathcal{T}\})^*$. Write $\Psi = \widehat{\Phi} \cup \{\widehat{\delta}_t \mid t \in \mathcal{T}\}$. Since $1 \in \mathcal{T}$ and $\mathrm{id} \in \Phi^*$, we can write each image of an element $\widehat{\delta} \in \Psi$ as $\tau(g)^{\widehat{\delta}} = \tau(tg^\delta\, t^{-1})$ for some $t \in \mathcal{T}$ and $\delta \in \Phi^*$ where t or δ is possibly trivial. Since $\widehat{\sigma} \in \Psi^*$, we can write $\widehat{\sigma} = \widehat{\sigma}_1 \cdots \widehat{\sigma}_n$ with each $\widehat{\sigma}_i \in \Psi$. The image $\tau(r)^{\widehat{\sigma}}$ has the form

$$\tau(r)^{\widehat{\sigma}} = \tau(t_n \cdots t_2^{\sigma_3\cdots\sigma_n}\, t_1^{\sigma_2\sigma_3\cdots\sigma_n} \cdot r^{\sigma_1\sigma_2\cdots\sigma_n} \cdot t_1^{-\sigma_2\sigma_3\cdots\sigma_n}\, t_2^{-\sigma_3\cdots\sigma_n} \cdots t_n^{-1}).$$

Since $\mathcal{T}$ is a transversal for UK in F, we can write $t_n \cdots t_2^{\sigma_3\cdots\sigma_n}\, t_1^{\sigma_2\sigma_3\cdots\sigma_n} = u\,t$ with $t \in \mathcal{T}$ and $u \in UK$. This yields that $\tau(r)^{\widehat{\sigma}} = \tau(u\,t\,r^{\sigma_1\sigma_2\cdots\sigma_n}\,t^{-1}\,u^{-1}) = \tau(u)\,\tau(t\,r^{\sigma_1\sigma_2\cdots\sigma_n}\,t^{-1})\,\tau(u)^{-1}$, which is a consequence of $\tau(t\,r^{\sigma_1\sigma_2\cdots\sigma_n}\,t^{-1})$. The latter relation $\tau(t\,r^{\sigma_1\sigma_2\cdots\sigma_n}\,t^{-1})$ is a relation of the group presentation in Eq. (7). In summary, each relation of the group presentation in Eq. (7) is a consequence of the finite L-presentation in Eq. (8) and vice versa. $\square$

In order to prove Reidemeister-Schreier Theorem 1.1 for finitely L-presented groups, we need to consider finite index subgroups that are not normal. For this purpose, we need to construct the relations $\tau(tr^\sigma\, t^{-1})$ with $t \in \mathcal{T}$, $r \in \mathcal{R}$, and $\sigma \in \Phi^*$ in Eq. (7). The overall strategy in this paper is to construct the relations as iterated images of the form $\tau(sr\,s^{-1})^{\widehat{\sigma}}$ for $s \in \mathcal{T}$ and some $\widehat{\sigma} \in \widehat{\Phi}^*$. If the subgroup H

is normal as in Proposition 6.3, the conjugation action $\delta_t\colon UK \to UK$ enables us to first construct the image $\tau(r^\sigma) = \tau(r)^{\widehat{\sigma}}$ and then to consider the conjugates $\tau(r^\sigma)^{\widehat{\delta}_t} = \tau(tr^\sigma t^{-1})$. However, in general, it is not sufficient to take as iterated relations those $\tau(trt^{-1})^\sigma = \tau(t^\sigma r^\sigma t^{-\sigma})$, with $t \in \mathcal{T}$ and $r \in \mathcal{R}$, as σ may not be invertible over $\{t^\sigma \mid t \in \mathcal{T}\}$. More precisely, we have the following

Remark 6.2. *Let $H = \langle a, b^2, ba^3b^{-1}, bab^{-2}a^{-1}b^{-1}, ba^{-1}b^{-2}ab^{-1}\rangle$ be a subgroup of the Basilica group G. The subgroup H is σ-invariant and thus we can consider the iterated images $\{\tau(r)^{\widehat{\sigma}} \mid r \in \mathcal{R}, \sigma \in \Phi^*\}$. A Schreier transversal $\mathcal{T}$ for H in G is given by $\mathcal{T} = \{1, b, ba, ba^2, bab, ba^2b\}$. We have $\mathcal{T}^\sigma = \{1, a, ab^2, ab^4, ab^2a, ab^4a\}$. Note that $\mathcal{T}^\sigma \subseteq UK$ holds. Thus we cannot ensure that the iterated images $\{\tau(trt^{-1})^{\widehat{\sigma}} \mid r \in \mathcal{R}, t \in \mathcal{T}, \sigma \in \Phi^*\}$ contain all relations in Eq. (7). As the subgroup H is not normal in G, we cannot consider the conjugate action as well. However, an invariant finite L-presentation for the subgroup H can be computed with Theorem 7.1 as the subgroup H is leaf-invariant (see Section 7 below).*

In the following, we use Theorem 6.1 to prove our variant of the Reidemeister-Schreier Theorem for invariantly finitely L-presented groups first.

Proposition 6.3. *Every finite index subgroup of an invariantly finitely L-presented group is finitely L-presented.*

Proof. Let H be a finite index subgroup of an invariantly finitely L-presented group $G = F/K$. By Corollary 5.15, the subgroup $H \cong UK/K$ contains a normal subgroup L/K with finite index in G that is Φ-invariant. By Theorem 6.1, the subgroup $L/K \leq F/K$ is finitely L-presented. The subgroup H is a finite extension of a finitely L-presented group and thus, by Corollary 2.4, the subgroup H is finitely L-presented. $\square$

Recall that we do not have a method to construct an invariant L-presentation for a finite extension of an invariantly L-presented group. Therefore, we cannot ensure invariance of the finite L-presentation obtained from Corollary 5.15. In Section 7, we study conditions on a subgroup of an invariantly L-presented group that ensure the invariance of the subgroup L-presentation. First, we complete our proof of Theorem 1.1:

