RINGS WHOSE SEMIGROUP OF RIGHT IDEALS IS J-TRIVIAL

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Received: 12 January 2011; Revised: 5 May 2011
Communicated by Abdullah Harmanci

Abstract. A semigroup $S$ is $J$-trivial if any two distinct elements of $S$ must generate distinct ideals of $S$. We investigate this condition for the semigroup of all right ideals of a ring under right ideal multiplication. There is a rich interplay between the underlying ring and the semigroup of all of its right ideals.

Mathematics Subject Classification (2010): 16D25, 16E25, 16N99, 20M10
Keywords: semigroup of right ideals, ring, $J$-trivial, idempotents, nilpotent, radicals, $\pi$-regular, strongly regular, right weakly regular, 0-cancellative

1. Introduction

Here $R$ is a ring. (Herein all rings are associative, not necessarily commutative, not necessarily with identity). Let $\mathbb{R}(R)$, $\mathbb{L}(R)$, and $\mathbb{I}(R)$ denote the multiplicative semigroups of right, respectively left, two-sided ideals of $R$. In previous works we considered these semigroups when they are bands (every element idempotent) [7, 8]. Rings for which every right ideal is idempotent are called right weakly regular (r.w.r.) rings, and have been studied in great detail. For a survey of r.w.r. rings, see [9].

In this paper we consider the $J$-trivial condition for the semigroups $\mathbb{R}(R)$ and $\mathbb{L}(R)$ and the consequences for the underlying ring $R$. A semigroup $S$ is said to be $J$-trivial if, whenever $a, b \in S$ such that $a$ and $b$ generate the same ideal in $S$, then $a = b$. (Here $S$ will always denote a semigroup and $S^1$ is the monoid obtained by adjoining an identity element 1 to $S$ [3, p.4].) Recall that the Green’s relation $J$ on $S$ is defined by: $aJb$ if $a, b \in S$ and $S^1aS^1 = S^1bS^1$; i.e., $a$ and $b$ are $J$-equivalent [3, p.48]. Semigroups which are finite and $J$-trivial have arisen in the study of formal languages [12], and in the context of full transformation semigroups [13]. Saito gives conditions for a periodic semigroup to be $J$-trivial [13, Lemma 1.1]. Observe that every semilattice (commutative semigroup in which every element is idempotent) is $J$-trivial, and that whenever $S$ is $J$-trivial, then so is $S^1$ and $S^0$. (Here $S^0$ is the semigroup with zero, 0, adjoined [3, p.4].) Not all bands are $J$-trivial. For example, let $S$ be a semigroup in which $ab = b$ for all $a, b \in S$; such a
A semigroup is called right zero [3, p.37]. Any right zero semigroup with more than one element is a band that is not $J$-trivial.

In this paper we show that $\mathbb{R}(R)$ is $J$-trivial if $R$ is either commutative, right duo (every right ideal of $R$ is two-sided), or nilpotent. The paper is arranged as follows. In Section 2 we consider conditions that imply $\mathbb{R}(R)$ is $J$-trivial. If $R$ is either right duo, commutative, nilpotent, or a skewfield, then $\mathbb{R}(R)$ is $J$-trivial. If $\mathbb{R}(R)$ is either 0-cancellative or has identity, then $\mathbb{R}(R)$ is $J$-trivial. In Sections 3, 4, and 5 we obtain results assuming $\mathbb{R}(R)$ is $J$-trivial, a hypothesis that is assumed for the remainder of this introduction. In Section 3 idempotent right ideals are shown to be ideals, maximal right ideals are considered, and the Jacobson and Brown-McCoy radicals of $R$ are shown to be equal. In Section 4 minimal right ideals are considered, subdirectly irreducible rings are classified, and it is shown that every idempotent is central. In Section 5 it is shown that $R$ r.w.r. implies $R$ is strongly regular and that $R$ $\pi$-regular implies $R$ is strongly $\pi$-regular.

