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HOCHSCHILD TWO-COCYCLES AND THE GOOD TRIPLE $(As, Hoch, Mag^{\infty})$

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Abstract. The aim of this paper is to introduce the category of Hoch-algebras whose objects are associative algebras equipped with an extra magmatic operation $\succ$ verifying the following relation motivated by the Hochschild two-cocycle identity:

$$\mathcal{R}_2: \quad (x \succ y) * z + (x * y) \succ z = x \succ (y * z) + x * (y \succ z).$$

Such algebras appear in mathematical physics with $\succ$ associative under the name of compatible products. Here, we relax the associativity condition. The free Hoch-algebra over a K-vector space is then given in terms of planar rooted trees and the triple of operads $(As, Hoch, Mag^{\infty})$ endowed with the infinitesimal relations is shown to be good. Hence, according to Loday's theory, we then obtain an equivalence of categories between connected infinitesimal *Hoch*-bialgebras and Mag^{∞}-algebras.

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Notation: In the sequel K is a field. We adopt Sweedler notation for the binary cooperation Δ on a K-vector space V and set $\Delta(x) = x_{(1)} \otimes x_{(2)}$. For a K-vector space V, we set $\bar{T}(V) := \bigoplus_{n>0} V^{\otimes n}$.

1. Introduction

The well-known Poincaré-Birkhoff-Witt and the Cartier-Milnor-Moore theorems together can be rephrased as follows:

Theorem 1.1. (CMM-PBW) [6] *For any cocommutative (associative) bialgebra $\mathcal{H}$, $Com^c - As$-bialgebra for short, the following are equivalent.*

1. *$\mathcal{H}$ is connected;*
2. *$\mathcal{H}$ is isomorphic to $U(Prim\ \mathcal{H})$ as a bialgebra;*
3. *$\mathcal{H}$ is isomorphic to $Com^c(Prim\ \mathcal{H})$ as a coalgebra,*

where U is the usual enveloping functor and Prim $\mathcal{H}$ the usual Lie algebra of the primitive elements of $\mathcal{H}$.

In the theory developed by J.-L. Loday [6], this result is rephrased by saying that the triple of operads (Com, As, Lie), endowed with the usual Hopf relation, is good, where Com, As, and Lie stand respectively for the operads of commutative, associative and Lie algebras. Other good triples of operads equipped with other relations than the usual Hopf one, have been found since. A summary can be found in [6], see also [3,4] for other examples.

It has been shown in [3] that the triple of operads $(As, Dipt, Mag^\infty)$ endowed with the semi-infinitesimal relations is good. The operad $Dipt$ is governed by dipterous algebras which are associative algebras equipped with an extra left module on themselves, see also [7], and Mag^∞ is governed by Mag^∞-algebras, i.e., K-vector spaces having one n-ary (magmatic) generating operation for each integer $n > 1$. We then obtained that the category of connected infinitesimal dipterous bialgebras, $As^c - Dipt$-bialgebras for short, was equivalent to the category of Mag^∞-algebras. In this paper, we propose another equivalence of categories involving Mag^∞: the category of connected infinitesimal $Hoch$-bialgebras is equivalent to the category of Mag^∞-algebras.

In Section 2, we introduce $Hoch$-algebras, give examples and an explicit construction of the free $Hoch$-algebra over a K-vector space. In Section 3, we introduce the notion of (connected) infinitesimal $Hoch$-bialgebras. In Section 4 we prove the announced equivalence of categories. In Section 5, we deal with unital $Hoch$-algebras and close by Section 6 with two other good triples involving the operad $Hoch$.

2. The free $Hoch$-algebra

A $Hoch$-algebra G is a K-vector space equipped with an associative operation $*$ and a magmatic operation $\succ$ verifying:

$$\mathcal{R}_2 : (x \succ y) * z + (x * y) \succ z = x \succ (y * z) + x * (y \succ z),$$

for all $x, y, z \in G$.

