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ON THE FINITENESS PROPERTIES OF GENERALIZED LOCAL COHOMOLOGY MODULES

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Abstract. In this paper we study certain properties of generalized local cohomology modules with respect to a Serre class. We have proved that the membership of the generalized local cohomology of finite modules M and N in a Serre subcategory in the upper range (lower rang) depends on the support of module M (N).

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1. Introduction

Throughout this paper $(R, \mathfrak{m})$ is a commutative Noetherian local ring. For unexplained terminology from homological and commutative algebra we refer to [8] and [7]. Generalized local cohomology was given in the local case by J. Herzog [9] and in the more general case by Bijan-Zadeh [5]. Let R be a commutative Noetherian ring with identity, $\mathfrak{a}$ an ideal of R and let M, N be two R-modules. For an integer $i \geq 0$, the i-th generalized local cohomology module $H^i_{\mathfrak{a}}(M, N) = \varinjlim_{n\in\mathbb{N}} Ext^i_R(M/\mathfrak{a}^n M, N)$ with $M = R$, we obtain the ordinary local cohomology module $H^i_{\mathfrak{a}}(N)$ of N with respect to $\mathfrak{a}$ which was introduced by Grothendieck. We recall some properties of generalized local cohomology modules which we need in this note. For any ideal $\mathfrak{a}$ of R and two R-modules M and N the following statements hold:

(i) If $0 \to N' \to N \to N'' \to 0$ is an exact sequence of R-modules, then there are long exact sequences

$$\begin{aligned} &0 \to H^0_{\mathfrak{a}}(M, N') \to H^0_{\mathfrak{a}}(M, N) \to H^0_{\mathfrak{a}}(M, N'') \to \dots \\ &\to H^n_{\mathfrak{a}}(M, N') \to H^n_{\mathfrak{a}}(M, N) \to H^n_{\mathfrak{a}}(M, N'') \to \dots \end{aligned}$$

and

$$0 \to H^0_{\mathfrak{a}}(N'', M) \to H^0_{\mathfrak{a}}(N, M) \to H^0_{\mathfrak{a}}(N', M) \to \dots$$
$$\to H^n_{\mathfrak{a}}(M, N') \to H^n_{\mathfrak{a}}(M, N) \to H^n_{\mathfrak{a}}(M, N'') \to \dots$$

of generalized local cohomology modules.

(ii) If N is an $\mathfrak{a}$-torsion R-module, then there is an isomorphism $H^n_{\mathfrak{a}}(M, N) \to \mathrm{Ext}^n_R(M, N)$ for all $n \geq 0$.

Recall that a class S of R-modules is a *Serre subcategory* of the category of R-modules, when it is closed under taking submodules, quotients and extensions. In this paper, we study some properties of generalized local cohomology modules by using Serre classes. In [1], the authors have discussed the connection between $H^i_{\mathfrak{a}}(N)$ and the Serre classes of R-modules.
Using the generalized local cohomology modules, we can define $t_{\mathfrak{a}}(M, N)$ (*resp.* $t^{\mathfrak{a}}(M, N)$) of a pair (M, N) of R-modules relative to the ideal $\mathfrak{a}$ by

$$t^S_{\mathfrak{a}}(M, N) = t_{\mathfrak{a}}(M, N) = \inf\{i \in \mathbb{N} \mid H^i_{\mathfrak{a}}(M, N) \textit{ is not in } S\}$$
$$(\text{resp. } t^{\mathfrak{a}}_S(M, N) = t^{\mathfrak{a}}(M, N) = \sup\{i \in \mathbb{N} \mid H^i_{\mathfrak{a}}(M, N) \textit{ is not in } S\})$$

with the usual convention that the infimum (resp. supremum) of the empty set of integers is interpreted as $+\infty$ (resp. $-\infty$). We denote $t_{\mathfrak{a}}(R, N) = t_{\mathfrak{a}}(N)$ (*resp.* $t^{\mathfrak{a}}(R, N)$ $= t^{\mathfrak{a}}(N)$). We study the behavior of $t_{\mathfrak{a}}(M, N)$ and $t^{\mathfrak{a}}(M, N)$ under changing one of the M and N, when we fixed the one others.
This paper recovers some results regarding the local cohomology R-modules that have appeared in different papers.