Proof of Theorem 1.1. Let $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R}\rangle$ be a finitely L-presented group and let H be a finite index subgroup of G. Denote the free group over $\mathcal{X}$ by F. Define the normal subgroups $K = \langle \mathcal{Q} \cup \bigcup_{\sigma\in\Phi^*} \mathcal{R}^\sigma\rangle^F$ and $M = \langle \bigcup_{\sigma\in\Phi^*} \mathcal{R}^\sigma\rangle^F$. Let $U \leq F$ be generated by the generators of H so that $H \cong UK/K$ holds. Then we have $M \trianglelefteq K \trianglelefteq F$ and $G = F/K$. Further, the group $J = F/M$ is invariantly finitely L-presented by $\langle \mathcal{X} \mid \emptyset \mid \Phi \mid \mathcal{R} \rangle$ and it naturally maps onto G. The subgroup $UK/M \leq F/M$ has finite index in J as $[F : UK]$ is finite. By Proposition 6.3, the subgroup UK/M of the invariantly finitely L-presented group $J = F/M$ is finitely L-presented. The exact sequence $1 \to K/M \to UK/M \to UK/K \to 1$ yields that $H \cong UK/K \cong (UK/M)/(K/M)$ where the kernel K/M is finitely generated as a normal subgroup by the image of the fixed relations in $\mathcal{Q}$. By Proposition 2.6, H is finitely L-presented as a factor group of the finitely L-presented group UK/M whose kernel is finitely generated as a normal subgroup. $\square$

7. Invariant Subgroup *L*-Presentations

The algorithms in [2,15] are much more efficient on invariant L-presentations. Therefore, we study conditions on a subgroup of an invariantly L-presented group to be invariantly L-presented itself. By Theorem 6.1, each Φ-invariant normal subgroup H of an invariantly finitely L-presented group $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ is invariantly finitely L-presented as soon as $[G : H]$ is finite.

Consider the notion introduced in Section 6 and let $\pi \colon F \to \mathrm{Sym}(UK\backslash F)$ be a permutation representation as usual. Recall that the subgroup H is *leaf-invariant*, if the π-*leafs*

$$\Psi = \{\psi\delta \mid \psi \in \Phi, \delta \in \mathcal{V}, \psi\delta \notin \mathcal{V}, \psi\delta\pi = \pi\},$$

of $\mathcal{V}$ satisfy $\Psi = \{\psi\delta \mid \psi \in \Phi, \delta \in \mathcal{V}, \psi\delta \notin \mathcal{V}\}$. This definition yields the following

Theorem 7.1. *Each leaf-invariant, finite index subgroup of an invariantly finitely L-presented group is invariantly finitely L-presented.*

Proof. Let $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ be invariantly finitely L-presented and let $H \leq G$ be a leaf-invariant finite index subgroup of G. Clearly, we can consider $\mathcal{Q} = \emptyset$ in the following. The π-leafs Ψ satisfy $\Psi = \{\psi\delta \mid \psi \in \Phi, \delta \in \mathcal{V}, \psi\delta \notin \mathcal{V}\}$. By Lemma 5.8, each π-leaf $\psi\delta \in \Psi \subseteq \Phi^*$ defines an endomorphism of the subgroup UK. Moreover, Lemma 5.8 shows that each $\sigma \in \Phi^*$ can be written as $\sigma = \vartheta\,\delta$ with $\vartheta \in \mathcal{V}$ and $\delta \in \Psi^*$. Consider the finite L-presentation

$$\langle \mathcal{Y} \mid \emptyset \mid \{\widehat{\psi\delta} \mid \psi\delta \in \Psi\} \mid \{\tau(tr^{\vartheta}\,t^{-1}) \mid \vartheta \in \mathcal{V}, r \in \mathcal{R}, t \in \mathcal{T}\} \rangle, \tag{9}$$

where $\mathcal{Y}$ denotes the Schreier generators of UK, $\widehat{\psi\sigma}$ denotes the endomorphism of the free group $F(\mathcal{Y})$ induced by the endomorphisms $\psi\sigma$ of UK, and $\mathcal{T}$ is a Schreier transversal for UK in F. For $t \in \mathcal{T}$, $\sigma \in \Phi^*$, and $r \in \mathcal{R}$, the relation $\tau(t\,r^{\sigma}\,t^{-1})$ of the group presentation in Eq. (7) can be obtained from the L-presentation in Eq. (9)

as follows: Since each $\sigma \in \Phi^*$ can be written as $\sigma = \vartheta\,\delta$ with $\vartheta \in \mathcal{V}$ and $\delta \in \Psi^*$, we claim that the relation $\tau(t\,r^\sigma\,t^{-1})$ is a consequence of the image $\tau(tr^\vartheta\,t^{-1})^{\widehat{\delta}}$. The latter image satisfies that $\tau(tr^\vartheta\,t^{-1})^{\widehat{\delta}} = \tau(t^\delta\,r^{\vartheta\delta}\,t^{-\delta}) = \tau(t^\delta\,r^\sigma\,t^{-\delta})$. As $\delta \in \Psi^*$, we can write $\delta = \delta_1 \cdots \delta_n$ with each $\delta_i \in \Psi$. Recall that $\delta_i \pi = \pi$ holds. Thus the right-coset $UK\,1$ satisfies that $UK\,1{\cdot}t^{\delta_i} = UK\,1{\cdot}t = UK\,t$ and therefore $UK\,t^{\delta_1\cdots\delta_n} = UK\,t$. Hence, there exists $u \in UK$ so that $t^\delta = ut$ and we obtain

$$\tau(tr^\vartheta\,t^{-1})^{\widehat{\delta}} = \tau(t^\delta\,r^\sigma\,t^{-\delta}) = \tau(ut\,r^\sigma\,t^{-1}\,u^{-1}) = \tau(u)\,\tau(t\,r^\sigma\,t^{-1})\,\tau(u)^{-1}$$

which is a consequence of $\tau(t\,r^\sigma\,t^{-1})$ and vice versa. Similarly, every relation of the L-presentation in Eq. (9) is a consequence of the relations in Eq. (7). Therefore, the invariant finite L-presentation in Eq. (9) defines the leaf-invariant finite index subgroup H. $\square$