Remark 2.1. 1) Recall that a formal deformation of an associative algebra $(A, *)$ is a $K[[t]]$-bilinear multiplication law $m_t : A[[t]] \otimes_{K[[t]]} A[[t]] \mapsto A[[t]]$ on the space $A[[t]]$ of formal power series in a variable t with coefficients in A, satisfying the following properties:

$$m_t(a, b) = a * b + m_1(a, b)t + m_2(a, b)t^2 + \dots,$$

for $a, b \in A$ where m_t is associative, that is: the equation $m_t(m_t(a, b), c) = m_t(a, m_t(b, c))$ for $a, b, c \in A$ holds. It is well known that m_1 satisfies $\mathcal{R}_2$ if and only if m_t is associative modulo t^2. In this case, $A[[t]]$ equipped with the initial associative operation $*$ and m_1 is a $Hoch$-algebra.

2) when $\succ$ turns out to be associative, such algebras appear in mathematical physics in the works of A. Odesskii and V. Sokolov [9,8], A.B. Goncharov [2] and V. Dotsenko [1]. See also [5] for others examples of such algebras on matrices. In this paper, we relax the associativity condition on $\succ$.

Let V be a K-vector space. The free $Hoch$-algebra over V is defined as follows. It is equipped with a linear map $i : V \to Hoch(V)$ and for any $Hoch$-algebra G and any linear map $f : V \to G$, there exists a unique $Hoch$-algebra morphim $\phi : Hoch(V) \to G$ such that $\phi \circ i = f$. We now give an explicit construction of the free $Hoch$-algebra over a K-vector space.
Denote by T_n the set of rooted planar trees (degrees at least 2) with n leaves. The cardinalities of T_n are registered under the name *A001003 little Schroeder numbers* of the Online Encyclopedy of Integer Sequences. For $n = 1, 2, 3$, we get:

$T_1 = \{ \mid \},\ T_2 = \{$  $\},\ T_3 = \{$  $\}.$

Define grafting operations by:

$$[\cdot, \ldots, \cdot] : T_{n_1} \times \ldots \times T_{n_p} \to T_{n_1 + \ldots + n_p}, \quad (t_1, \ldots, t_p) \mapsto [t_1, \ldots, t_p] := t_1 \vee \ldots \vee t_p,$$

where the tree $t_1 \vee \ldots \vee t_p$ is the tree whose roots of the t_i have been glued together and a new root has been added. Observe that any rooted planar tree t can be decomposed in a unique way via the grafting operation as $t_1 \vee \ldots \vee t_p$. Set $T_\infty := \bigoplus_{n>0} KT_n$. Define over $\bar{T}(T_\infty)$, the following binary operations, first on trees, then by bilinearity:

$$Concatenation: \quad (t_1 \ldots t_p) * (s_1 \ldots s_q) := t_1 \ldots t_p s_1 \ldots s_q,$$

$$(t_1 \ldots t_p) \succ (s_1 \ldots s_q) := \sum_{k=1}^{q} \sum_{i=1}^{p} t_1 \ldots t_{p-i}[t_{p-(i-1)}, \ldots, t_p, s_1, \ldots, s_k] s_{k+1} \ldots s_q.$$

For instance we get:

$\mid \mid \mid \succ \mid := \mid \mid$ 

$\mid \succ \mid$ 

Theorem 2.2. *The K-vector space $\bar{T}(T_\infty)$ endowed with the operations $*$ and $\succ$ is the free $Hoch$-algebra over K.*