2. Conditions which imply that $\mathbb{R}(R)$ is $J$-trivial

We first consider conditions on the ring $R$ which will imply that $\mathbb{R}(R)$ is $J$-trivial. For any skewfield $K$, the semigroup $\mathbb{R}(K)$ has only two elements, 0 and $K$, and $K$ is the identity for the semigroup. So $\mathbb{R}(K)$ is $J$-trivial.

Recall that a ring $R$ is right (left) duo if every right (respectively, left) ideal of $R$ is a two-sided ideal. Proposition 2.1. Let $R$ be a ring. Then we have the following.

(i) If $A, B \in \mathbb{I}(R)$ and $A \neq B$, then $A$ and $B$ are not $J$-equivalent in either $\mathbb{R}(R)$ or $\mathbb{L}(R)$.

(ii) $\mathbb{I}(R)$ is $J$-trivial.

(iii) If $R$ is right (left) duo, then $\mathbb{R}(R)$ (respectively, $\mathbb{L}(R)$) is $J$-trivial.

(iv) If $R$ is commutative, then $\mathbb{R}(R)$ and $\mathbb{L}(R)$ are both $J$-trivial.

Proof. Suppose $A, B \in \mathbb{I}(R)$ and $A$ and $B$ are $J$-equivalent in $\mathbb{R}(R)$. Then either $A = B$, $A = XB$, $A = BX$, or $A = XBY$ for some $X, Y \in \mathbb{R}(R)$. In each case $A \subseteq B$. Similarly, $B \subseteq A$, so $A = B$. Proceed similarly if $A, B$ are $J$-equivalent in $\mathbb{L}(R)$. This establishes part (i). Parts (ii) and (iii) follow immediately from (i), and (iv) follows immediately from (iii).

Note that for any commutative ring $A$ and any set $\Omega$ of commuting indeterminates, the polynomial ring $A[\Omega]$ and the ring of formal power series $A < \Omega >$ are each commutative and hence both $\mathbb{R}(A[\Omega])$ and $\mathbb{R}(A < \Omega >)$ are $J$-trivial.
Proposition 2.2. If \( R \) is nilpotent, then \( \mathbb{R}(R) \) and \( \mathbb{L}(R) \) are \( \mathcal{J} \)-trivial.

**Proof.** Let \( H, K \in \mathbb{R}(R) \) with \( HK = K \mathcal{J} R \). For convenience in calculation we operate in the semigroup with identity, 1, adjoined to \( \mathbb{R}(R) \). So \( H = XKY \) and \( K = BHT \), where \( X, Y, B, T \) are each in \( \mathbb{R}(R) \cup \{1\} \). A routine calculation establishes that \( H = (XB)^n H(TY)^n \), for all \( n \in \mathbb{N} \). If any one of \( X, B, T, \) or \( Y \) is not 1, then since \( H \) is nilpotent, by choosing \( n \) large enough we get \( H = 0 \). So \( K = 0 \). If \( X = Y = 1 \) we get \( H = K \). Thus \( \mathbb{R}(R) \) is \( \mathcal{J} \)-trivial. Similarly, \( \mathbb{L}(R) \) is \( \mathcal{J} \)-trivial. \( \square \)

Let \( \text{char } R = n > 1 \). Recall that \( R \) can be embedded as an ideal in the ring \( R^1 \), where \( R^1 \) is the set \( \mathbb{Z}_n \times R \) with the operations \((\alpha, r) + (\beta, t) = (\alpha + \beta, r + t), (\alpha, r)(\beta, t) = (\alpha\beta, \alpha t + \beta r), \) \( \alpha, \beta \in \mathbb{Z}_n, r, t \in R, \) and that \( R^1 \) has identity with \( \text{char } R^1 = n \) [2]. Observe that right ideals of \( R \) map onto right ideals of \( R^1 \) under the embedding mapping \( r \to (0, r) \). Identifying \( R \) with its image \( R^1 \) we see that \( \mathbb{R}(R) \subseteq \mathbb{R}(R^1) \). We refer to this embedding process as the Dorroh extension of \( R \) using \( \mathbb{Z}_n \), since it follows a procedure first used by J. Dorroh in [5].