For finite L-presentations $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ with $\Phi = \{\sigma\}$, the leaf-invariance of the subgroup H yields the existence of a positive integer j so that $\sigma^j \pi = \pi$ holds. If we assume the positive integer j to be minimal, then $\mathcal{V} = \{\mathrm{id}, \sigma, \ldots, \sigma^{j-1}\}$ and $\Psi = \{\sigma^j\}$. In this case, the invariant finite L-presentation in Eq. (9) becomes

$$H \cong \langle \mathcal{Y} \mid \emptyset \mid \{\widehat{\sigma^j}\} \mid \{\tau(tr^{\sigma^i}\,t^{-1}) \mid t \in \mathcal{T}, r \in \mathcal{R}, 0 \leq i < j\} \rangle.$$

Note that the subgroup H in Theorem 7.1 is not necessarily normal in G. However, leaf-invariance of a subgroup is a restrictive condition on the subgroup. We try to weaken this condition with the following

Definition 7.2. Let $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ be a finitely L-presented group and let $H \leq G$ be a finite index subgroup with permutation representation π. The subgroup H is *weakly leaf-invariant*, if

$$\Psi = \{\psi\delta \mid \psi \in \Phi, \delta \in \tilde{\mathcal{V}}, \psi\delta \notin \tilde{\mathcal{V}}, \psi\delta \rightsquigarrow_\pi \mathrm{id}\}$$

satisfies $\Psi = \{\psi\delta \mid \psi \in \Phi, \delta \in \tilde{\mathcal{V}}, \psi\delta \notin \tilde{\mathcal{V}}\}$.

The notion of a weakly leaf-invariant subgroup is less restrictive than leaf-invariance as the low-index subgroups of the Basilica group suggest: Among the 4 956 low-index subgroups of the Basilica group with index at most 20 there are 2 539 weakly leaf-invariant subgroups; only 156 of these subgroups are leaf-invariant. More precisely, Table 1 shows the number of subgroups ($\leq$) that are normal ($\trianglelefteq$), maximal (max), leaf-invariant (l.i.), weakly leaf-invariant (w.l.i.), and the number of subgroups that are weakly leaf-invariant and normal ($\trianglelefteq$ + w.l.i.). For finite L-presentations $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ with $\Phi = \{\sigma\}$, each leaf-invariant subgroup is weakly

TABLE 1. Subgroups of the Basilica group with index at most 20.

index	$\leq$	$\trianglelefteq$	max	l.i.	w.l.i	$\trianglelefteq +$ w.l.i
1	1	1	1	1	1	1
2	3	3	3	0	3	3
3	7	4	7	4	4	4
4	19	7	0	0	19	7
5	11	6	11	6	6	6
6	39	13	0	0	14	12
7	15	8	15	8	8	8
8	163	19	0	0	139	19
9	115	13	9	49	52	13
10	83	19	0	0	22	18
11	23	12	23	12	12	12
12	355	31	0	0	98	28
13	27	14	27	14	14	14
14	115	25	0	0	30	24
15	77	24	0	24	24	24
16	1843	47	0	0	1531	43
17	35	18	35	18	18	18
18	1047	44	0	0	366	40
19	39	20	39	20	20	20
20	939	45	0	0	158	42

leaf-invariant by Lemma 5.13, (iii). On the other hand, a weakly leaf-invariant subgroup with $\Phi = \{\sigma\}$ such that $\mathrm{id} \rightsquigarrow_\pi \sigma^\ell$ holds, is leaf-invariant by Lemma 5.13, (iv). There are subgroups of a finitely L-presented group that are weakly leaf-invariant but not leaf-invariant; see Lemma 5.13, (v). If Φ contains more than one generator, we may ask the following

Question 3. *Is every leaf-invariant subgroup weakly leaf-invariant?*

The problem is that Definitions 5.7 and 7.2 depend on the minimal sets $\mathcal{V}$ and $\tilde{\mathcal{V}}$ which satisfy $\tilde{\mathcal{V}} \subseteq \mathcal{V}$ but which may differ in general. We do not have an answer to this question. Moreover, the sets $\mathcal{V}$ and $\tilde{\mathcal{V}}$ in the Definitions 5.7 and 7.2 may also depend on choice of the ordering $\prec$ in Algorithm 1. However we have the following

Lemma 7.3. *The conditions leaf-invariance and weak leaf-invariance do not depend on the choice of the ordering $\prec$ in Algorithm 1.*