Proof. Let $x := x_1 \dots x_m$, $y := y_1 \dots y_n$ and $z := z_1 \dots z_p$. We get:

$$\begin{aligned}
(x \succ y) * z + (x * y) \succ z &= \sum_{k=1}^{n} \sum_{i=0}^{m-1} x_1 \dots x_{m-(i+1)} [x_{m-i}, \dots, x_m, y_1, \dots, y_k] y_{k+1} \dots y_n z_1 \dots z_p \\
&+ \sum_{k=1}^{p} \sum_{i=0}^{n-1} x_1 \dots x_m y_1 \dots y_{n-(i+1)} [y_{n-i}, \dots, y_n, z_1, \dots, z_k] z_{k+1} \dots z_p \\
&+ \sum_{k=1}^{p} \sum_{i=0}^{m-1} x_1 \dots x_{m-(i+1)} [x_{m-i}, \dots, x_m, y_1, \dots, y_n, z_1 \dots, z_k] z_{k+1} \dots z_p
\end{aligned}$$

$$\begin{aligned}
x \succ (y * z) + x * (y \succ z) &= \sum_{k=1}^{n} \sum_{i=0}^{m-1} x_1 \dots x_{m-(i+1)} [x_{m-i}, \dots, x_m, y_1, \dots, y_k] y_{k+1} \dots y_n z_1 \dots z_p \\
&+ \sum_{k=1}^{p} \sum_{i=0}^{m-1} x_1 \dots x_{m-(i+1)} [x_{m-i}, \dots, x_m, y_1, \dots, y_n, z_1, \dots, z_k] z_{k+1} \dots z_p \\
&+ \sum_{k=1}^{p} \sum_{i=0}^{n-1} x_1 \dots x_m y_1 \dots y_{n-(i+1)} [y_{n-i}, \dots, y_n, z_1, \dots, z_k] z_{k+1} \dots z_p,
\end{aligned}$$

showing that

$$(x \succ y) * z + (x * y) \succ z = x \succ (y * z) + x * (y \succ z),$$

holds for all forests of planar rooted trees x, y, z. Observe that any rooted planar tree $t := [t_1, \dots, t_n]$ can be rewritten as:

$$t = (t_1 * (t_2 \dots t_{n-1})) \succ t_n - t_1 * ((t_2 \dots t_{n-1}) \succ t_n)$$

Let G be a $Hoch$-algebra and $g \in G$ and $f : K \to G$ be a linear map. Consider the embedding $i : K \hookrightarrow \bar{T}(T_\infty)$ defined by $i(1_K) := |$ and define by induction the map $\phi : \bar{T}(T_\infty) \to G$ as follows:

$$\phi(|) = g,$$

$$\phi(t_1 \dots t_n) = \phi(t_1) *_G \phi(t_2) *_G \dots *_G \phi(t_n),$$

$$\phi(t) = (\phi(t_1) *_G (\phi(t_2) \dots \phi(t_{n-1}))) \succ_G \phi(t_n) - \phi(t_1) *_G ((\phi(t_2) \dots \phi(t_{n-1})) \succ_G \phi(t_n))$$

for any $t := [t_1, \dots, t_n]$ and extend ϕ by linearity. By construction, ϕ is a morphism of associative algebras. Using the fact that:

$$(x * y) \succ z - x * (y \succ z) = x \succ (y * z) - (x \succ y) * z,$$

and changes of indices in the involving sums, one shows that ϕ is also a morphism for the magmatic operations. It is then the only $Hoch$-morphism such that $\phi \circ i = f$. $\square$

As the operad $Hoch$ is nonsymmetric, the following holds. Let V be a K-vector space. The free $Hoch$-algebra over V is the K-vector space:

$$Hoch(V) := \bigoplus_{n>0} Hoch_n \otimes V^{\otimes n},$$

with $Hoch(K) := \bigoplus_{n>0} Hoch_n \simeq \bar{T}(T_\infty)$ (hence $Hoch_n$ is explicitely described in terms of forests of rooted planar trees) equipped with the operations $*$ and $\succ$ defined as follows:

$$((t_1 \ldots t_n) \otimes \omega) * ((s_1 \ldots s_p) \otimes \omega') = (t_1 \ldots t_n s_1 \ldots s_p) \otimes \omega\omega',$$

$$((t_1 \ldots t_n) \otimes \omega) \succ ((s_1 \ldots s_p) \otimes \omega') = ((t_1 \ldots t_n) \succ (s_1 \ldots s_p)) \otimes \omega\omega',$$

for any $\omega \in V^{\otimes n}, \omega' \in V^{\otimes p}$. The embedding map $i : V \hookrightarrow Hoch(V)$ is defined by: $v \mapsto | \otimes v$.