Corollary 2.3. Let \( R \) be a nilpotent ring with \( \text{char } R = p \), where \( p \) is a prime. Then \( \mathbb{R}(R^1) = \mathbb{R}(R) \cup \{R^1\} \). Consequently, if \( \mathbb{R}(R) \) is \( \mathcal{J} \)-trivial, then \( \mathbb{R}(R^1) \) is \( \mathcal{J} \)-trivial.

**Proof.** As described above form the Dorroh extension of \( R \) using \( \mathbb{Z}_p \). Then \( \mathbb{R}(R) \cup \{R^1\} \subseteq \mathbb{R}(R^1) \). Let \( B \) be a nonzero right ideal of \( R^1 \) and let \( \alpha 1 + r = x \in B \), where \( \alpha \in \mathbb{Z}_p, r \in R \). If \( \alpha \not= 0 \), then \( \alpha^{-1} 1 + x = 1 + \alpha^{-1} r \). Since \( r \) is nilpotent, so is \( \alpha^{-1} r \). Consequently \( \alpha^{-1} x \) is a unit in \( R^1 \) and hence \( B = R^1 \). Thus \( \mathbb{R}(R) \cup \{R^1\} = \mathbb{R}(R^1) \). Using this and that \( \mathbb{R}(R) \) is \( \mathcal{J} \)-trivial it follows immediately that \( \mathbb{R}(R^1) \) is \( \mathcal{J} \)-trivial. \( \square \)

We next give an example to show that if \( \mathbb{R}(R) \) is \( \mathcal{J} \)-trivial, then \( R \) need not be right duo.

**Example 2.4.** Let \( K \) be any skewfield, and let \( R = \begin{bmatrix} 0 & K & K \\ 0 & 0 & K \\ 0 & 0 & 0 \end{bmatrix} \). Since \( R \) is nilpotent, then \( \mathbb{R}(R) \) is \( \mathcal{J} \)-trivial by Proposition 2.2. Further, the right ideal \( \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & K \\ 0 & 0 & 0 \end{bmatrix} \) is not two-sided, so that \( R \) is not right duo.
If the skewfield in Example 2.4 has characteristic $p$ for some prime $p$, then we can use Corollary 2.3 to embed the ring of Example 2.4 in a ring $R^1$ with identity and having that $R(R^1)$ is $J$-trivial.

We use $[B]$ for the ideal in the semigroup $R(R)$ generated by $B \in R(R)$.

**Proposition 2.5.** If $R(R)$ is $J$-trivial and $\overline{R}$ is a homomorphic image of the ring $R$, then $R(\overline{R})$ is $J$-trivial.

**Proof.** Let $\phi : R \to \overline{R}$ be a surjective ring homomorphism with $Ker \phi = I$. For notational convenience let $S = R(\overline{R})$. For any $C \in R(R)$ we use $\overline{C}$ for its image under $\phi$. Consider $H, K \in S$ with $HJK$. In general, from $HJK$ we have that $H = \alpha K \beta$ and $K = \gamma H \sigma$, where $\alpha, \beta, \gamma, \sigma \in S^1$. First consider the case where $H = X KY$ and $K = BH \overline{T}$. Then $H + I = (X + I)(K + I)(Y + I)$ and $K + I = (B + I)(H + I)(T + I)$. So $H + I \in [K + I]$ in $R(R)$, and $K + I \in [H + I]$ in $R(R)$. Since $R(R)$ is $J$-trivial, this yields $H + I = K + I$. Consequently $\overline{H} = \overline{K}$.

The other cases, where one or more of $\alpha, \beta, \gamma$, or $\sigma$ is 1, are either similar to the first case or easier.