Proof. We prove this lemma by constructing the set $\mathcal{V}$ returned by Algorithm 1 (the set $\tilde{\mathcal{V}}$ from Lemma 5.12) independently from the ordering $\prec$ provided that the subgroup is (weakly) leaf-invariant. Let $\pi\colon F \to \mathrm{Sym}(UK\backslash F)$ be the permutation representation as usual and assume that the subgroup is leaf-invariant. For each $j \geq 0$, we write $\Phi^{(j)} = \{\sigma \in \Phi^* \mid \|\sigma\| = j\}$. Define $\mathcal{W}_0 = \{\mathrm{id}\}$ and recursively $\mathcal{W}_{n+1} = \{\sigma \in \Phi\mathcal{W}_n \mid \sigma\pi \neq \pi\} \subseteq \Phi^{(n+1)}$. Let $\mathcal{W} = \bigcup_{n\geq 0} \mathcal{W}_n$. Clearly, the construction of $\mathcal{W}$ does not depend on the ordering $\prec$ in Algorithm 1. We show that the sets $\mathcal{W}$ and $\mathcal{V}$ coincide. Write $\mathcal{S}_j = \mathcal{V} \cap \Phi^{(j)}$ and $\mathcal{T}_j = \mathcal{W} \cap \Phi^{(j)}$. Then $\mathcal{S}_0 = \{\mathrm{id}\} = \mathcal{T}_0$. In order to prove that $\mathcal{W} = \mathcal{V}$ holds, it suffices to show that $\mathcal{S}_j = \mathcal{T}_j$ for each $j \geq 0$. Suppose that, for $n \in \mathbb{N}_0$, we have $\mathcal{S}_j = \mathcal{T}_j$ for all $j < n$ while $\mathcal{S}_n \neq \mathcal{T}_n$. If $\sigma \in \mathcal{S}_n = \mathcal{V} \cap \Phi^{(n)}$, it is contained in $\mathcal{V}$ and hence it satisfies $\sigma\pi \neq \pi$. Moreover, we have $\sigma \in \Phi\mathcal{S}_{n-1} = \Phi\mathcal{T}_{n-1}$ and thus $\sigma \in \mathcal{T}_n$. If $\sigma \in \mathcal{T}_n = \mathcal{W} \cap \Phi^{(n)}$ but $\sigma \notin \mathcal{S}_n$, then $\sigma = \psi\delta$ with $\psi \in \Phi$ and $\delta \in \mathcal{T}_{n-1} = \mathcal{S}_{n-1} \subseteq \mathcal{V}$. Note that σ satisfies $\sigma = \psi\delta$ with $\delta \in \mathcal{V}$, $\psi \in \Phi$, and $\sigma = \psi\delta \notin \mathcal{V}$. Hence σ is a π-leaf. Since the subgroup H is leaf-invariant we have $\sigma\pi = \pi$. This is a contradiction to $\sigma \in \mathcal{T}_n$.

For proving the statement for weak leaf-invariance, the same arguments as above and the construction $\tilde{\mathcal{S}}_0 = \{\mathrm{id}\}$ and $\tilde{\mathcal{S}}_n = \{\sigma \in \Phi\tilde{\mathcal{S}}_n \mid \sigma \not\rightsquigarrow_\pi \mathrm{id}\}$ apply. □

The subgroup $J = \langle x_1, x_2, x_3, x_4\, x_1\, x_4^{-1}, x_4^3\rangle$ of the subgroup H in Section 4 is weakly leaf-invariant but it is not leaf-invariant. The notion of a weakly leaf-invariant subgroup yields the following

Lemma 7.4. *A normal subgroup $UK \trianglelefteq F$ is σ-invariant if and only if $\sigma \rightsquigarrow_\pi \mathrm{id}$.*

Proof. Since $UK \trianglelefteq F$, we have $UK = \mathrm{Core}_F(UK) = \ker(\pi)$. Thus $\mathrm{im}(\pi) \cong F/\ker(\pi) = F/UK$. If UK is σ-invariant, σ induces an endomorphism $\bar{\sigma}\colon F/UK \to F/UK$ and, as $F/UK \cong \mathrm{im}(\pi)$, it induces an endomorphism $\gamma\colon \mathrm{im}(\pi) \to \mathrm{im}(\pi)$ so that the diagram

$$\begin{array}{ccc} F & \xrightarrow{\sigma} & F \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathrm{im}(\pi) & \xrightarrow[\gamma]{} & \mathrm{im}(\pi) \end{array}$$

commutes. Thus $\sigma \rightsquigarrow_\pi \mathrm{id}$. If, on the other hand, $\sigma\pi = \pi\gamma$ holds for a homomorphism $\gamma\colon \mathrm{im}(\pi) \to \mathrm{im}(\sigma\pi)$, each $g \in UK = \ker(\pi)$ satisfies $1 = 1^\gamma = (g^\pi)^\gamma = g^{\pi\gamma} = g^{\sigma\pi} = (g^\sigma)^\pi$. Hence $g^\sigma \in \ker(\pi) = UK$ and thus, UK is σ-invariant. □

Lemma 7.4 yields that a Φ-invariant normal subgroup is weakly leaf-invariant. However, there exist subgroups which are weakly leaf-invariant but not Φ-invariant (e.g. the subgroup $H = \langle a, bab^{-1}, b^3\rangle$ of the Basilica group in Section 4 satisfies $\sigma^2 \rightsquigarrow_\pi \mathrm{id}$ but not $\sigma \rightsquigarrow_\pi \mathrm{id}$; thus, it is weakly leaf-invariant but not Φ-invariant). The condition $UK \trianglelefteq F$ in Lemma 7.4 is necessary, as we have the following

Remark 7.5. *The condition $UK \trianglelefteq F$ in Lemma 7.4 is necessary, as the subgroup $H = \langle a, b^2, ba^3b^{-1}, bab^{-2}a^{-1}b^{-1}, ba^{-1}b^{-2}ab^{-1}\rangle$ of the Basilica group G is not normal in G, it satisfies $(UK)^\sigma \subseteq UK$; however, it does not satisfy $\sigma \rightsquigarrow_\pi \mathrm{id}$.*

On the other hand, the subgroup $H = \langle a, bab, ba^{-1}b, b^4\rangle$ of the Basilica group G satisfies $\sigma \rightsquigarrow_\pi \mathrm{id}$ but it does not satisfy $(UK)^\sigma \subseteq UK$ as $[F : \mathrm{Core}_F(UK)] = [F : \tilde{L}] = 8 \neq 4 = [F : UK]$.