2. Study of $t_{\mathfrak{a}}(M, N)$

Lemma 2.1. *Let N be in S and M a finitely generated R-module. Then for any $i \in \mathbb{N}_0$, the R- modules $Tor^R_i(M, N)$ and $Ext^i_R(M, N)$ are in S.*

Proof. Let $\cdots \to F_i \to F_{i-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ be a minimal free resolution of M. Then $Tor^R_i(M, N)$ is an R-subquotient of $F_i \bigotimes N \cong N^{rk(F_i)}$ and hence is in S, where $rk(F)$ means the rank of a free module F. The similar argument shows that $Ext^i_R(M, N)$ is in S. □

Theorem 2.2. *Let S be a Serre subcategory of the category of R-modules. Let $\mathfrak{a}$ be an ideal of R and N a finite R-module. Suppose that L is an R-module in S. If M is a finite R-module with $SuppM \subseteq SuppN$, then $t_{\mathfrak{a}}(M, L) \geq t_{\mathfrak{a}}(N, L)$, where the support of T is denoted by $SuppT$ for an R-module T. In particular if $SuppN = SuppM$, then $t_{\mathfrak{a}}(N, L) = t_{\mathfrak{a}}(M, L)$*

Proof. It is enough to show that $H^i_{\mathfrak{a}}(M,L)$ is in S for all $i < t_{\mathfrak{a}}(N,L)$ and all finitely generated R-modules M such that Supp$M \subseteq$ Supp$N$. To this end, we argue by induction on i. In view of hypothesis $\Gamma_{\mathfrak{a}}(L)$ is in S. Therefore, since $H^0_{\mathfrak{a}}(M,L) \cong Hom(M,\Gamma_{\mathfrak{a}}(L))$, we see, by Lemma 2.1, that $H^0_{\mathfrak{a}}(M,L)$ is in S. Now, suppose, inductively, that $i > 0$ and that the result has been proved for $i-1$. By Gruson's theorem (see [12, 4.1]), there is a chain $0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$ of submodules of M such that each of the factor M_i/M_{i-1} is a homomorphic image of a direct sum of finitely many copies of N. In view of the long exact sequence of generalized local cohomology modules that induced by the short exact sequence

$$0 \to M_{j-1} \to M_j \to M_j/M_{j-1} \to 0 \qquad j = 1,\dots,l,$$

it suffices to treat with only the case $l = 1$. So we have an exact sequence

$$0 \to K \to \bigoplus_{i=1}^{t} N \to M \to 0,$$

where $t \in \mathbb{N}$ and K is a finitely generated R-module. This induces the long exact sequence

$$0 \to H^0_{\mathfrak{a}}(M,L) \to H^0_{\mathfrak{a}}(\bigoplus_{i=1}^{t} N,L) \to H^o_{\mathfrak{a}}(K,L) \to \cdots$$

$$H^{i-1}_{\mathfrak{a}}(K,L) \to H^i_{\mathfrak{a}}(M,L) \to H^i_{\mathfrak{a}}(\bigoplus_{i=1}^{t} N.L).$$

By induction hypothesis, $H^{i-1}_{\mathfrak{a}}(K,L)$ is in S. Also, $H^i_{\mathfrak{a}}(\bigoplus_{i=1}^{t} N,L)$ is in S, because $H^i_{\mathfrak{a}}(\bigoplus_{i=1}^{t} N,L) \cong \bigoplus_{i=1}^{t} H^i_{\mathfrak{a}}(N,L)$ and $H^i_{\mathfrak{a}}(N,L)$ is in S, so that, in view of the above exact sequence, the R-module $H^i_{\mathfrak{a}}(M,L)$ is in S. □

Lemma 2.3. (i) *Let M be a finitely generated R-module, N an R-module and $t_{\mathfrak{a}}(N) > 0$. Then*