**Example 2.6.** The homomorphic image of a $J$-trivial semigroup need not be $J$-trivial. Let $F = \langle 1, x, y \rangle$ be the free monoid generated by $x$ and $y$. This monoid is $J$-trivial. Let $B = \langle p, q \mid pq = 1 \rangle$ be the bicyclic semigroup. Then $B$ is a simple monoid, and hence any two right ideals are $J$-related. In particular, $B$ is not $J$-trivial. Define $\phi : F \to B$ by $\phi(1) = 1$, $\phi(x) = p$, $\phi(y) = q$. Then $B$ is a homomorphic image of $F$.

**Proposition 2.7.** If $R(R)$ has identity, then the identity is $R$ and $R(R)$ is $J$-trivial.

**Proof.** Let $X$ be the identity of $R(R)$. Let $H$ be a right ideal of $R$. Then $H = HX \subseteq HR \subseteq H$ which implies that $H = HR$ and hence $R$ is a right identity for $R(R)$. So $X = R$. In this case $R$ is right duo, and hence $R(R)$ is $J$-trivial by Proposition 2.1 (iii). \[\square\]

Note that in Proposition 2.7 one cannot replace “$R(R)$ has identity” with “$R$ has identity”. Any simple ring with identity and which is not a skewfield has that $R(R)$ is not $J$-trivial.

The converse of Proposition 2.7 is false. In the ring of Example 2.4, the right ideal...
Let \( H \) hence Example 2.11. Then the ideal \( r \) is not two-sided, so that \( R \) is not the identity of \( \mathbb{R}(R) \). Similarly, for \( n \geq 3 \) one can show that, in the \( n \times n \) strictly upper triangular matrix ring \( U \) over any skewfield, we have that \( \mathbb{R}(U) \) is \( J \)-trivial, but \( U \) is not the identity of \( \mathbb{R}(U) \).

We say that a semigroup \( S \) is left (right) \( 0 \)-cancellative if \( sx = sy \) \( (xs = ys) \) implies \( x = y \) for all non-zero \( s, x, y \in S \). The semigroup \( S \) is \( 0 \)-cancellative if \( S \) is both left and right \( 0 \)-cancellative. See \([3, p.3]\).

**Proposition 2.8.** If \( \mathbb{R}(R) \) is \( 0 \)-cancellative, then \( \mathbb{R}(R) \) and \( L(R) \) are each \( J \)-trivial.

**Proof.** Let \( H, K \in \mathbb{R}(R) \) with \( HJK \) in \( \mathbb{R}(R) \). If either \( H \) or \( K \) is zero, then both must be zero. So take \( H \) and \( K \) to be nonzero. From \( HJK \) we get that there exist \( X, Y, B, T \in \mathbb{R}(R) \) such that \( XHY = K \) and \( BKT = H \). Then \( K = XHY = X(BKT)Y \subseteq XKY = X(XHY)Y = X^2HY^2 \subseteq XHY = K \). So \( K = XKY \). Thus \( XKY = XHY \). Note that if either \( X \) or \( Y \) is zero, then \( K = 0 \).

So \( X \) and \( Y \) are nonzero. If \( X, Y \in \mathbb{R}(R) \), then using that \( \mathbb{R}(R) \) is \( 0 \)-cancellative and \( XHY = XKY \) we get \( K = H \). If \( X = Y = 1 \), then \( K = H \). If \( X = 1 \) and \( Y \in \mathbb{R}(R) \), then \( KY = HY \) and hence \( H = K \). Similarly, if \( X \in \mathbb{R}(R) \) and \( Y = 1 \), we get \( K = H \). Thus \( \mathbb{R}(R) \) is \( J \)-trivial. Proceed similarly to get \( L(R) \) is \( J \)-trivial. \( \square \)

Note that the converse of Proposition 2.8 is false, as the next example illustrates.

**Example 2.9.** Let \( A \) be any commutative ring and let \( R = A \oplus A \). Then \( \mathbb{R}(R) \) is not \( 0 \)-cancellative but \( \mathbb{R}(R) \) is \( J \)-trivial.