A weakly leaf-invariant normal subgroup satisfies the following variant of our Reidemeister-Schreier Theorem:

Theorem 7.6. *A weakly leaf-invariant normal subgroup which has finite index in an invariantly finitely L-presented group is invariantly finitely L-presented.*

Proof. Let $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R}\rangle$ be invariantly finitely L-presented and let $H \cong UK/K$ be a finite index normal subgroup of G. As usual, we may consider $\mathcal{Q} = \emptyset$ as G is invariantly L-presented. Let $\tilde{\mathcal{V}} \subseteq \mathcal{V}$ be the set from Lemma 5.12. Since H is weakly leaf-invariant, the weak-leafs Ψ in Definition 7.2 satisfy $\Psi = \{\psi\delta \mid \psi \in \Phi, \delta \in \tilde{\mathcal{V}}, \psi\delta \notin \tilde{\mathcal{V}}\}$. By Lemma 7.4, each $\psi\delta \in \Psi$ induces an endomorphism of the normal subgroup $UK \trianglelefteq F$. Let $\mathcal{T}$ be a Schreier transversal for UK in F and let $\mathcal{Y}$ denote the Schreier generators of the subgroup UK. Then each endomorphism $\psi\delta \in \Psi$ of UK translates to an endomorphism $\widehat{\psi\delta}$ of the free group $F(\mathcal{Y})$. Consider the invariant finite L-presentation

$$\langle \mathcal{Y} \mid \emptyset \mid \{\widehat{\psi\delta} \mid \psi\delta \in \Psi\} \cup \{\widehat{\delta_t} \mid t \in \mathcal{T}\} \mid \{\tau(r^\sigma) \mid r \in \mathcal{R}, \sigma \in \tilde{\mathcal{V}}\}\rangle, \tag{10}$$

where δ_t denotes the endomorphism of UK which is induced by conjugation by $t \in \mathcal{T}$. The finite L-presentation in Eq. (10) defines the normal subgroup H. This statement follows with the same techniques as above; in particular, it follows from rewriting the presentation in Eq. (7). □

The subgroup H in Section 4 is a normal subgroup satisfying $\sigma^2 \rightsquigarrow_\pi \mathrm{id}$. Hence, Theorem 7.6 shows that this subgroup is invariantly finitely L-presented. Even non-invariant L-presentations may give rise to invariant subgroup L-presentations as the following shows:

Remark 7.7. *There are non-invariant L-presentation $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R}\rangle$ and finite index subgroups $H \leq G$ that satisfy $(UK)^\sigma \subseteq UK$ for each $\sigma \in \Phi^*$. For instance, the finite L-presentation of Baumslag's group G in* [14] *is non-invariant (see the proof of Proposition 2.1) while its index-*3 *subgroup $H = \langle a^3, b, t\rangle$ satisfies $(UK)^\sigma \subseteq UK$ for each $\sigma \in \Phi$. The subgroup H even admits an invariant L-presentation over the generators $x = a^3$ and $y = a^2ta^{-2}$ given by*

$$\langle \{x, y\} \mid \emptyset \mid \{\delta_t, \delta_{t^2}\} \mid \{y^{-1}xyx^{-4}\}\rangle$$

where δ_t is induced by the map $x \mapsto x$ and $y \mapsto xyx^{-3}$ and δ_{t^2} is induced by the map $x \mapsto x$ and $y \mapsto xyx^{-2}$.

The finite L-presentations for finite index subgroups in Proposition 6.3, Theorem 7.1, and Theorem 7.6, are derived from the group's L-presentation $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R}\rangle$ by restricting to those endomorphisms in Φ^* which restrict to the subgroup. However, there are subgroups of an invariantly L-presented group so that no endomorphism from Φ^* restricts to the subgroup. In this case the finite L-presentation for the finite index subgroup needs to be constructed as a finite extension of the finitely L-presented stabilizing core L as in the proof of Theorem 1.1. The following remark gives an example of a subgroup of the invariantly finitely L-presented Basilica group so that no endomorphism from Φ^* restricts to the subgroup:

Remark 7.8. *Let $H = \langle b^2, a^3, ab^2a^{-1}, a^{-1}b^2a, bab^{-1}a\rangle$ denote a subgroup of the Basilica group G. Then H is a normal subgroup with index* 6 *in G. We are not able to find an invariant finite L-presentation for H.*

The subgroup H admits the permutation representation $\pi\colon F \to \mathrm{Sym}(UK\backslash F)$. We have

$$\pi\colon \begin{cases} a & \mapsto & (1,2,3)(4,6,5) \\ b & \mapsto & (1,4)(2,5)(3,6) \end{cases} \quad \textit{and} \quad \sigma\pi\colon \begin{cases} a & \mapsto & (\,) \\ b & \mapsto & (1,2,3)(4,6,5) \end{cases}$$

as well as

$$\sigma^2\pi\colon \begin{cases} a & \mapsto & (1,3,2)(4,5,6) \\ b & \mapsto & (\,) \end{cases} \quad \textit{and} \quad \sigma^3\pi\colon \begin{cases} a & \mapsto & (\,) \\ b & \mapsto & (1,3,2)(4,5,6). \end{cases}$$

Clearly, $\sigma^3 \rightsquigarrow_\pi \sigma$ but, for each $0 < \ell < 3$, we do not have $\sigma^\ell \rightsquigarrow_\pi \mathrm{id}$. The homomorphism $\gamma\colon \mathrm{im}(\sigma\pi) \to \mathrm{im}(\sigma^3\pi)$ with $\sigma^3\pi = \sigma\pi\gamma$ is bijective. Suppose there existed $\sigma^n \in \Phi^*$ so that the subgroup UK is σ^n-invariant. By Lemma 7.4, the normal subgroup UK is σ^n-invariant if and only if $\sigma^n \rightsquigarrow_\pi \mathrm{id}$ holds. Clearly $n > 3$. Since $\sigma^n \rightsquigarrow_\pi \mathrm{id}$, there exists a homomorphism $\psi\colon \mathrm{im}(\pi) \to \mathrm{im}(\sigma^n\pi)$ so that $\sigma^n\pi = \pi\psi$. We obtain $\pi\psi = \sigma^n\pi = \sigma^{n-3}\,\sigma^3\pi = \sigma^{n-3}\,\sigma\pi\gamma = \sigma^{n-2}\pi\gamma$. Iterating *this rewriting process eventually yields a positive integer* $0 \leq \ell < 3$ *so that* $\pi\psi = \sigma^n\pi = \sigma^\ell\pi\gamma^m$ *for some* $m \in \mathbb{N}$*. As* γ *is bijective, this yields that* $\sigma^\ell\pi = \pi\psi\gamma^{-m}$ *and hence* $\sigma^\ell \rightsquigarrow_\pi \mathrm{id}$*; a contradiction. Thus there is no positive integer* $n \in \mathbb{N}$ *so that* $\sigma^n \rightsquigarrow_\pi \mathrm{id}$*. Hence, no substitution in* Φ^* *restricts to the subgroup* UK*.*

Our method to compute a finite L-presentation for the subgroup H in Remark 7.8 is therefore given by our explicit proof of Theorem 1.1. If the subgroup H in Remark 7.8 admits an invariant finite L-presentation, the substitutions may not be related to the substitutions Φ of the finite L-presentation $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ of the Basilica group in Proposition 4.1. It is neither clear to us whether H admits an invariant finite L-presentation at all nor do we know how to possibly prove that H does not admit such invariant finite L-presentation.