(1) $t^{\mathfrak{a}}(M,N/\Gamma_{\mathfrak{a}}(N)) = t^{\mathfrak{a}}(M,N)$

(2) $t_{\mathfrak{a}}(M,N) = t_{\mathfrak{a}}(M,N/\Gamma_{\mathfrak{a}}(N))$

(ii) *Let $x \in \mathfrak{a}$ be a regular element on N. Then*

(1) $t_{\mathfrak{a}}(M,N/xN) \geq t_{\mathfrak{a}}(M,N) - 1$

(2) $t^{\mathfrak{a}}(M,N) \geq t^{\mathfrak{a}}(M,N/xN)$.

Proof. Since $H^i_{\mathfrak{a}}(M,\Gamma_{\mathfrak{a}}(N)) = Ext^i_R(M,\Gamma_{\mathfrak{a}}(N))$, it follows from Lemma 2.1 that $H^i_{\mathfrak{a}}(M,\Gamma_{\mathfrak{a}}(N))$ is in S. Now, the claim is clear by the long exact sequence

$$\cdots \to H^i_{\mathfrak{a}}(M,\Gamma_{\mathfrak{a}}(N)) \to H^i_{\mathfrak{a}}(M,N) \to H^i_{\mathfrak{a}}(M,N/\Gamma_{\mathfrak{a}}(N)) \to \cdots.$$

(ii) It is clear by the long exact sequence

$$\cdots \to H^i_{\mathfrak{a}}(M,N) \to H^i_{\mathfrak{a}}(M,N) \to H^i_{\mathfrak{a}}(M,N/xN) \to \cdots.$$ □

Theorem 2.4. *Let $\mathfrak{a}$ be an ideal of R and L, M and N finitely generated R-modules.*

(i) *If $0 \to L \to M \to N \to 0$ is an exact sequence, then for any R-module C, we have $t_{\mathfrak{a}}(M, C) = \inf\{t_{\mathfrak{a}}(L, C), t_{\mathfrak{a}}(N, C)\}$*

(ii) $t_{\mathfrak{a}}(R, N) = \inf\{t_{\mathfrak{a}}(C, N) \mid C$ *is finitely generated over* $R\}$

(iii) *If $r < t_{\mathfrak{a}}(R/P, N)$ for all $P \in SuppM$, then $r < t_{\mathfrak{a}}(M, N)$.*

(iv) $t_{\mathfrak{a}}(M, L) = \inf\{t_{\mathfrak{a}}(R/P, L) \mid P \in Supp\ M\}$

(v) *If $l = pd(N) < \infty$, then $t_{\mathfrak{a}}(M, N) \geq t_{\mathfrak{a}}(M, R) - pd(N)$.*

Proof. (i), (ii) are clear by definition.
(iii) There is a prime filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$ of submodules of M, such that $M_i/M_{i-1} \cong R/P_i$ where $P_i \in$Supp$M$. We use induction on t. When $t = 1$, $H^r_{\mathfrak{a}}(M, N) = H^r_{\mathfrak{a}}(R/P, N)$ is in S. Now suppose that $t > 1$ and that the result has been proved for $t-1$. The exact sequence $0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0$ induces the long exact sequence $H^r_{\mathfrak{a}}(M_t/M_{t-1}, N) \to H^r_{\mathfrak{a}}(M_t, N) \to H^r_{\mathfrak{a}}(M_{t-1}, N)$. It follows that $H^r_{\mathfrak{a}}(M_t, N)$ is in S. This completes the proof of the theorem.
(iv) By using Theorem 2.2, $t_{\mathfrak{a}}(R/P, N) \geq t_{\mathfrak{a}}(M, N) = r$ for all $P \in$Supp$M$ and so we assume that $r = t_{\mathfrak{a}}(M, N) < t_{\mathfrak{a}}(R/P, N)$. Note that, in view of (iii) $H^r_{\mathfrak{a}}(M, N)$ is in S. This contradiction completes the proof.
(v) We use induction on $l = pd(N)$. If $l = 0$, then there is a nothing to prove. Now, assume that $l > 0$ and that the assertion holds for $l - 1$. We can construct exact sequence $0 \to T \to F \to N \to 0$ of finitely generated R-modules such that F is free and $pd(T) = l - 1$. By the induction hypothesis, $t_{\mathfrak{a}}(M, T) \geq t_{\mathfrak{a}}(M, R) - l + 1$. Let $i < t_{\mathfrak{a}}(M, R) - l$. Then, it follows from the exact sequence $H^i_{\mathfrak{a}}(M, F) \to H^i_{\mathfrak{a}}(M, N) \to H^{i+1}_{\mathfrak{a}}(M, T)$, that $H^i_{\mathfrak{a}}(M, N)$ is in S, and the result follows. □