**Proposition 2.10.** Let \( R \) be a simple ring with \( R^2 \neq 0 \). Then either \( R \) is a skewfield or \( \mathbb{R}(R) \) is not \( J \)-trivial.

**Proof.** Assume \( R \) is not a skewfield and let \( H \in \mathbb{R}(R) \) with \( 0 \neq H \neq R \). If \( RH = 0 \), then the ideal \( r(R) = \{ x \mid Rx = 0 \} \) is nonzero and hence \( R = r(R) \), contrary to \( R^2 \neq 0 \). So \( RH = R \). Similarly \( HR \neq 0 \). Then \( H^2 \subseteq HR = H(RH) \subseteq H^2 \) and hence \( H^2 = HR \). Consequently \( H^2 \in [R] \). Also, \( R = RH^2 \), so \( R \in [H^2] \). Then \( R \not\in JH^2 \). Since \( H^2 \) is not \( R \) we have that \( \mathbb{R}(R) \) is not \( J \)-trivial. \( \square \)

**Example 2.11.** In Proposition 2.2 the hypothesis “\( R \) is nilpotent” cannot be replaced by “\( R \) is nil”. If \( R \) is a simple nil ring which is not nilpotent, then by Proposition 2.10 \( \mathbb{R}(R) \) is not \( J \)-trivial. Examples of such rings were first given by Smoktunowicz, see \([14]\).
As an immediate consequence of Proposition 2.10 we have that if $R$ is a simple ring with identity and $M_n(R)$ is the full $n \times n$ matrix ring over $R$, then $\mathbb{R}(M_n(R))$ is not $J$-trivial for $n > 1$.

Note that for any commutative ring $A$ and any set $\Omega$ of commuting indeterminates, the polynomial ring $A[\Omega]$ and the ring of formal power series $A < \Omega >$ are each commutative and hence both $\mathbb{R}(A[\Omega])$ and $\mathbb{R}(A < \Omega >)$ are $J$-trivial.

**Proposition 2.12.** If for some $m \in N$, $\mathbb{R}(R^m)$ is $J$-trivial, then $\mathbb{R}(R)$ is $J$-trivial.

**Proof.** For convenience of notation let $S = \mathbb{R}(R)$ and consider $H, K \in \mathbb{R}(R)$ with $[H] = [K]$ in $S$. Then there exist $X, Y, B, T \in S^1$ such that $H = XKY$ and $K = BHT$. A routine calculation shows that $H = (XB)^nH(TY)^n$ for $n \in N$. Choose $n = m$ to get $H \in \mathbb{R}(R^m)$. Similarly $K \in \mathbb{R}(R^m)$. Also, $H = (XB)^mH(TY)^m = ([XB]^mX)K(Y(TY))^m$, so $H$ is in the ideal in $\mathbb{R}(R^m)$ generated by $K$. Similarly, $K$ is in the ideal in $\mathbb{R}(R^m)$ generated by $H$. So $H, K$ in $\mathbb{R}(R^m)$. But $\mathbb{R}(R^m)$ is $J$-trivial, so $H = K$.

**Corollary 2.13.** If, for some $m \in N$, $R^m$ is right duo or commutative, then $\mathbb{R}(R)$ is $J$-trivial.

**Proposition 2.14.** Let $R = R_1 \oplus R_2$, where $R_1$ is a ring with $\mathbb{R}(R_1)$ $J$-trivial and $R_2$ is a nilpotent ring. Then $\mathbb{R}(R)$ is $J$-trivial.

**Proof.** The argument is similar to that for Proposition 2.12. Since $R_2$ is nilpotent, some power of $R$ is in $R_1$. Then $H$ and $K$ will be $J$-equivalent in $\mathbb{R}(R_1)$, and since $\mathbb{R}(R_1)$ is $J$-trivial we have $H = K$.

**Corollary 2.15.** Let $R = R_1 \oplus R_2$, where $R_1$ is a ring such that $\mathbb{R}(R_1^m)$ is $J$-trivial for some $m \in N$, and $R_2$ is nilpotent. Then $\mathbb{R}(R)$ is $J$-trivial.