8. Examples of Subgroup L-Presentations

In this section, we consider the subgroup $H = \langle a, bab^{-1}, b^3 \rangle$ of the Basilica group G as in Section 4. We demonstrate how our methods apply to this subgroup and, in particular, how to compute the L-presentation in Section 4.

Coset-enumeration for finitely L-presented groups [16] allows us to compute the permutation representation $\pi\colon F \to \mathrm{Sym}(UK\backslash F)$ for the group's action on the right-cosets. A Schreier transversal for H in G is given by $\mathcal{T} = \{1, b, b^2\}$ and we have

$$\pi\colon F \to \mathcal{S}_n, \left\{ \begin{array}{lcl} a & \mapsto & (\,) \\ b & \mapsto & (1,2,3). \end{array} \right.$$

Moreover, H is a normal subgroup with index 3 in G and it satisfies $\sigma^2 \rightsquigarrow_\pi \mathrm{id}$. By Lemma 5.13, there exists an integer $k \geq 2$ so that $\sigma^k \sim \mathrm{id}$; we can verify that $\sigma^4\pi = \pi$ holds. Thus $\sigma^4 \sim \mathrm{id}$. In particular, the subgroup H is (weakly) leaf-invariant and normal. Therefore the following techniques apply to this subgroup:

- As the subgroup H is a finite index subgroup of an invariantly finitely L-presented group G, the general methods of Proposition 6.3 and Theorem 6.1 apply.
- As the subgroup H is leaf-invariant, the methods in Theorem 7.1 apply.
- As the subgroup H is weakly leaf-invariant and normal, the methods in Theorem 7.6 apply.

We demonstrate these different techniques for the subgroup H. First, we consider the general method from Proposition 6.3. Note that the stabilizing subgroup L and stabilizing core $\tilde{L}$ coincide by Corollary 5.17. The stabilizing subgroups $L = \tilde{L}$ have index 9 in F and a Schreier generating set for $L = \tilde{L}$ is given by

$$\begin{array}{llll} x_1 = a^3 & x_4 = abab^{-1}a^{-2} & x_7 = a^2bab^{-1} & x_{10} = b^2a^2ba^{-2}. \\ x_2 = bab^{-1}a^{-1} & x_5 = ab^2a^{-1}b^{-2} & x_8 = a^2b^2a^{-2}b^{-2} & \\ x_3 = b^3 & x_6 = b^2aba^{-1} & x_9 = b^2a^3b^{-2} & \end{array}$$

Let F denote the free group over $\{a, b\}$ and let E denote the free group over $\{x_1, \ldots, x_{10}\}$. The Reidemeister rewriting $\tau\colon F \to E$ allows us to rewrite the iterated relation $r = [a, a^b]$. We obtain $\tau(r) = x_1^{-1}x_{10}^{-1}x_6\,x_{10}^{-1}x_9\,x_3$. Furthermore, the rewriting τ allows us to translate the substitution σ of the Basilica group to an endomorphism of the free group E. The homomorphism $\widehat{\sigma}\colon E \to E$ is induced by the map

$$\begin{array}{lcllcl} x_1 & \mapsto & x_3^2, & x_6 & \mapsto & x_8\,x_9, \\ x_2 & \mapsto & x_5, & x_7 & \mapsto & x_3\,x_2\,x_5\,x_6, \\ x_3 & \mapsto & x_1, & x_8 & \mapsto & x_3\,x_2\,x_4\,x_{10}^{-1}x_8^{-1}, \\ x_4 & \mapsto & x_6\,x_2^{-1}x_3^{-1}, & x_9 & \mapsto & x_8\,x_{10}\,x_8\,x_{10}, \\ x_5 & \mapsto & x_8^{-1}, & x_{10} & \mapsto & x_8\,x_{10}\,x_7\,x_3^{-1}. \end{array}$$

Similarly, the conjugation actions δ_a and δ_b which are induced by conjugation with a and b, respectively, translate to endomorphisms $\widehat{\delta_a}$ and $\widehat{\delta_b}$ of the free group E. By Proposition 6.3, the stabilizing subgroups $L = \tilde{L}$ are finitely L-presented by

$$M = L/K \cong \langle \{x_1, \ldots, x_{10}\} \mid \emptyset \mid \{\widehat{\sigma}, \widehat{\delta_a}, \widehat{\delta_b}\} \mid \{x_1^{-1}x_{10}^{-1}x_6\,x_{10}^{-1}x_9\,x_3\}\rangle.$$

The subgroup H satisfies the short exact sequence $1 \to M \to H \to \mathbb{Z}_3 \to 1$ with a cyclic group $\mathbb{Z}_3 = \langle \alpha \mid \alpha^3 = 1\rangle$ of order 3. Corollary 2.4 yields the following finite L-presentation for the subgroup H:

$$\langle \{\alpha, x_1, \ldots, x_{10}\} \mid \{\alpha^3 x_1^{-1}\} \cup \{(x_i^{-1})^\alpha x_i^{\delta_a}\}_{1\le i\le 10} \mid \tilde{\Psi} \mid \{x_1^{-1}x_{10}^{-1}x_6\,x_{10}^{-1}x_9\,x_3\}\rangle.$$

where the substitutions $\widehat{\Psi} = \{\widehat{\sigma}, \widehat{\delta_a}, \widehat{\delta_b}\}$ of M's finite L-presentation are dilated to endomorphisms $\tilde{\Psi} = \{\widetilde{\sigma}, \widetilde{\delta_a}, \widetilde{\delta_b}\}$ of the free group over $\{\alpha, x_1, \ldots, x_{10}\}$ as in the proof of Proposition 2.3.