Definition 2.5. An R-module N is said to be *Weakly Laskerian* if the set of associated primes of any quotient module of N is finite.

Remark 2.6. *If N is weakly Laskerian, then AssN is finite. This holds, by employing a method of proof which is similar to that used in [7, 2.1.1], N is $\mathfrak{a}$-torsion-free if and only if $\mathfrak{a}$ contains a non-zerodivisor on N.*

Theorem 2.7. *Let S be a Serre subcategory of the category of R-modules. Let $\mathfrak{a}$ be an ideal of R and M a finite R-module. Suppose that N a weakly Laskerian R-module of dimension n. If $t_{\mathfrak{a}}(N) > 0$, then the module $Hom(R/\mathfrak{a}, H^{t_{\mathfrak{a}}(M,N)}_{\mathfrak{a}}(M, N))$ is in S. Furthermore, if L is a finite R-module such that $SuppL \subseteq V(\mathfrak{a})$, where $V(\mathfrak{a})$ is the set of prime ideals of R containing $\mathfrak{a}$, then $Hom(L, H^{t_{\mathfrak{a}}(M,N)}_{\mathfrak{a}}(M, N))$ is in S.*

Proof. Set $t_{\mathfrak{a}}(M,N) = t$ and we use induction on $dim(N) = n$. If $n = 0$, then $N = \Gamma_m(N)$ and hence $H^i_{\mathfrak{a}}(M,N) = Ext^i_R(M, \Gamma_{\mathfrak{a}}(N))$ for all i. Therefore, since $t_{\mathfrak{a}}(N) > 0$, the R-module $\Gamma_{\mathfrak{a}}(N)$ is in S, it follows from Lemma 2.1, that $Hom(R/a, H^i_{\mathfrak{a}}(M,N)$ is in S. So suppose that $n > 0$ and that the result has been proved for smaller values of n. Since, $H^i_{\mathfrak{a}}(M,\Gamma_{\mathfrak{a}}(N)) \cong Ext^i_R(M,\Gamma_{\mathfrak{a}}(N))$ for all i, it follows from Lemma 2.1, that $Hom(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M,N))$ is in S if and only if $Hom(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M, N/\Gamma_{\mathfrak{a}}(N)))$ is in S. Thus we may assume that $\Gamma_{\mathfrak{a}}(N) = 0$. Then there exists $x \in \mathfrak{a}$ such that x is an N-sequence. The exact sequence $0 \to N \to N \to N/xN \to 0$ implies the following long exact sequence of generalized local cohomology modules

$$H^{t-1}_{\mathfrak{a}}(M,N) \overset{x}{\to} H^{t-1}_{\mathfrak{a}}(M,N) \overset{\theta}{\to} H^{t-1}_{\mathfrak{a}}(M,N/xN) \overset{\varphi}{\to} H^{t}_{\mathfrak{a}}(M,N) \overset{x}{\to} H^{t}_{\mathfrak{a}}(M,N) \overset{\psi}{\to} H^{t}_{\mathfrak{a}}(M,N/xN).$$