Observe that $R = R_1 \oplus R_2$ will have $\mathbb{R}(R)$ is $J$-trivial when $R_2$ is nilpotent and $R_1^m$ is either commutative or right duo, for some $m$.

### 3. Maximal right ideals and radicals

Unless otherwise specified, for the remainder of the paper $R$ will have identity.

**Proposition 3.1.** Let $\mathbb{R}(R)$ be $J$-trivial.

(i) If $H \in \mathbb{R}(R)$ and $H = H^2$, then $H \in \mathbb{I}(R)$.

(ii) If $R$ is r.w.r., then $\mathbb{R}(R) = \mathbb{I}(R)$.

(iii) If $\mathbb{R}(R)$ is regular, then $\mathbb{R}(R) = \mathbb{I}(R)$. 
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Proof. (i) We have that $H = H^2 \subseteq HR \subseteq H$, which implies that $H = HR$. Since $H = HR$ we have $H = H^2 = (HR)H = H(RH)$. Thus $H \in [RH]$, and trivially $RH \in [H]$. So $[RH] = [H]$ and since $\mathbb{R}(R)$ is $J$-trivial we have $RH = H$.

(ii) This part follows immediately from part (i).

(iii) Every regular ring is r.w.r. [16, p.173].

Recall that a semigroup $S$ is periodic if for each $s \in S$ there exists $n, m \in \mathbb{N}, n > m$, such that $s^n = s^m$ [3, p.20].

Corollary 3.2. Let $\mathbb{R}(R)$ be $J$-trivial and periodic. If $H \in \mathbb{R}(R)$, then for some $k \in \mathbb{N}$, $H^k$ is an idempotent ideal. Consequently, each nonzero right ideal of $R$ is either nilpotent or contains a nonzero idempotent ideal of $R$.

Proof. Recall that each element in a periodic semigroup has a power which is an idempotent [3, p.20]. The desired result follows from this and from Proposition 3.1 (i).

Proposition 3.3. (i) If $M$ is a maximal right ideal of $R$, then either $M^2 = M$ or $M$ is an ideal of $R$.

(ii) If $\mathbb{R}(R)$ is $J$-trivial, then every maximal right ideal of $R$ is an ideal of $R$.

Proof. (i) Since $RM$ is an ideal of $R$ and $M \subseteq RM$ we have that either $RM = M$, and hence $M$ is a two-sided ideal of $R$, or $RM = R$. If the latter holds, then $M^2 = (MR)M = M(RM) = MR = M$.

(ii) Let $\mathbb{R}(R)$ be $J$-trivial and let $M$ be a maximal right ideal of $R$. Suppose $M$ is not an ideal of $R$. Then $RM = R$. Hence $R \in [M]$. So $[R] \subseteq [M]$, but, because $R$ has identity, $M = MR \in [R]$, which implies $[M] \subseteq [R]$. So $[R] = [M]$, and since $\mathbb{R}(R)$ is $J$-trivial we have $R = M$, a contradiction.

It is worth noting that from Proposition 3.3 (i) we see that in a ring with identity a maximal right ideal which is nilpotent must be a two-sided ideal.

Recall that because $R$ has identity the Jacobson radical of $R$, denoted by $J(R)$, is the intersection of all maximal right ideals of $R$, and the Brown-McCoy radical of $R$, denoted by $B(R)$, is the intersection of all maximal ideals of $R$ [15]. Neither of these results need hold for rings without identity [15].

Corollary 3.4. If $\mathbb{R}(R)$ is $J$-trivial, then $J(R) = B(R)$. If $J(R) = 0$, then $R$ is isomorphic to the subdirect product of skewfields.

Proof. That $J(R) = B(R)$ follows immediately from Proposition 3.3(ii). If $J(R) = 0$, then $B(R) = 0$ and $R$ is isomorphic to a subdirect product of rings with identity.
of the form \(R/M\), where the ideal \(M\) is also maximal as a right ideal of \(R\). So \(R/M\) has no proper nonzero right ideals and hence is a skewfield.

\[\square\]

4. Minimal right ideals

Recall that an idempotent \(e\) is left semicentral if \(ere = re\) for all \(r \in R\) [1].