Secondly, the subgroup H is (weakly) leaf-invariant and normal. Therefore, the methods in Section 7 apply. First, we consider the construction in Theorem 7.1 for leaf-invariant subgroups: A Schreier generating set for the subgroup UK is given by $x_1 = a$, $x_2 = bab^{-1}$, $x_3 = b^2ab^{-2}$, and $x_4 = b^3$. Since $\sigma^4\pi = \pi$, the subgroup H is σ^4-invariant and its suffices to rewrite the relation $r = [a, b]$ and its images

$\tau(tr^{\sigma^i}t^{-1})$, $0 \le i < 4$. These have the form:

i	$t=1$	$t=b$	$t=b^2$
0	$x_1^{-1}x_4^{-1}x_3^{-1}x_4\,x_1\,x_4^{-1}x_3\,x_4$	$x_2^{-1}x_1^{-1}x_2\,x_1$	$x_3^{-1}x_2^{-1}x_3\,x_2$
1	$x_4^{-1}x_2^{-1}x_4^{-1}x_3\,x_4\,x_2^{-1}x_4\,x_1$	$x_4^{-1}x_3^{-1}x_1\,x_3^{-1}x_4\,x_2$	$x_1^{-1}x_4^{-1}x_2\,x_4\,x_1^{-1}x_3$
2	$x_1^{-2}x_4^{-1}x_2^{-2}x_4\,x_1^2\,x_4^{-1}x_2^2\,x_4$	$x_2^{-2}x_4^{-1}x_3^{-2}x_4\,x_2^2\,x_4^{-1}x_3^2\,x_4$	$x_3^{-2}x_1^{-2}x_3^2\,x_1^2$
3	$x_4^{-2}x_3^{-2}x_4^{-1}x_2^2\,x_4\,x_3^{-2}x_4^2\,x_1^2$	$x_4^{-1}x_1^{-2}x_4^{-2}x_3^2\,x_4^2\,x_1^{-2}x_4\,x_2^2$	$x_4^{-1}x_2^{-2}x_4^{-1}x_1^2\,x_4\,x_2^{-2}x_4\,x_3^2$

Denote the set of relations above by $\mathcal{S}$. The endomorphism σ^4 translates, via τ, to an endomorphism of the free group over $\{x_1,\dots,x_4\}$ which is induced by the map

$$\widehat{\sigma^4}\colon \left\{\begin{array}{lcl} x_1 & \mapsto & x_1^4, \\ x_2 & \mapsto & x_4\,x_2^4\,x_4^{-1}, \\ x_3 & \mapsto & x_4^2\,x_3^4\,x_4^{-2}, \\ x_4 & \mapsto & x_4^4. \end{array}\right.$$

By Theorem 7.1, an L-presentation for the subgroup H is given by

$$H \cong \langle \{x_1,\dots,x_4\} \mid \emptyset \mid \{\widehat{\sigma^4}\} \mid \mathcal{S}\rangle.$$

Finally, the subgroup H is weakly leaf-invariant and normal. Therefore, the methods in Theorem 7.6 apply. As $\sigma^2 \rightsquigarrow_\pi \mathrm{id}$, it suffices to consider the relations $\tau(r)$, $\tau(r^\sigma)$, and their images under the substitutions $\widehat{\sigma^2}$ and $\widehat{\delta_b}$ (because a Schreier transversal is given by $\mathcal{T} = \{1, b, b^2\}$). The substitutions $\widehat{\sigma^2}$ and $\widehat{\delta_b}$ are induced by the maps

$$\widehat{\sigma^2}\colon \left\{\begin{array}{lcl} x_1 & \mapsto & x_1^2, \\ x_2 & \mapsto & x_3^2, \\ x_3 & \mapsto & x_4\,x_2^2\,x_4^{-1}, \\ x_4 & \mapsto & x_4^2, \end{array}\right. \quad \text{and} \quad \widehat{\delta_b}\colon \left\{\begin{array}{lcl} x_1 & \mapsto & x_2, \\ x_2 & \mapsto & x_3, \\ x_3 & \mapsto & x_4\,x_1\,x_4^{-1}, \\ x_4 & \mapsto & x_4. \end{array}\right.$$

Theorem 7.6 yields the finite L-presentation

$$H \cong \langle \{x_1,\dots,x_4\} \mid \emptyset \mid \{\widehat{\sigma^2}, \widehat{\delta_b}\} \mid \{\tau(r), \tau(r^\sigma)\}\rangle$$

for the subgroup H as in Section 4.

8.1. An Application to the Grigorchuk Group. As a finite L-presentation of a group allows the application of computer algorithms, we may use our constructive proof of Theorem 1.1 to investigate the structure of a self-similar group by its finite index subgroups as in [18]. As an application, we consider the Grigorchuk group, see [11], $\mathfrak{G} = \langle a,b,c,d\rangle$ and its normal subgroup $D = \langle d\rangle^{\mathfrak{G}}$. We show that the subgroup $D = \langle d\rangle^{\mathfrak{G}}$ has a minimal generating set with 8 elements and thereby we correct a mistake in [3,10].