Using this exact sequence by the induction hypothesis and Lemmas 2.1, 2.3, it follows that the R-module $Hom(R/\mathfrak{a}, H^{t-1}_{\mathfrak{a}}(M,N/xN))$ is in S. Note that, by Lemma 2.1, $Ext^1_R(R/\mathfrak{a}, im\theta)$ is in S. Now, using the exact sequence

$$0 \to Hom(R/\mathfrak{a}, im\theta) \to Hom(R/\mathfrak{a}, H^{t-1}_{\mathfrak{a}}(M,N/xN)) \to Hom(R/\mathfrak{a}, im\varphi) \to Ext^1_R(R/\mathfrak{a}, im\theta),$$

we get $Hom(R/\mathfrak{a}, im\varphi) = Hom(R/\mathfrak{a}, (0 :_{H^t_{\mathfrak{a}}(M,N)} x)) = Hom(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M,N))$ is in S. The last part follows from Gruson's Theorem and the similar argument in Theorem 2.2. □

Theorem 2.8. *Let S be a Serre subcategory of the category of R-modules. Let $\mathfrak{a}$ be an ideal of R and N a weakly Laskerian R-module of finite krull dimension, such that $t_{\mathfrak{a}}(N) > 0$ and $H^i_{\mathfrak{a}}(M,N)$ is in S for all $i < t$. Let X be a submodule of $H^t_{\mathfrak{a}}(M,N)$ such that X is in S. Then $Hom(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M,N)/X)$ is in S.*

Proof. Let X be a submodule of $H^t_{\mathfrak{a}}(M,N)$ such that X is in S. The short exact sequence

$$0 \to X \to H^t_{\mathfrak{a}}(M,N) \to H^t_{\alpha}(M,N)/X \to 0$$

induces the following exact sequence

$$0 \to Hom(R/\mathfrak{a}, X) \to Hom(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M,N)) \to Hom(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M,N)/X) \to Ext^1_R(R/\mathfrak{a}, X).$$

Since $Hom(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M,N))$ and $Ext^1_R(R/\mathfrak{a}, X)$ are in S by Theorem 2.7 and Lemma 2.1, we have $Hom(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M,N)/X)$ is in S. □

The previous theorem recovers the [4, 3, 6, 10].

Example 2.9. *Let $r = f$-depth$(\mathfrak{a}, N) = 0$ and S be class of Artinian R-modules. Then $Hom(R/\mathfrak{a}, H^r_{\mathfrak{a}}(N))$ is not Artinian (cf. [11]). This example shows that if we delete the assumption $t_{\mathfrak{a}}(N) > 0$, then it may happen that $Hom(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M, N)/X)$ is not in S.*

Remark 2.10. *The following classes of modules are Serre subcategories and they are true in above theorems.*

(1) *The class of zero modules and $t_{\mathfrak{a}}(M, N) = grade(\mathfrak{a} + Ann(M), N)$.*
(2) *The class of Artinian modules and $t_{\mathfrak{a}}(M, N) = f$-depth$(\mathfrak{a} + Ann(M), N)$.*
(3) *The class of Noetherian modules and $t_{\mathfrak{a}}(M, N) = f_{\mathfrak{a}}(M, N)$, where $f_{\mathfrak{a}}(M, N)$, the $\mathfrak{a}$-finiteness dimension of a pair (M, N) of R-modules relative to the ideal $\mathfrak{a}$, is the least non-negative integer i such that $H^i_{\mathfrak{a}}(M, N)$ is not finitely generated.*
(4) *The class of R- modules with finite support and $t_{\mathfrak{a}}(M, N) = g$-depth$(\mathfrak{a} + Ann(M), N)$.*

3. Study of $t^{\mathfrak{a}}(M, N)$

Notation. The cohomological dimension $cd_{\mathfrak{a}}(M, N)$ of M and N with respect to $\mathfrak{a}$ is defined as $cd_{\mathfrak{a}}(M, N) = \sup\{i \geq 0 \mid H^i_{\mathfrak{a}}(M, N) \neq 0\}$. Note that if $pd_R(M)$ is finite, then, by easy induction, we can show that $cd_{\mathfrak{a}}(M, N) < \infty$.