Since the generating function associated with the Schur functor $\bar{T}$ is $f_{\bar{T}}(x) := \frac{x}{1-x}$ and with the Schur functor T_∞ is $f_{T_\infty}(x) := \frac{1+x-\sqrt{(1-6x+x^2)}}{4} = x + x^2 + 3x^3 + 11x^4 + 45x^5 + \ldots$, the generating function of the operad $Hoch$ is $f_{\bar{T}} \circ f_{T_\infty}$, that is:

$$f_{Hoch}(x) := \frac{1 + x - \sqrt{1 - 6x + x^2}}{3 - x + \sqrt{1 - 6x + x^2}} = x + 2x^2 + 6x^3 + 22x^4 + \ldots.$$

The sequence $(1, 2, 6, 22, 90, \ldots)$ is registered as $A006318$ under the name *Large Schroeder numbers* on the Online Encyclopedy of Integer Sequences.

Remark 2.3. When $\succ$ is associative, the corresponding free algebra has been constructed in [1].

3. Infinitesimal $Hoch$-bialgebras

By definition, an infinitesimal $Hoch$-bialgebra (or an $As^c - Hoch$-bialgebra for short) $(\mathcal{H}, * \succ, \Delta)$ is a $Hoch$-algebra equipped with a coassociative coproduct Δ verifying the following so-called nonunital infinitesimal relations:

$$\Delta(x \succ y) := x_{(1)} \otimes (x_{(2)} \succ y) + (x \succ y_{(1)}) \otimes y_{(2)} + x \otimes y.$$

$$\Delta(x * y) := x_{(1)} \otimes (x_{(2)} * y) + (x * y_{(1)}) \otimes y_{(2)} + x \otimes y.$$

It is said to be connected when $\mathcal{H} = \bigcup_{r>0} F_r\mathcal{H}$ with the filtration $(F_r\mathcal{H})_{r>0}$ defined as follows:

$$\text{(The primitive elements)} \quad F_1\mathcal{H} := Prim\ \mathcal{H} = \ker \Delta,$$

Set $\Delta^{(1)} := \Delta$ and $\Delta^{(n)} := (\Delta \otimes id_{n-1})\Delta^{(n-1)}$ with $id_{n-1} = \underbrace{id \otimes \ldots \otimes id}_{times\ n-1}$. Then,

$$F_r\mathcal{H} := \ker\ \Delta^{(r)}.$$

Theorem 3.1. *Let V be a K-vector space. Define on $Hoch(V)$, the free Hoch-algebra over V, the cooperation $\Delta : Hoch(V) \to Hoch(V) \otimes Hoch(V)$ recursively as follows:*

$$\Delta(i(v)) := 0, \text{ for all } v \in V,$$

$$\Delta(x \succ y) := x_{(1)} \otimes (x_{(2)} \succ y) + (x \succ y_{(1)}) \otimes y_{(2)} + x \otimes y.$$

$$\Delta(x \star y) := x_{(1)} \otimes (x_{(2)} \star y) + (x \star y_{(1)}) \otimes y_{(2)} + x \otimes y,$$

for all $x, y \in Hoch(V)$. Then $(Hoch(V), \Delta)$ is a connected infinitesimal Hoch-bialgebra.

Proof. This result can be proved by hand or can be seen as a corollary of the Theorem 4.2 in the next section. □

4. A good triple of operads

It can be usefull to have the following result when searching for good triples.