The Grigorchuk group $\mathfrak{G}$ satisfies the well-known

Proposition 8.1 (Lysënok [24]). *The group $\mathfrak{G}$ is invariantly finitely L-presented by $\mathfrak{G} \cong \left\langle \{a,b,c,d\} \mid \{a^2,b^2,c^2,d^2,bcd\} \mid \{\sigma\} \mid \{(ad)^4,(adacac)^4\}\right\rangle$, where σ is the endomorphism of the free group over $\{a,b,c,d\}$ induced by the mapping $a \mapsto aca$, $b \mapsto d$, $c \mapsto b$, and $d \mapsto c$.*

It was claimed in [3, Section 4.2] and in [10, Section 6] that the normal subgroup $D = \langle d \rangle^{\mathfrak{G}}$ is generated by $\{d, d^a, d^{ac}, d^{aca}\}$. In the following, we show that the Reidemeister Schreier Theorem 1.1 allows us to prove that a generating set for $D = \langle d \rangle^{\mathfrak{G}}$ contains as least 8 elements. The coset-enumeration for finitely L-presented groups [16] and the solution to the subgroup membership problem for finite index subgroups [16] show that the subgroup

$$H = \langle\, d, d^a, d^{ac}, d^{aca}, d^{acac}, d^{acaca}, d^{acacac}, d^{acacaca} \,\rangle \tag{11}$$

has index 16 in $\mathfrak{G}$. It is a normal subgroup of $\mathfrak{G}$ so that $\mathfrak{G}/H$ is a dihedral group of order 16. In particular, the subgroup H and the normal subgroup $D = \langle d \rangle^{\mathfrak{G}}$ coincide. A permutation representation $\pi\colon F \to \mathcal{S}_n$ for the group's action on the right-cosets $UK\backslash F$ is given by

$$\pi\colon F \to \mathcal{S}_{16}, \quad \left\{ \begin{array}{lcl} a & \mapsto & (1,2)(3,5)(4,6)(7,9)(8,10)(11,13)(12,14)(15,16) \\ b & \mapsto & (1,3)(2,4)(5,7)(6,8)(9,11)(10,12)(13,15)(14,16) \\ c & \mapsto & (1,3)(2,4)(5,7)(6,8)(9,11)(10,12)(13,15)(14,16) \\ d & \mapsto & (\,). \end{array} \right.$$

Our variant of the Reidemeister-Schreier Theorem and the techniques introduced in Section 7 enable us to compute a subgroup L-presentation for D. For this purpose, we first note that $\sigma^3 \rightsquigarrow_\pi \mathrm{id}$. Hence, the normal core $D = \mathrm{Core}_F(UK) = \ker(\pi)$ is σ^3-invariant. The core $\mathrm{Core}_F(UK)$ is a free group with rank 49 and a Schreier transversal for D in $\mathfrak{G}$ is given by

$$1, a, b, ab, ba, aba, bab, (ab)^2, (ba)^2, a(ba)^2, b(ab)^2, (ab)^3, (ba)^3, a(ba)^3, b(ab)^3, (ab)^4.$$

A finite L-presentation with generators $d_0 = d$, $d_1 = d^a$, $d_2 = d^{ac}$, $d_3 = d^{aca}$, $d_4 = d^{acac}$, $d_5 = d^{acaca}$, $d_6 = d^{acacac}$, and $d_7 = d^{acacaca}$ is given by

$$D \cong \langle \{d_0, \ldots, d_7\} \mid \emptyset \mid \{\widehat{\sigma}, \delta_a, \delta_b\} \mid \mathcal{R}\,\rangle,$$

where the iterated relations are

$$\mathcal{R} = \left\{ d_0^2, [d_1,d_0], [d_1,d_4], [d_7, d_3\, d_4]^4, [d_7\, d_0, d_3\, d_4], (d_3\, d_7\, d_4\, d_0)^2, (d_7\, d_4^{d_3}\, d_0\, d_3^{d_4})^2 \right\}$$

and the endomorphisms $\{\widehat{\sigma}, \delta_a, \delta_b\}$ are induced by the maps

$$\delta_a\colon \left\{\begin{array}{lcl} d_0 & \mapsto & d_1, \\ d_1 & \mapsto & d_0, \\ d_2 & \mapsto & d_3, \\ d_3 & \mapsto & d_2, \\ d_4 & \mapsto & d_5, \\ d_5 & \mapsto & d_4, \\ d_6 & \mapsto & d_7, \\ d_7 & \mapsto & d_6, \end{array}\right. ,\quad \delta_b\colon \left\{\begin{array}{lcl} d_0 & \mapsto & d_0, \\ d_1 & \mapsto & d_2, \\ d_2 & \mapsto & d_1, \\ d_3 & \mapsto & d_4^{d_0}, \\ d_4 & \mapsto & d_3^{d_0}, \\ d_5 & \mapsto & d_6, \\ d_6 & \mapsto & d_5, \\ d_7 & \mapsto & d_7^{d_0}, \end{array}\right. ,\quad \text{and}\quad \widehat{\sigma}\colon \left\{\begin{array}{lcl} d_0 & \mapsto & d_0, \\ d_1 & \mapsto & d_0^{d_7^{d_3}}, \\ d_2 & \mapsto & d_0^{d_7^{d_4}}, \\ d_3 & \mapsto & d_0^{d_7^{d_4} d_7^{d_3}}, \\ d_4 & \mapsto & d_0^{d_7^{d_3} d_7^{d_4}}, \\ d_5 & \mapsto & d_0^{d_7^{d_3} d_7^{d_4} d_7^{d_3}}, \\ d_6 & \mapsto & d_0^{d_7^{d_4} d_7^{d_3} d_7^{d_4}}, \\ d_7 & \mapsto & d_0^{d_7^{d_4} d_7^{d_3} d_7^{d_4} d_7^{d_3}}. \end{array}\right.$$

The latter L-presentation of the normal subgroup D allows us to compute the abelianization $D/[D,D]$ using the methods from [2]. These computations yield that $D/[D,D] \cong (\mathbb{Z}_2)^8$ is 2-elementary abelian of rank 8. Hence, the normal subgroup D has a minimal generating set of length at least 8. Because a generating set with 8 generators was already given in Eq. (11), a minimal generating set of D has precisely 8 elements. In particular, this shows that $D \neq \langle d, d^a, d^{ac}, d^{aca}\rangle$. The latter mistake could have been detected also by computing the abelianization of the image of $D = \langle d\rangle^{\mathfrak{G}}$ in a finite quotient of $\mathfrak{G}$ (e.g. the quotient $G/\mathrm{Stab}(n)$ for $n \geq 4$) by hand or using a computer algebra system such as Gap.

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