Theorem 3.1. *Let S be a Serre subcategory of the category of R-modules. Let $\mathfrak{a}$ be an ideal of R, N a finite R-module and M a finite R-module with $pd(M) < \infty$, where we denote by $pd(T)$ the projective dimension over R of T for an R-module T. If L is a finite R-module with Supp$L \subseteq$Supp$N$, then $t^{\mathfrak{a}}(M, L) \leq t^{\mathfrak{a}}(M, N)$. In particular, if Supp$L$=Supp$N$, then $t^{\mathfrak{a}}(M, L) = t^{\mathfrak{a}}(M, N)$.*

Proof. It is enough to show that $H^i_{\mathfrak{a}}(M, L)$ belongs to S for all finite R-module L with Supp$L \subseteq$Supp$N$ and for all $i > t^{\mathfrak{a}}(M, N)$. Since $pd(M) < \infty$, so $cd_{\mathfrak{a}}(M, L)$ is finite, we have $H^i_{\mathfrak{a}}(M, L) = 0$ is in S for all $i > cd_{\mathfrak{a}}(M, L)$. We now argue by descending induction on i. Now, assume that $t^{\mathfrak{a}}(M, N) < i$ and that the claim holds for $i + 1$. By Gruson's Theorem (see [12, 4.1]), there is a chain $0 = L_0 \subset L_1 \subset \cdots \subset L_l = L$ of submodules of L such that each of the factor L_i/L_{i-1} is a homomorphic image of a direct sum of finitely many copies of N. In view of the long exact sequence of generalized local cohomology modules that induced by short exact sequence $0 \to L_{i-1} \to L_i \to L_i/L_{i-1} \to 0$, for $i = 1, \ldots, l$, it suffices to treat with only the case $l = 1$. So, we have an exact sequence $0 \to K \to \bigoplus_{i=1}^t N \to L \to 0$, where $t \in \mathbb{N}$ and K is a finitely generated R-module. This induces the long exact sequence

$$H_{\mathfrak{a}}^{i-1}(M,L) \to H_{\mathfrak{a}}^{i}(M,K) \to H_{\mathfrak{a}}^{i}(M,\bigoplus_{i=1}^{t} N) \to H_{\mathfrak{a}}^{i}(M,L) \to H_{\mathfrak{a}}^{i+1}(M,K).$$

By the induction hypothesis, $H_{\mathfrak{a}}^{i+1}(M,K)$ belongs to S. Also, $H_{\mathfrak{a}}^{i}(M,\bigoplus_{i=1}^{t}(N)) \cong \bigoplus_{i=1}^{t} H_{\mathfrak{a}}^{i}(M,N)$ belongs to S, because $i > t^{\mathfrak{a}}(M,N)$. Therefore, by the above exact sequence $H_{\mathfrak{a}}^{i}(M,L)$ belongs to S. □

Theorem 3.2. *Let $\mathfrak{a}$ be an ideal of R and L, C and N finitely generated R-modules.*

(i) *If $0 \to L \to N \to C \to 0$ is an exact sequence, then for any finitely generated R-module M, we have $t^{\mathfrak{a}}(M,N) = \sup\{t^{\mathfrak{a}}(M,C), t^{\mathfrak{a}}(M,L)\}$.*

(ii) $t^{\mathfrak{a}}(M,R) = \sup\{t^{\mathfrak{a}}(M,C) \mid C$ *is finitely generated over* R $\}$.

(iii) *Let $H_{\mathfrak{a}}^{r}(M,R/P)$ be in S for all $P \in SuppN$. Then $H_{\mathfrak{a}}^{r}(M,N)$ is in S.*

(iv) *If $pd_R(M) < \infty$, then $t^{\mathfrak{a}}(M,N) = \sup\{t^{\mathfrak{a}}(M,R/P) \mid P \in SuppN\}$.*

(v) *If $l = pd(M) < \infty$, then $t^{\mathfrak{a}}(M,N) - l \le t^{\mathfrak{a}}(R,N) = t^{\mathfrak{a}}(N)$.*

Proof. (i) and (ii) are clear by definition.