Lemma 4.1. *Let $\mathcal{C}, \mathcal{A}, \mathcal{Z}, \mathcal{Q}$ and $Prim$ be operads. Suppose the triples of operads $(\mathcal{C}, \mathcal{A}, \ Prim)$ and $(\mathcal{C}, \mathcal{Z}, Vect)$ equipped with the same compatibility relations, between products and coproducts, to be good. Suppose $\mathcal{A} = \mathcal{Z} \circ \mathcal{Q}$, then $Prim = \mathcal{Q}$.*

Proof. Since $(\mathcal{C}, \mathcal{Z}, Vect)$ is good, the notion of $\mathcal{C}^c - \mathcal{Z}$-bialgebra has a meaning and the following are equivalent:

(1) The $\mathcal{C}^c - \mathcal{Z}$-bialgebra $\mathcal{H}$ is connected.
(2) As $\mathcal{Z}$-algebra, $\mathcal{H}$ is isomorphic to the free $\mathcal{Z}$-algebra over its primitive elements.
(3) As $\mathcal{C}^c$-coalgebra, $\mathcal{H}$ is isomorphic to the cofree $\mathcal{C}^c$-coalgebra over its primitive elements.

As $(\mathcal{C}, \mathcal{A}, \ Prim)$ is good, the isomorphism of Schur functors $\mathcal{A} \simeq \mathcal{C}^c \circ Prim$ holds. Therefore, if V is a K-vector space, then $\mathcal{A}(V) = \mathcal{C}^c(Prim(V))$. As $\mathcal{A}(V) = \mathcal{C}^c(Prim(V))$, it obeys the third item, consequently the second item. But by hypothesis $\mathcal{A}(V) = \mathcal{Z}(\mathcal{Q}(V))$. As $\mathcal{A}(V)$ is isomorphic to the free $\mathcal{Z}$-algebra over its primitive elements, we get $\mathcal{Q}(V) = Prim(V)$. Consequently, $Prim = \mathcal{Q}$. □

Theorem 4.2. *The triple of operads $(As, Hoch, Mag^{\infty})$ endowed with the infinitesimal relation is good.*

Proof. Fix an integer $n > 0$. By $[n] - Mag$ we mean the nonsymmetric binary operad generated by n magmatic (binary) operations. In [4, Theorems 4.4 (and 4.5)], it has been shown that for each integer $n > 0$ the triples of operads $(As, [n] -$

$Mag, Prim\ [n]-Mag$) endowed with the infinitesimal relations were good. For $n=2$, the operadic ideal J generated by the primitive operations:

$$*(\succ \otimes id)+\succ(*\otimes id)-\succ(id\otimes *)-*(id\otimes \succ),$$

$$*(*\otimes id)-*(id\otimes *),$$

yields another good triple of operads $(As, [2]-Mag/J, Prim\ ([2]-Mag/J))$ (cf. [6, Proposition 3.1.1] on quotient triples), which turns out to be the triple $(As, Hoch, Prim\ Hoch)$. As $(As, As, Vect)$ endowed with the infinitesimal relation is good (cf [7]) and $Hoch = As\circ Mag^{\infty}$ using Section 2, we get $Prim\ Hoch = Mag^{\infty}$ by using Lemma 4.1. □

Remark 4.3. We give here another proof. The triple $(As, Hoch, Prim_{Hoch})$ can be shown to be good via [6, Theorem 2.5.1] checking hypotheses $H0$, $H1$ and $H2epi$ of this theorem. The two first hypotheses are straightforward. Let V be a K-vector space. For the last one, recall that the projection map $Hoch(V)\to V$ determines a unique coalgebra map $\phi(V): Hoch(V)\to As^c(V)$ mapping any tree of $Hoch(n)$, with $n\in\mathbb{N}^*$ to $1_n\in As^c(n)$. Consider $s(V): As^c(V)\to Hoch(V)$ mapping 1_n to the tree $|\ldots|$ (n times). It is also a coalgebraic morphism and $\phi(V)\circ s(V) = id_{As^c(V)}$. Hence $H2epi$ holds. Hence, using [6, Theorem 2.5.1], $(As, Hoch, Prim_{Hoch})$ is good. By construction $Hoch(K) = As(T_\infty)$ and $Hoch(V)$ is isomorphic to $As(Mag^{\infty}(V))$ using Section 2. As $(As, As, Vect)$, endowed with the same compatibility relations betwen products and coproducts, that is the infinitesimal one, is good, we get $Prim_{Hoch} = Mag^{\infty}$.