**Proposition 4.1.** If \(\mathbb{R}(R)\) is \(J\)-trivial, then any idempotent in \(R\) is central.

**Proof.** Let \(e \in E(R)\). Since \(e \in ReR\) we have \(eR \subseteq ReR\) and hence \(eR \subseteq eReR \subseteq eR\), so \(eR = (eR)^2\). Then by Proposition 3.1(i) we have \(eR = ReR\). Then \(Re = Ree \subseteq ReR = eR\). So for each \(r \in R\) there exists \(y \in R\) such that \(re = ey\). Then \(ere = e^2y = ye = re\). Thus \(e\) is left semicentral and consequently \(1 - e\) is left semicentral. Let \(f \in E(R)\). Then \((ef - fe)e = 0\) and \((ef - fe)(1 - e) = ef - fe - (ef - fe)e = ef - fe\). Thus \(ef - fe = (ef - fe)(1 - e) = (1 - e)(ef - fe)(1 - e) = 0\). So \(e\) commutes with every idempotent of \(R\). It is well-known that this implies \(e\) is central in \(R\).

\[\square\]

**Proposition 4.2.** Let \(\mathbb{R}(R)\) be \(J\)-trivial. If \(B\) is a minimal right ideal of \(R\) and \(B^2 \neq 0\), then we have the following.

(i) \(B\) is an ideal of \(R\),

(ii) there exists a central idempotent \(e \in R\) such that \(B = eR\) and \(eR = Re = eRe\),

(iii) \(R = eR \oplus (1 - e)R = eRe \oplus (1 - e)R\) and \(eRe\) is a skewfield, so \((1 - e)R\) is an ideal of \(R\) which is maximal as a right (left) ideal of \(R\).

**Proof.** (i) Since \(0 \neq B^2 \subseteq B\), by minimality of \(B\) we get \(B^2 = B\). So by Proposition 3.1(i), \(B\) is an ideal of \(R\).

(ii) It is well-known that any non-nilpotent minimal right ideal is generated by an idempotent [11, Section 31]. So there exists \(e \in E(R)\) such that \(B = eR\). By Proposition 4.1, \(e\) is central.

(iii) Since \(eR\) is a minimal right ideal of \(R\) we have that \(eRe\) is a skewfield [11, Theorem 3.16]. Using the Pierce decomposition with \(e\) we have \(R = eR \oplus (1 - e)R\), and this is a direct sum of two-sided ideals of \(R\). From \(eRe = eR \cong R/(1 - e)R\), and since \(eRe\) is a skewfield, then \((1 - e)R\) is maximal as a right (left) ideal of \(R\).

\[\square\]

**Corollary 4.3.** Let \(\mathbb{R}(R)\) be \(J\)-trivial. If \(R\) has a minimal right ideal which is not nilpotent, then \(R = R_1 \oplus R_2\) where \(\mathbb{R}(R_1)\) and \(\mathbb{R}(R_2)\) are \(J\)-trivial.
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Proof. From Proposition 4.2(iii) we have $R = eR \oplus (1-e)R$, where $eR$ and $(1-e)R$ are ideals of $R$. Use $R/eR \cong (1-e)R$ and Proposition 2.5 to get that $R((1-e)R)$ is $J$-trivial. Similarly, $R(eR)$ is $J$-trivial.

Proposition 4.4. Let $R$ be a subdirectly irreducible ring (not necessarily having identity) and let $H$ be the heart of $R$. Assume $H^2 \neq 0$ and that $\mathbb{R}(R)$ is $J$-trivial. If $R$ contains a minimal right ideal $B$ of $R$ with $B \subseteq H$, then $R$ is a skewfield.