(iii) There is a prime filtration $0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_t = N$ of submodules of N, such that $N_i/N_{i-1} \cong R/P_i$ where $P_i \in \mathrm{Supp}N$. We use induction on t. When $t = 1$, $H_{\mathfrak{a}}^{r}(M,R/P) = H_{\mathfrak{a}}^{r}(M,N)$ is in S, where we put $P = P_i$. Now suppose that $t > 1$ and that the result has been proved for $t-1$. The exact sequence $0 \to N_{t-1} \to N_t \to N_t/N_{t-1} \to 0$ induces the long exact sequence

$$H_{\mathfrak{a}}^{r}(M,N_{t-1}) \to H_{\mathfrak{a}}^{r}(M,N_t) \to H_{\mathfrak{a}}^{r}(M,R/P_t).$$

It follows that $H_{\mathfrak{a}}^{r}(M,N_t)$ is in S. This completes the proof.

(iv) By using Theorem 3.1 $t^{\mathfrak{a}}(M,R/P) \le t^{\mathfrak{a}}(M,N) = r$ for all $P \in \mathrm{Supp}N$ and so we assume that $t^{\mathfrak{a}}(M,R/P) < t^{\mathfrak{a}}(M,N) = r$. By using (iii), $H_{\mathfrak{a}}^{r}(M,N)$ is in S. This contradiction completes the proof.

(v) We use induction on l. If $l = 0$, then there is nothing to prove. We can construct an exact sequence $0 \to M' \to F \to M \to 0$ of finitely generated R-modules such that F is free and $pd(M') = l-1$. By the induction hypothesis, $t^{\mathfrak{a}}(M',N) \le t^{\mathfrak{a}}(R,N) + l - 1$. Let $i > t^{\mathfrak{a}}(R,N) + l$. Then, it follows from the exact sequence

$$H_{\mathfrak{a}}^{i-1}(M',N) \to H_{\mathfrak{a}}^{i}(M,N) \to H_{\mathfrak{a}}^{i}(F,N) \to H_{\mathfrak{a}}^{i}(M',N)$$

that $H_{\mathfrak{a}}^{i}(M,N)$is in S. Hence $t^{\mathfrak{a}}(M,N) \le t^{\mathfrak{a}}(R,N) + l = t^{\mathfrak{a}}(N) + l$. □

Theorem 3.3. *Let S be a Serre subcategory of the category of R-modules. Let $\mathfrak{a}$ be an ideal of R, N a finitely generated R-module and M an R-module, $t = t^{\mathfrak{a}}(M, N)$. Assume that one of the following conditions is satisfied:*

(i) $t_{\mathfrak{a}}(N) > 0$,
(ii) $t^{\mathfrak{a}}(M, N) > pd(M)$.

Then the $H^t_{\mathfrak{a}}(M, N)/\mathfrak{a}H^t_{\mathfrak{a}}(M, N)$ belongs to S.

Proof. We prove by induction on $dimN = n$. If $n = 0$, then N is $\mathfrak{m}$-torsion, and hence $\mathfrak{a}$-torsion module. Therefore $H^t_{\mathfrak{a}}(M, \Gamma_{\mathfrak{a}}(N)) = Ext^t_R(M, N)$ is in S by Lemma 2.1. Thus the claim holds for $n = 0$. Now, suppose, inductively, that $n > 0$ and the result has been proved for all finitely generated R-module of dimension smaller than n. Since $t_{\mathfrak{a}}(N) > 0$, view of the long exact sequence of generalized local cohomology modules that is induced by the exact sequence $0 \to \Gamma_{\mathfrak{a}}(N) \to N \to N/\Gamma_{\mathfrak{a}}(N) \to 0$, we may assume that $\Gamma_{\mathfrak{a}}(N) = 0$. Then there exists $x \in a$ such that x is an N-sequence. The exact sequence $0 \to N \to N \to N/xN \to 0$ implies the following long exact sequence of generalized local cohomology modules

$$H^{i-1}_{\mathfrak{a}}(M, N/xN) \to H^i_{\mathfrak{a}}(M, N) \to H^i_{\mathfrak{a}}(M, N) \to H^i_{\mathfrak{a}}(M, N/xN).$$