We then obtain another equivalence of categories involving the operad Mag^{∞}.

Corollary 4.4. *The category of connected infinitesimal As^c-Hoch-bialgebras and the category of Mag^{∞}-algebras are equivalent.*

$$\{conn.\ As^c-Hoch-bialg.\} \underset{Primitive}{\overset{U}{\leftrightarrows}} \{Mag^{\infty}-alg.\},$$

where U and $Primitive$ are respectively the universal enveloping functor and the primitive functor.

Proof. Apply [6, Theorem 2.6.3]. □

Remark 4.5. *The functor $Primitive$ is obviously given as follows. If $(\mathcal{H},*,\succ)$ is a connected infinitesimal Hoch-bialgebra, then for all integer $n>1$ and for all primitive elements $x_1,\ldots,x_n\in\mathcal{H}$, the element:*

$$[x_1,\ldots,x_n]_n := (x_1*\ldots*x_{n-1})\succ x_n - x_1*((x_2*\ldots*x_{n-1})\succ x_n),$$

will be primitive. The functor U acts as follows. Let $(M, ([, \dots,]_n)_{n>1})$ be a Mag^{∞}-algebra with the $[, \dots,]_n$ being its generating n-ary operations. Then $U(M)$ is given by $Hoch(M)/\sim$, where the equivalence relation $\sim$ consists in identifying,

$$(x_1 * \dots * x_{n-1}) \succ x_n - x_1 * ((x_2 * \dots * x_{n-1}) \succ x_n),$$

with $[x_1, \dots, x_n]_n$, for all $x_1 \dots, x_n \in M$.

5. Extension to a unit

Unital $Hoch$-algebras are $Hoch$-algebras equipped with a unit 1 whose compatibility with operations are defined as follows:

$$x \succ 1 = x = 1 \succ x, \quad x * 1 = x = 1 * x.$$

For instance, $Hoch_+(V) := K.1_K \oplus Hoch(V)$, where $Hoch(V)$ is the free $Hoch$-algebra over a K-vector space V is a unital $Hoch$-algebra with unit 1_K. This gives birth to unital Mag^{∞}-algebras which are Mag^{∞}-algebras such that the generating operations are related with the unit as follows:

$$[1, \cdot, \dots, \cdot]_n = 0,$$

$$[\cdot, \dots, 1, \dots, \cdot]_n = [\cdot, \dots, \cdot]_{n-1},$$

$$[\cdot, \dots, \cdot, 1]_n = 0.$$

Over $Hoch_+(V)$, one has a unital infinitesimal coproduct δ defined via the former coproduct Δ as follows:

$$\delta(x) = 1_K \otimes x + x \otimes 1_K + \Delta(x),$$

for any $x \in Hoch(V)$. The compatibility relations are the so-called unital infinitesimal relations defined as follows:

$$\Delta(x \succ y) := x_{(1)} \otimes (x_{(2)} \succ y) + (x \succ y_{(1)}) \otimes y_{(2)} - x \otimes y.$$

$$\Delta(x * y) := x_{(1)} \otimes (x_{(2)} * y) + (x * y_{(1)}) \otimes y_{(2)} - x \otimes y.$$

We then obtain the good triple of operads $(As, Hoch, Mag^{\infty})$ equipped with the unital infinitesimal relations.

6. Other triples of operads

The triple of operads $(As, Hoch, Mag^{\infty})$ endowed with the infinitesimal relations are not the only one involving the operad $Hoch$. By changing the compatibility relations, two other good triples of operads $(Com, Hoch, Prim_{Com}\ Hoch)$ and $(As, Hoch, Prim_{As}\ Hoch)$ endowed respectively with the Hopf relations and the semi-Hopf relations can be proposed. But contrarily to the case of the triple $(As, Hoch, Mag^{\infty})$ the explicit descriptions of operads of the primitive elements of these two other triples are open problems.

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