Proof. It is well-known that the non-nilpotent heart of a subdirectly irreducible ring must itself be a simple ring [4, p.135]. So $H$ is a simple ring. If $B^2 = 0$, then the ring $H$ must contain a non-zero nilpotent ideal. Consequently this ideal is $H$ itself, contrary to $H^2 \neq 0$. So $B^2 \neq 0$. Use Proposition 4.2 to get that $H$ is a skewfield. So the ring $H$ has an identity element, which forces $H = R$, and hence $R$ is a skewfield.

Corollary 4.5. Let $R$ be a subdirectly irreducible ring (not necessarily having identity) with heart $H$, $H^2 \neq 0$. If $\mathbb{R}(R)$ is $J$-trivial and $R$ is right Artinian, then $R$ is a skewfield.

Proof. The chain condition yields the existence of a minimal right ideal $B$ of $R$ with $B \subseteq H$.

Example 4.6. The ring in Example 2.4 is subdirectly irreducible with heart $H = \begin{bmatrix} 0 & 0 & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

5. Regularity conditions

Let $E(R)$ denote the set of idempotents of $R$. Recall that a ring $R$ is strongly regular if $R$ is regular and every idempotent of $R$ is central [6].

Theorem 5.1. If $R$ is r.w.r. and $\mathbb{R}(R)$ is $J$-trivial, then $R$ is strongly regular.

Proof. Let $B \in \mathbb{R}(R)$. Then $B = B^2 = (BR)R = B(RB)$. So $B \in [RB]$. Since trivially $RB$ is in $[B]$, we then have $[B] = [RB]$ and consequently $B = RB$. So each right ideal of $R$ is a two-sided ideal. It is known that a r.w.r. ring with this property is a regular ring [7]. By Proposition 4.1 we have that every idempotent of $R$ is central. Therefore, $R$ is strongly regular.
Corollary 5.2. The following are equivalent:

(i) $R$ is r.w.r. and $\mathbb{R}(R)$ is $J$-trivial,
(ii) $R$ is regular and $\mathbb{R}(R)$ is $J$-trivial,
(iii) $R$ is strongly regular,
(iv) $\mathbb{R}(R)$ is a semilattice.

Proof. The equivalence of (i), (ii), and (iii) is clear from the proof of Theorem 5.1. The equivalence of (iii) and (iv) is given in [7]. Any semilattice is a band and is $J$-trivial, so (iv) implies (i), completing the logical circuit.

Note that for a skewfield $K$, the ring is $M_n(K)$ is regular, and hence r.w.r., but for $n > 1$, $\mathbb{R}(M_n(K))$ is not $J$-trivial.

Recall that $R$ is $\pi$-regular if for each $r \in R$ there exists $b \in R$ such that $r^n br^n$, and $R$ is strongly $\pi$-regular if for each $r \in R$ there exists $m \in N$ such that $r^n = r^{n+1} y$ for some $y \in R$ [16, Section 23]. It is known that every strongly $\pi$-regular ring is $\pi$-regular, but there are $\pi$-regular rings that are not strongly $\pi$-regular [16, Theorem 23.4].

Proposition 5.3. Let $\mathbb{R}(R)$ be $J$-trivial. Then $R$ is $\pi$-regular if and only if $R$ is strongly $\pi$-regular.

Proof. Since all strongly $\pi$-regular rings are $\pi$-regular, it suffices to show that $\pi$-regular implies strongly $\pi$-regular when $\mathbb{R}(R)$ is $J$-trivial. Let $R$ be $\pi$-regular and let $r \in R$. Then $r^n = r^n br^n$, for some $n \in N$, $b \in R$. Observe that $r^n b$ is idempotent, so by Proposition 4.1, $r^n b$ is central and hence $r^n = r^{2n} b \in r^{n+1} R$. So $R$ is strongly $\pi$-regular.

Note that the hypothesis that $R$ is $\pi$-regular and $\mathbb{R}(R)$ is $J$-trivial does not imply that $R$ is r.w.r., as the example of any nonzero nilpotent ring shows.

References


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