It yields that $H^i_{\mathfrak{a}}(M, N/xN)$ belongs to S for all $i > t$. By using Lemma 2.3, $t^{\mathfrak{a}}(M, N/xN) \le t^{\mathfrak{a}}(M, N)$. If $t^{\mathfrak{a}}(M, N/xN) < t^{\mathfrak{a}}(M, N)$, then $H^t_{\mathfrak{a}}(M, N/xN)$ belongs to S. If $t^{\mathfrak{a}}(M, N/xN) = t^{\mathfrak{a}}(M, N)$, thus $H^t_{\mathfrak{a}}(M, N/xN)\Big/\mathfrak{a}H^t_{\mathfrak{a}}(M, N/xN)$ belongs to S by induction hypothesis. Now, consider the exact sequence

$$H^t_{\mathfrak{a}}(M, N) \overset{x}{\to} H^t_{\mathfrak{a}}(M, N) \overset{\theta}{\to} H^t_{\mathfrak{a}}(M, N/xN) \overset{\varphi}{\to} H^{t+1}_{\mathfrak{a}}(M, N),$$

which induces the following two exact sequences

$$H^t_{\mathfrak{a}}(M, N) \overset{x}{\to} H^t_{\mathfrak{a}}(M, N) \to im\theta \to 0 \quad \text{and} \quad 0 \to im\theta \to H^t_{\mathfrak{a}}(M, N/xN) \to im\varphi \to 0,$$

where we denote by $im\psi$ the image of a map ψ. Therefore, we can obtain the following two exact sequences:

$$H^t_{\mathfrak{a}}(M, N)/\mathfrak{a}H^t_{\mathfrak{a}}(M, N) \overset{x}{\to} H^t_{\mathfrak{a}}(M, N)/\mathfrak{a}H^t_{\mathfrak{a}}(M, N) \to im\theta/\mathfrak{a}im\theta \to 0, \qquad (**)$$

$$Tor^R_1(R/\mathfrak{a}, im\varphi) \to im\theta/\mathfrak{a}im\theta \to H^t_{\alpha}(M, N/xN)\Big/\mathfrak{a}H^t_{\mathfrak{a}}(M, N/xN) \to im\varphi/\mathfrak{a}im\varphi \to 0 \ (*)$$

Since $x \in \mathfrak{a}$, from (**) exact sequence, we deduce that, $H^t_{\mathfrak{a}}(M, N)/\mathfrak{a}H^t_{\mathfrak{a}}(M, N) \cong im\theta/\mathfrak{a}im\theta$. Now, $Tor^R_1(R/\mathfrak{a}, im\varphi)$ and $H^t_{\mathfrak{a}}(M, N/xN)\Big/\mathfrak{a}H^t_{\mathfrak{a}}(M, N/xN)$ belong in

S by Lemma 2.1. The claim follows by (*). In addition, in view of (ii) the assertion follows by repeating the above argument. □

The previous theorem recovers the [2, 3.1].

Example 3.4. *Let* $(R, \mathfrak{m})$ *be a commutative local ring* $t = cd_{\mathfrak{m}}(N) = 0, N \neq 0$ *and* S *be class of zero modules. Then* $H^t_{\mathfrak{m}}(N)/\mathfrak{m}H^t_{\mathfrak{m}}(N) = N/\mathfrak{m}N \neq 0$. *This example shows that if we delete the assumption* $t^{\mathfrak{a}}(M, N) > pd(M)$ *and* $t_{\mathfrak{a}}(N) > 0$, *then it may happen that* $H^t_{\mathfrak{a}}(M, N)/\mathfrak{a}H^t_{\mathfrak{a}}(M, N)$ *is not* S.

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